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APPROXIMATE ALGORITHM FOR CONSTRUCTING OF OPTIMAL RELIABLE SYSTEM WITH ARBITRARY STRUCTURE*

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Abstract

Some functioning quality index (effectiveness index) for systems with arbitrary structure is defined. It is proposed an approximate algorithm of the optimal distribution of a system "weight" (or another factor of restriction) between its elements for effectiveness index maximization. The algorithm is based on the steepest descent method which is used in tasks of the convex programming with linear restriction.

Problems concerning the optimal distribution of resource appears if one wants to receive the highest effectiveness index of complex systems. Quality of complex systems functioning is evaluated by indexes depending on a type of the system, on its purposes and on the criteria of a satisfactory system functioning. We usually have some restrictions. For example, the system cost may be the restricted resource for one type of systems and the system weight may be for another. Let us call any restricted resource by name "weight" independently on its concrete substance.

A system consisted of n elements is considered. It is assumed that every i-th element may have the only two states: the state of the good

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working order (let us mark it by $S_i=1$) and the state of failure (let us mark it by $S_i=0$).

Then at any arbitrary fixed moment the system is in one of 2^n different states $S=(S_1, S_2, \cdots, S_n)$, where S_i is 0 or 1.

The general method of an evaluation of the effectiveness index of complex systems, in respect that its elements are subjected to failures, is given in [1]:

$$F=\sum_S H_S \phi_S \tag{1}$$

Here H_S is a probability that the system is in the S-th state and ϕ_S is a system conditional effectiveness index inherent in the S-th state. (The summation is all over 2^n means of symbol S).

One can simply calculate the probability H_S if all elements are supposed independent

$$H_S=\prod_{i=1}^{n} r_i^{S_i}(I-r_i)^{I-S_i} \tag{2}$$

Here r_i is a probability that at any fixed moment of time the i-th element is in a working state.

The choice of an index ϕ_S provides us with the system effectiveness index F, whichever one likes. Naturally, the indexes ϕ_S do not depend on values r_i.

Let us consider a system which elements one can make in some different variants. For example, the i-th element one can make in the variants $i_1, i_2 \cdots, i_j, \cdots$. The j-th variant of the i-th element is characterized by two indexes: the first one is a reliability $r_i(i_j)$ and the second one is a "weight" $g_i(i_j)$. On principle, the number of different variants may be infinite. So redundancy leads to the infinite countable set of the variants.

One can improve the effectiveness index of systems by some essential different ways: the first way is a change of the system structure or principles of its functioning and the second one is an improvement of

the elements reliability without any system structure change. The latest way is more simple one and in most cases it is the only way at the completing phase of system design. In this case it is interesting to consider the task of the optimal allocation of the full system "weight" between its elements to maximize the system effectiveness index [2].

First let us define how the effectiveness index F depends on the reliability of each system element. The set of the system states one can divided to two subsets: the first one includes all states such that $S_i=1$ for some fixed i-th element and the second one includes all states such that $S_i=0$ for the same element. Then using the expression (2) we have from (1)

$$F=r_i(\sum_{S^*} H_{S^*}(\phi_{S^*,1}-\phi_{S^*,0})+\sum_{S^*} H_{S^*}\phi_{S^*,0}\,. \tag{3}$$

Here S^* is a system state without respect to the i-th element, $\{S^*,0\}$ and $\{S^*,1\}$ are the system states characterized by S^* and $S_i=0$ or $S_i=1$ respectively.

Thus from (3) we can make a conclusion that the effectiveness index F is a linear function of r_i.

The next preliminary step of the investigation is to study a dependence of the element reliability from its "weight". All variants of the i-th element design may be arranged in the set accordingly with their values of $r_i(i_j)$.

We call a variant as expedient one if the following condition is satisfied

$$r_i(i_{j+1})>r_i(i_j) \quad \text{if} \quad g_i(i_{j+1})>g_i(i_j)$$

(In other words, the more weight of the element variant the more reliability). If the condition is not satisfied the variant is called as non-expedient one. We shall consider only the expedient variants. Under certain conditions function $r_i(g)$ for expedient variants is strictly convex. (For example, it takes place when one uses a redundancy for improvement of the element reliability). If for some j and 1 the set of

the expedient variants is characterized by the following inequality

$$\frac{r_i(i_{j+1})-r_i(i_j)}{g_i(i_{j+1})-g_i(i_j)} > \frac{r_i(i_{j+m})-r_i(i_j)}{g_i(i_{j+m})-g_i(i_j)}$$

for all $0<m<1$, *i.e.* $r_i(g)$ has local concavities, one can expel points i_{j+m} for practical evaluations. In this case the function $r_i(g)$ is the convex broken-line "drawn on" the set of the points $r_i(g)$.

In this assumption a new function F is a strictly convex function for all arguments g_i.

Thus, we can pose two optimal tasks:

a) to find

$$\text{Sup} \quad F(g_i(i_j); \quad i=1,2,\cdots,n; \quad j=1,2,\cdots).$$

$$g_i: \quad \sum_{i=1}^{n} g_i \leqslant g_0$$

b) to find

$$\text{Inf} \quad g(r_i(i_j); \quad i=1,2,\cdots,n; \quad j=1,2,\cdots). \tag{5}$$

$$r_i: \quad F(r_{i;}\, i=1,2,\cdots,n) \geqslant F_0$$

Here g_0 and F_0 mark maximum admitted "weight" of the system and minimum admitted effectiveness index of the system, respectively.

These two tesks are typical tasks of the convex programming with the linear restrictions. Therefore one can use the steepest descent method to solve the problems. We suggest the following practical algorithm of the optimal procedure.

First the system variant having minimum of the index $F^{(0)}$ is chosen. This variant has, in our suppositions, the minimum "weight"

$$g^{(0)}=\sum_{i=1}^{n} g_i(i_0)$$

Then we have to find the element which reliability improvement is the most expedient for maximization of the system effictiveness index. At the first step of the optimal process we calculate values

$$\gamma_i^{(I)}=\frac{F_i^{(1)}-F^{(0)}}{g_i^{(1)}-g^{(0)}}, \quad i=1,2,\cdots,n,$$

where $F_i^{(1)}$ and $g_i^{(1)}$ are values of the effectiveness index and the system "weight" respectively under the condition that the variant i_0 substitutes for the variant $i_1(i=1,2,\cdots,n)$.

Then the k-th element having

$$\gamma_k^{(1)}=\underset{1\leq i\leq n}{\mathrm{Max}} \quad \gamma_i^{(1)}; \quad (i=1,2,\cdots,n)$$

is found. The variant k_0 of that element substitutes for the variant k_1. It is supposed that the initial state before the second step of the optimal process is characterized by $F^{(1)}=F_k^{(1)}$ and $g^{(1)}=g_k^{(1)}$. The same process is continued, *i.e.* we obtain the values

$$\gamma_i^{(2)}=(F_i^{(2)}-F^{(1)}/(g_i^{(2)}-g^{(1)}) \text{ for all } i=1,,\cdots 2,n \text{ and so on.}$$

To the N-th step of the process the system consists of the elements variants: $1_{j_1(N)}, 2_{j_2(N)}, \cdots n_{j_n(N)}$ and its "weight" is equal to

$$g^{(N)}=\sum_{i=1}^{n} g_i(i_{j(N)}). \tag{7}$$

It is clear that $N=\sum_{i=1}^{n} j_i(N)$.

We calculate the values of $\gamma_i^{(N+1)}$ by using of the expressions (3) and (7):

$$\gamma_i^{(N+1)}=\frac{r_i(i_{j(N+1)}-r_i(i_{j(N)})}{g_i(i_{j(N+1)})-g_i(i_{j(N)})}\sum_{S^*}(\phi_{S^*,1}-\phi_{S^*,0})H_{S^*} \tag{8}$$

If the reliability of the system elements is closed to unit (*i.e.* $1-r_i\ll 1/n$ for all $i=1,2,\cdots,n$), one can obtain the approximate expression

$$\sum_{S^*}(\phi_{S^*,1}-\phi_{S^*,0})H_{S^*}\approx \sum(\phi_E-\phi_{E^*,S_k=0})\ (I-r_k) \tag{9}$$

Here ϕ_E is the conditional system effectiveness index under the condition that all elements are working, *i.e.* $S_1=1, S_2=1,\cdots,S_n=1, \phi_{E^*}, S_k=0$ is the

conditional system effectiveness index under the condition that all elements are working with the exception of the k-th one, *i.e.* $S_1=1, S_2=1, \cdots, S_{k-1}=1, S_k=0, S_{k+1}=1, \cdots, S_n=1$. To solve this optimal problem we suggest the following practical method.

a) We describe all variants of design for the i-th element $(i=1, 2, \cdots, n)$:

$$i_1, i_2, \cdots, i_j, \cdots$$

b) We calculate the indexes of reliability and "weight" ($r_i(i_j)$ and $g_i(i_j)$ respectively) for all variants of any elements.

c) We make the following table where the expedient variants are arranged in certain order

Table 1.

number of variant	element 1		element 2		$\cdots$	element n	
	$r_1(1_j)$	$g_1(1_j)$	$r_2(2_j)$	$g_2(2_j)$	$\cdots$	$r_n(n_j)$	$g_n(n_j)$
0	$r_1(1_0)$	$g_1(1_0)$	$r_2(2_0)$	$g_2(2_0)$	$\cdots$	$r_n(n_0)$	$g_n(n_0)$
1	$r_1(1_1)$	$g_1(1_1)$	$r_2(2_1)$	$g_2(2_1)$	$\cdots$	$r_n(n_1)$	$g_n(n_1)$
2	$r_1(1_2)$	$g_1(1_2)$	$r_2(2_2)$	$g_2(2_2)$	$\cdots$	$r_n(n_2)$	$g_n(n_2)$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$

d) The gradients $\gamma_1^{(1)}, \gamma_2^{(1)}, \cdots$ for the first step are calculated:

$$\gamma_1^{(1)} \approx \frac{r_1(1_1)-r_1(1_0)}{g_1(1_1)-g_1(1_0)} \sum_{k=2}^{n} (\phi_E - \phi_{E^*, S_k=0}) \ (I-r_k(k_0)),$$

$$\gamma_2^{(1)} \approx \frac{r_2(2_1)-r_2(2_0)}{g_2(2_1)-g_2(2_0)} \sum_{\substack{k=1 \\ k \neq 2}}^{n} (\phi_E - \phi_{E^*, S_k=0}) \ (1-r_k(k_0))$$

and so on.

e) We find the maximum gradient. For example, if it is $\gamma_1^{(1)}$, the variant 1_0 of the element is substituted by the variant 1_1.

f) We calculate the gradients $\gamma_1^{(2)}, \gamma_2^{(2)}, \cdots$ for the second step.

g) We determine the maximum of the obtained gradients. Then we substitute the last variant of the respective element by the next one from the table.

The values of the system effectiveness index F and system "weight" g are calculated on every step of the process. The process continues to the moment when the conditions (4) or (5) are satisfied.

Conclusion

Let us note that likewise optimal problems of the reliability theory were solved in several papers (for example, [3-7]) for less general systems (usually for an ordinary redundancy) but for several restrictions. The most of these papers give the exact algorithms.

The considered problem concerns with the more general type of systems and the more general type of the effectiveness index but the algorithm is approximate.

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