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ON BULK QUEUES WITH STATE DEPENDANT PARAMETERS

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Abstract

The problem considered in this paper is the single server bulk service queuing system characterized by (i) Hyper-Poisson arrival time distribution with k branches, the mean arrival rates depending upon the state of the system; (ii) first come, first served queue discipline; (iii) exponential service time distribution, the mean service rate depending upon the state of the system; and (iv) finite waiting space. The functions defining the dependance of the mean arrival and service rates upon the state of the system are assumed to be arbitrary functions. The server serves units in batches of fixed size S or the whole queue, whichever is less. Assuming steady state conditions to exist, a recurrence relation connecting the various probabilities introduced is found out. By specifying a few particular forms of the state functions, in order to study the effect of reneging and balking, graphs depicting two queue characteristics, viz., (i) probability of no delay, and (ii) mean number of units in the queue are drawn. The corresponding recurrence relations for the queuing system M/M/1 with bulk service and state dependant parameters have also been deduced.

Introduction

Bulk service queuing problems have been considered by a number of workers, the first to consider such a problem being Bailey [5]. Among other workers who studied similar problems, mention may be made of Downton [7], Miller [15], Jaiswal [10, 11, 12], Foster and Nyunt [8], Takace [17], Kailson [14], Arora [4] etc. But all these workers have assumed that the mean arrival and service rates are constant. The main purpose of the present paper is to relax this assumption. We thus assume that the mean arrival and service rates depend arbitrarily upon the state of the system. The input distribution is supposed to be Hyper-Poisson with k independent branches.

State dependant parameters are encountered in a number of queuing processes. For instance, the effect of assuming a finite source from which the units arrive at a service facility to be served is to introduce state dependant parameters, *e.g.* ' machine interference problem,' (see [1]). Also if we introduce more than one server, or allow balking and/or reneging, we are essentially introducing state dependant parameters (see [16], [1, 2, 3], [13]). Among other workers who have considered state dependant parameters, we may mention Conway and Maxwell [6]; Hiller, Conway and Maxwell [9] etc. But in all these papers one finds that the functions definding the dependance of the mean arrival and/or service rates upon the state of the system are particular functions. In this paper we assume that these functions are arbitrary functions.

It may also be of some interest to note here that in almost all queuing processes considered so far, an assumption is intentionally made that the interarrival times and the service times are distributed independently of one another. By allowing the queue parameters to depend upon the state of the system we are in a way relaxing this assumting also.

One other important difference to be noted is that while studying bulk service, we cannot deduce the results for multiple server bulk service system even after we have obtained results with arbitrary state functions whereas in the ordinary case when units are served singly, this is obviously possible.

To solve the problem (formulated precisely in the next session) we no more can use the powerful method of generating functions. We thus resort to a rather heuristic technique and obtain a recurrence relation from which the various probabilities may be calculated and hence the queue characteristics. By assuming some particular forms of the state functions, in order to study the effect of reneging in the general case and of balking and reneging in case of M/M/1 queuing system with bulk service, we have calculated these probabilities in a number of cases and present here only the graphs for the queue characteristics (i) probability of no delay, and (ii) mean number of units in the queue.

Formuration of the Problem

Suppose that units arrive at a service facility which serves the units in batches of fixed size S or the whole queue, whichever is less, at random. The service time distribution of the batches is assumed to be negative exponential with parameter $\mu(n)$ when there are n units in the queue, *i.e.* the first order probability that a batch is served in time dt is $\mu(n)\ dt$ when there are n units in the queue. The arrival channel consists of k independent branches. A unit arriving for service enters the rth branch a fraction σ_r of the time on the average, so that $\sum_{r=1}^{k}\sigma_r=1$. Only one unit can enter any branch at a time and if a unit is present in any one of the k branches, no other unit can enter any other branch. We further assume the existence of a reservoir of infinite capacity attached with the arrival channel which emits a unit as soon as the arrival channel is free, so that the arrival channel is never free. The unit in the rth branch enters the system (queue or service) at the rate $\lambda_r(n)$ per unit time when there are n units already in the queue, *i.e.* the first order probability that a unit enters the system from the rth branch of the arrival channel in time dt is $\lambda_r(n)\ dt$ when there are n units in the queue. The queue discipline is assumed to be first come, first served. The maximum number of units in the system is allowed to be N, so that when a unit comes from the arrival channel and finds N units in the system, it is not allowed to join the queue and it overflows, *i.e.* is lost to the system.

Continuity Equations and their Solution

Let $p(n, r)$ denote the steady state probability that there are n units in the queue, the unit in the arrival channel being in the rth branch. Also let $q(0, r)$ denote the probability that there is no unit in the system, the unit in the arrival channel being in the rth branch. Assuming steady state conditions to obtain, these probabilities satisfy the following continuity equations:

$$-[\lambda_r(n)+\mu(n)]\,p(n, r)+\sigma_r\sum_{j=1}^{k}\lambda_j(n-1)\,p(n-1, j)+\mu(n+S)\,p(n+S, r)=0 \qquad (1)$$

$$(n=1, 2, \cdots, N-S)$$

$$-[\lambda_r(0)+\mu(0)]\,p(0, r)+\sigma_r\sum_{s=1}^{k}\lambda_s(0)\,q(0, s)+\sum_{l=1}^{S}p(l, r)\,\mu(l)=0 \qquad (2)$$

$$-\lambda_r(0)\,q(0, r)+\mu(0)\,p(0, r)=0 \qquad (3)$$

$$-[\lambda_r(m)+\mu(m)]\,p(m, r)+\sigma_r\sum_{j=1}^{k}\lambda_j(m-1)\,p(m-1, j)=0 \qquad (4)$$

$$(m=N-S+1, \cdots, N-1)$$

$$-[\lambda_r(N)+\mu(N)]\,p(N, r)+\sigma_r\sum_{j=1}^{k}\lambda_j(N-1)\,p(N-1, j)+\sigma_r\sum_{j=1}^{k}\lambda_j(N)\,p(N, j)=0 \qquad (5)$$

where the equations (1) through (5) are valid for $r=1, 2, \cdots, k$.

From equation (3), we get

$$q(0, r)=\frac{\mu(0)\,p(0, r)}{\lambda_r(0)} \qquad (6)$$

Substituting the value of $q(0, r)$ from (6) in (2), we get the matrix equation

$$AP=Q \qquad (7)$$

where A is the matrix given by

$$A=||a_{ij}||$$

such that

$$a_{ij}=-\mu(0)\,\sigma_i, \qquad i\neq j$$

$$a_{ii}=\lambda_i(0)+\mu(0)\,(1-\sigma_i)$$

and $\boldsymbol{P}$ and $\boldsymbol{Q}$ are the column matrices given by

$$\boldsymbol{P}=[p(0,1), p(0,2), \cdots p(0,k)]$$

and

$$\boldsymbol{Q}=[\sum_{l=1}^{S} \mu(l)\,p(l,1), \sum_{l=1}^{S} \mu(l)\,p(l,2), \cdots, \sum_{l=1}^{S} \mu(l)\,p(l,k)]$$

respectively.

We observe that $\boldsymbol{A}^{-1}$ is given by

$$A^{-1}=||a'_{ij}||$$

such that

$$a'_{ij}=\frac{\mu(0)\,\sigma_i}{B A_i A_j}, \qquad i\neq j$$

$$a'_{ii}=\frac{B+\sigma_i/A_i}{BA_i}$$

where

$$A_i=\lambda_i(0)+\mu(0)$$

and

$$B=1-\mu(0)\sum_{r=1}^{k}\sigma_r/A_r.$$

Pre-multiplying both sides of equation (7) by $\boldsymbol{A}^{-1}$, we obtain for $i=1,2,\cdots,k$.

$$p(0, i)=\frac{1}{A_i}\left(\sum_{l=1}^{S}\mu(l)\,p(l, i)+\frac{\mu(0)\,\sigma_i\sum_{l=1}^{S}\sum_{j=1}^{k}\mu(l)\,p(l, j)/A_j}{B}\right) \tag{8}$$

Now, summing up equation (1) over r and rearranging terms, we get

$$\mu(n)\sum_{r=1}^{k}p(n, r)-\sum_{r=1}^{k}\lambda_r(n-1)\,p(n-1, r)=\mu(n+S)\sum_{r=1}^{k}p(n+S, r)-\sum_{r=1}^{k}\lambda_r(n)\,p(n, r) \tag{9}$$

Summing up equation (9) for $n=1, 2, \cdots, n-1$, we get

$$\sum_{r=1}^{k}\lambda_r(n-1)\,p(n-1, r)=\sum_{i=n+1}^{n+S-1}\sum_{r=1}^{k}\mu(i)\,p(i, r)+\sum_{r=1}^{k}\lambda_r(n)\,p(n, r) \tag{10}$$

where we have used the fact that

$$\sum_{l=1}^{S}\sum_{r=1}^{k}\mu(l)\,p(l, r)=\sum_{r=1}^{k}\lambda_r(0)\,p(0, r) \tag{11}$$

in view of (2) and (3).

Using the value of $\sum_{r=1}^{k}\lambda_r(n-1)\,p(n-1, r)$ from (10) in (1), we get

$$[\lambda_r(n)+\mu(n)]\,p(n, r)-\mu(n)\,\sigma_r\sum_{r=1}^{k}p(n, r)=B_{r, n} \tag{12}$$

where

$$B_{r, n}=\sigma_r\sum_{i=n+1}^{n+S-1}\sum_{r=1}^{k}\mu(i)\,p(i, r)+\mu(n+S)\,p(n+S, r).$$

The matrix formed by the coefficients of $p(n, i)$ in (12) is of the same form as the matrix A in the calculation of $p(0, i)$, and therefore, as there, we have the solution

$$p(n, i)=\frac{1}{C_{i, n}}\left(B_{i, n}+\frac{\mu(n)\,\sigma_i\sum_{j=1}^{k}(B_{j, n}/C_{j, n})}{D_n}\right) \tag{13}$$

$$(n=1, 2, \cdots, N-S;\ i=1, 2, \cdots, k)$$

where

$$C_{i,\,n}=\lambda_i(n)+\mu(n),$$

and

$$D_n=1-\mu(n)\sum_{j=1}^{k}\sigma_j/C_{j,\,n}.$$

Now summing up equation (4) over r, we get

$$\sum_{j=1}^{k}\lambda_j(m-1)\,p(m-1,j)=\sum_{j=1}^{k}C_{j,\,m}\,p(m,j) \tag{14}$$

Substituting the value of $\sum_{j=1}^{k}\lambda_j(m-1)\,p(m-1,j)$ from (14) in (4), we get

$$-C_{r,\,m}\,p(m,r)+\sigma_r\sum_{j=1}^{k}C_{j,\,m}\,p(m,j)=0 \tag{15}$$

$$(m=N-S+1,\cdots,N-1)$$

For $m=N-S+1$, equation (15) becomes

$$\boldsymbol{CR}=-C_{k,\,N-S+1}\,p(N-S+1,k)\,\boldsymbol{S} \tag{16}$$

where $\boldsymbol{C}$ is the $(k-1)\times(k-1)$ matrix

$$\boldsymbol{C}=||c_{ij}||$$

such that

$$c_{ij}=\sigma_i\,C_{j,\,N-S+1}\qquad i\neq j$$

$$c_{ii}=(\sigma_i-1)\,C_{i,\,N-S+1}$$

and $\boldsymbol{R}$ and $\boldsymbol{S}$ are the column matrices

$$\boldsymbol{R}=[p(N-S+1,1),\cdots,p(N-S+1,k-1)]$$

and

$$\boldsymbol{S}=[\sigma_1,\sigma_2,\cdots\sigma_{k-1}].$$

Now, C^{-1} is given by

$$C^{-1}=||c'_{ij}||$$

such that

$$c'_{ij}=-\frac{\sigma_i}{\sigma_k C_{i,\,N-S+1}}, \qquad i\neq j$$

$$c'_{ii}=-\frac{(\sigma_i+\sigma_k)}{\sigma_k C_{i,\,N-S+1}}$$

Pre-multiplying (16) by C^{-1}, we obtain

$$p(N-S+1,r)=\frac{\sigma_r C_{k,\,N-S+1}\, p(N-S+1,k)}{\sigma_k C_{r,\,N-S+1}} \tag{17}$$
$$(r=1,2,\cdots,k-1)$$

Now, adding all the equations represented by (1), (2) and (4), we get

$$\sum_{r=1}^{k}\lambda_r(N-1)\,p(N-1,r)=\mu(N)\sum_{r=1}^{k}p(N,r) \tag{18}$$

Substituting back the value of $\sum_{r=1}^{k}\lambda_r(N-1)\,p(N-1,r)$ from (18) in (5), we get

$$\sigma_r\sum_{j=1}^{k}C_{j,\,N}\,p(N,j)-C_{r,\,N}\,p(N,r)=0 \tag{19}$$

Equation (19) may be solved for $p(N,r)$ exactly as we solved $p(N-S+1,r)$ above, because the matrix formed by the coefficients is similar. The solution, therefore, is

$$p(N,r)=\frac{\sigma_r C_{k,\,N}\, p(N,k)}{\sigma_k C_{r,\,N}} \tag{20}$$
$$(r=1,2,\cdots,k-1)$$

Also from equation (5), we have

$$p(N,k)=\frac{\sigma_k\sum_{j=1}^{k}\lambda_j(N-1)\,p(N-1,j)}{C_{k,N}-C_{k,N}\sum_{i=1}^{k}\lambda_j(N)\sigma_j/C_{j,N}} \tag{21}$$

Also from equation (4), we have

$$p(m,r)=\frac{\sigma_r\sum_{j=1}^{k}\lambda_j(m-1)\,p(m-1,j)}{C_{r,m}} \tag{22}$$
$$(m=N-S+1,\cdots,N-1)$$

Thus, we have expressed all the probabilities, $p(n,r)$ and $q(0,r)$ for $n=1,2,\cdots,N$; $r=1,2,\cdots,k$ in terms of one unknown probability, namely $p(N-S+1,k)$. This unknown probability may now be evaluated by using the normalizing condition, *i.e.*

$$\sum_{n=0}^{N}\sum_{r=1}^{k}p(n,r)+\sum_{r=1}^{k}q(0,r)=1 \tag{23}$$

and hence all the probabilities are completely known.

The recurrence relations contained in equations (6), (8), (13), (17), (20), (21), (22) and (23) are the required recurrence relations which help us evaluate the probabilities uniquely.

Queuing System M/M/1

If in the recurrence relations obtained above, we substitute

$$\sigma_r=\delta_{rk},\qquad \sum_{i=1}^{k}p(n,i)=p(n),\qquad q(0)=\sum_{r=1}^{k}q(0,r)$$

we obtain the recurrence relations for the queuing system M/M/1 with bulk service and state dependant parameters. These relations are:

$$q(0)=\frac{\mu(0)\,p(0)}{\lambda(0)} \tag{24}$$

$$p(0)=\frac{\sum_{l=1}^{s}\mu(l)\,p(l)}{\lambda(0)} \tag{25}$$

$$p(n)=\frac{\sum_{i=n+1}^{n+s}\mu(i)\,p(i)}{\lambda(n)} \tag{26}$$

$(n=1, 2, \cdots, N-S)$

$$p(m)=\frac{\lambda(m-1)\,p(m-1)}{\lambda(m)+\mu(m)}=p(N-S)\prod_{i=N-S}^{m-1}\alpha_i \tag{27}$$

$(m=N-S+1, \cdots, N-1)$

where

$$\alpha_i=\frac{\lambda(i-1)}{\mu(i)+\lambda(i)}$$

$$p(N-S)=\frac{\mu(N)\,p(N)}{\lambda(N-s)-\sum_{m=N-S+1}^{N-1}\left(\mu(m)\prod_{i=N-S}^{m-1}\alpha_i\right)} \tag{28}$$

$$\sum_{i=0}^{N}p(i)+q(0)=1 \tag{29}$$

Numerical Calculations

The recurrence relations for both the queuing systems were tried on IBM 1620 digital computer and in a number of cases the probabilities were evaluated. Here we present the graph for

(i) probability of no delay, *i.e.* $\sum_{r=1}^{k}q(0, r)$

(ii) mean number of units in the queue

by assuming that

$\lambda_r(n)=k\,\sigma_r\,\lambda(1-n/N)$, balking

$\mu(n)=\mu+(n-1)\,\alpha$, reneging

or

$$\left.\begin{aligned}\mu(n)&=\mu \text{ for } n=1,2,3,4,5\\ &=\mu+(n-1)\,\alpha \text{ for } n>5\end{aligned}\right\}\quad\text{partial reneging,}$$

and $\mu=6, \alpha=1,2, N=20, k=3, S=5, \sigma_1=.22, \sigma_2=.33, \sigma_3=.45$.

From this numerical work, it is observed that for small values of λ the probability of no delay, p_0 say, is always high, whether the queue parameters depend upon the state of the system or not. For large values of λ, p_0 is too small in all cases. A difference is noted in the value of P_N, *i.e.* probability that the system is full. When the parameters depend upon the state of the system it is negligible whether λ is small or large. But P_N is appreciable when the parameters do not depend upon the state of the system, *i.e.* are constants. Thus with variable parameters, the probabilities fall of very rapidly and there is no necessity of fixing an upper bound on queue length.

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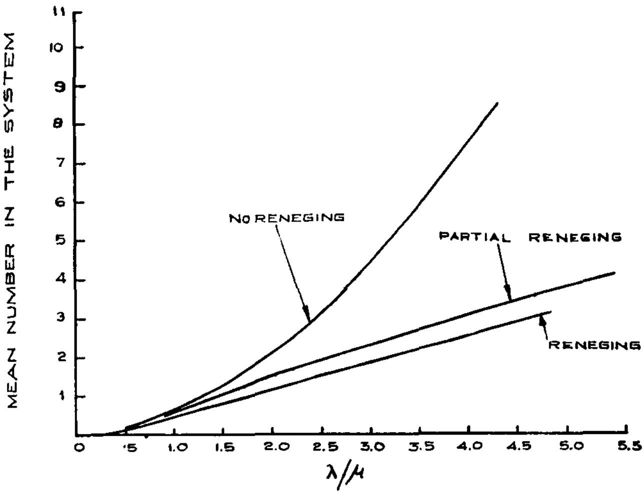
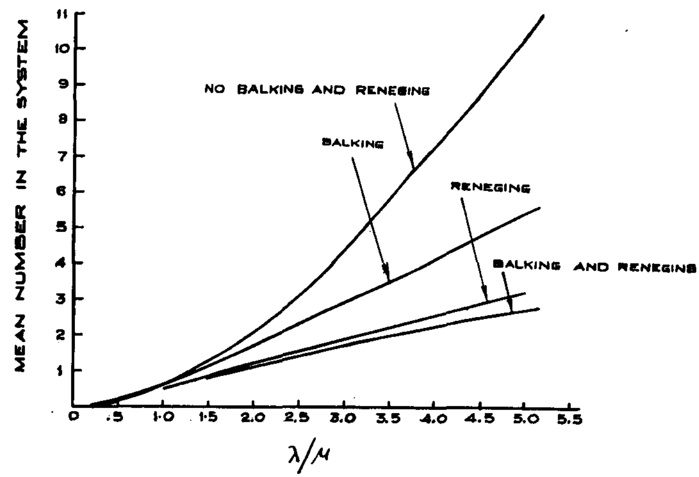

MEAN NUMBER IN THE SYSTEM
NO RENEGING
PARTIAL RENEGING
RENEGING
λ/μ

Fig. 1.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BALKING
RENEGING
BALKING AND RENEGING
λ/μ

Fig. 2,

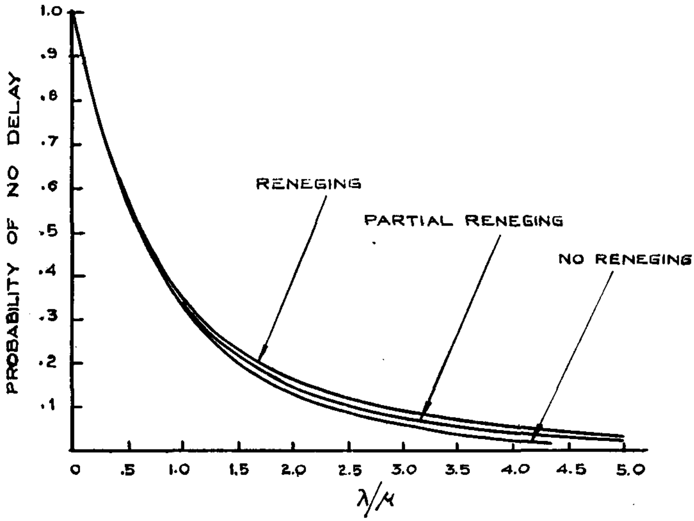

Fig. 3.

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