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DETERMINATION OF THE OPTIMUM TRANSPORT CAPACITY

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Summary

A method of estimating mean and standard deviation of population from truncated sample data is introduced. A decision method for the determination of a traffic capacity is discussed. A simple graphical method is worked out when the traffic demand is normal.

Introduction

Suppose an airline company is operating one scheduled flight on a route by a transport plane which has a certain number of passenger seats, say 100. The break even point of this particular plane is assumed to be 60%, for example. The manager of the airline company knows that the average passenger load factor is 80% and that quite frequently the seats are filled up. Since there exists a chance loss of missing customers he might wish to increase the number of flight in a day. However, if he increases one more flight, the avarage load factor per flight will be 40% which is less than the break even point. Therefore it might not be beneficial to increase the number of flights. But, since the break even point is 60%, the average load factor divided by the number of flights can not exceed this value even if he has 100% load factor for one flight per day. It is obvious that this consideration involves some mistakes. Since the present traffic demand is truncated by the maximum capacity of the passenger plane, the mean, calculated from the statistics of the carried passengers, does not represent the real average of the traffic demand. For example if there is a 140% intrinsic average demand, then the average passenger load factor will be still 70%, when the number of flights is increased by one.

This simple example indicates the importance of estimating the real mean from the truncated statistics. This problem will be discussed in the following section. The problems of determining the optimum passenger capacity or the optimum number of flights will also be considered in the consecutive sections.

The method is applicable not only for the capacity of an airline but for any type of transportation.

Estimation of Traffic Volume

Let us consider the problem of estimating the traffic demand for a route from the statistics of transportation on it. When there exists the maximum for the capacity of transportation, the rate of no vacancy α is expressed in terms of the probability density function $p(x)$ of the traffic demand x. That is

$$\int_a^\infty p(x)dx=\alpha \tag{1}$$

where a stands for the maximum transportation capacity. We denote the mean value and the standard deviation of the actual transportation in the past unit period as $m'a$ and $d'a$. Then, m' and d' are the mean and the standard deviation of the load factor. Since we are interested in the case of scheduled transportation, we can assume that there are enough number of sample data for the transportation so that we might not need to take into account of the effects of sampling error. Since the distribution of actual transportation reflects the real traffic demand cut down out of the maximum transport capacity, we have

$$m'=\frac{1}{a}\int_0^a xp(x)dx+\alpha \tag{2}$$

and

$$d'^2 = \frac{1}{a^2}\int_0^a x^2 p(x)dx - \alpha^2 \tag{3}$$

While for the mean ma and the standard deviation da of the real traffic demand, we have

$$m = \frac{1}{a}\int_0^\infty xp(x)dx \tag{4}$$

and

$$d^2 = \frac{1}{a^2}\int_0^\infty x^2 p(x)dx - m^2. \tag{5}$$

Therefore we have

$$m - m' = \frac{1}{a}\int_a^\infty xp(x)dx - \alpha \tag{6}$$

and

$$d^2 - d'^2 = \frac{1}{a^2}\int_a^\infty x^2 p(x)dx - (m^2 - \alpha^2). \tag{7}$$

If we take a simple distribution for the mathematical model of traffic demand, then the integral terms in the equations (6) and (7) can be expressed in terms of m, d and a. Then our estimation problem is to determine m and d from those equations knowing the values of m', d', α and a.

For example, if we assume $N(ma, d^2a^2)$, we have

$$m - m' = m\alpha + \frac{d}{\sqrt{2\pi}}\, exp\left\{-\frac{(1-m)^2}{2d^2}\right\} - \alpha \tag{6'}$$

and

$$\begin{aligned} d^2 - d'^2 = m^2\alpha + \frac{d}{\sqrt{2\pi}}(1+m)\, exp\left\{-\frac{(1-m)^2}{2d^2}\right\} \\ + d^2\alpha - (m^2 - \alpha^2). \end{aligned} \tag{7'}$$

We notice that the maximum transport capacity a is not involved in these equations. The values of m and d may be obtained by a numerical computation of (6′) and (7′).

Optimum Transport Capacity

Now, we shall examine the optimum capacity of a transportation. The profit of the transportation is

$$bx-c \quad \text{for} \quad x \leqq a,$$

and

$$ba-c \quad \text{for} \quad x > a$$

where b is a carriage charge for a unit transportation and c is the direct operating cost of the present transportation means.

Therefore the expected profit is

$$
\begin{aligned}
I &= \int_0^a (bx-c)p(x)dx + \int_a^\infty (ba-c)p(x)dx \\
&= b\int_0^a xp(x)dx + ab\int_a^\infty p(x)dx - c\,. \qquad (8)
\end{aligned}
$$

If we assume that the direct operating cost changes from c to c' when we switch to a new transportation means having the maximum capacity of a' from the present one, then we have

$$I' = b\int_0^{a'} xp(x)dx + a'b\int_{a'}^{\infty} p(x)dx - c' \qquad (8')$$

if the real traffic volume is not influenced by the change of transportation means.

Obviously, the best policy is to increase the capacity if $I<I'$ for $a<a'$ or to decrease it if $I<I'$ for $a'<a$. The expressions (8) and (8′) can be easily carried out when we know the mathematical expression of $p(x)$.

When the direct operating cost c is expressed as a known function of the capacity a, as

$$c = f(a) \qquad (9)$$

the optimum capacity a_* can be determined from the condition of

$dI/da=0$ which yields

$$f'(a_*)=b\int_a^\infty p(x)dx\,. \tag{10}$$

However, in general, this is not the case, and usually, only a few selection of transportation means is feasible.

Optimum Transport Frequency

Next, let us assume that we can change the capacity of transportation only by increasing or decreasing the frequency of transportation.

The profit for n times transportation is

$$I_n=b\int_0^{na} xp(x)dx+nab\int_{na}^\infty p(x)dx-nc \tag{11}$$

if a and c are the capacity and the direct operating cost for one carriage.

If we increase the frequency to $(n+1)$, then we have

$$I_{n+1}=b\int_0^{(n+1)a} xp(x)dx+(n+1)ab\int_{(n+1)a}^\infty p(x)dx-(n+1)c\,. \tag{11'}$$

Hence, we have

$$\begin{aligned}\Delta I\equiv I_{n+1}-I_n=b\int_{na}^{(n+1)a} xp(x)dx+ab\int_{(n+1)a}^\infty p(x)dx\\ -nab\int_{na}^{(n+1)a} p(x)dx-c\,.\end{aligned} \tag{12}$$

Then, if we have $\Delta I>0$, it will be beneficial to increase the number of transportation service. For example if the traffic volume is normal $N(\mu a, \sigma^2 a^2)$ then introducing a standardized scale

$$u=\frac{x-\mu a}{\sigma a}=\frac{x/a-\mu}{\sigma} \tag{13}$$

and denoting

$$u_n=\frac{n-\mu}{\sigma}, \quad \text{and} \quad u_{n+1}=\frac{(n+1)-\mu}{\sigma} \tag{14}$$

the equation (12) yields

$$\Delta I = b\int_{na}^{(n+1)a} xp(x)dx + ab\Phi^*(u_{n+1}) \\ -nab[\Phi^*(u_n)-\Phi^*(u_{n+1})]-c \tag{15}$$

leaving the first term in the right-hand side unchanged, where $\Phi^*(a)$ is the error integral

$$\Phi^*(u)=\int_a^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{u^2}{2}}du\,. \tag{16}$$

From (15), we have

$$\frac{\Delta I}{ab}=\frac{1}{a}\int_{na}^{(n+1)a} xp(x)dx+(n+1)\Phi^*(u_{n+1}) \\ -n\Phi^*(u_n)-\frac{c}{ab}\,. \tag{17}$$

The integral of the first term can be carried out if we know the exact expression of $p(x)$. However, we shall introduce a simple evaluation of the integral. In general we might assume that

$$na>\mu a$$

then since the normal distribution $p(x)$ is monotonously decreasing in $na<x<(n+1)a$, we have

$$na\int_{na}^{(n+1)a} p(x)dx<\int_{na}^{(n+1)a} xp(x)dx<(n+1)a\int_{na}^{(n+1)a} p(x)dx$$

or

$$na[\Phi^*(u_n)-\Phi^*(u_{n+1})]<\int_{na}^{(n+1)a} xp(x)dx<(n+1)a[\Phi^*(u_n)-\Phi^*(u_{n+1})]\,. \tag{18}$$

Subsequently, when we put

$$\Delta^*\equiv\Phi^*(u_{n+1})-\frac{c}{ab} \tag{19}$$

and if $\Delta^*>0$ then we have $\Delta>0$ where c/ab means the break even point.

In other words, $\Delta^*>0$ is the sufficient condition for $\Delta>0$. From the integral curve of $\Phi^*(u_{n+1})$ and the value of c/ab, we can easily determine whether or not we should increase the number of frequency from n to $n+1$, as is indicated in Figure 1. A point in the shaded area corresponds to the policy of increasing frequency of the transportation service.

In addition, we can express u_n and u_{n+1} in terms of m and d defined by (4) and (5). Since there are

$$nam=\mu a, \quad \text{and} \quad nad=\sigma a$$

we have

$$u_n=\frac{1-m}{d}, \quad u_{n+1}=\frac{\left(1+\frac{1}{n}\right)-m}{d} \tag{20}$$

from (14).

Additional Remarks and Conclusions

The appropriateness of our criterion is evident, since $\Delta^*>0$ might hold when

(i) the average load factor is increased, or

(ii) the break even point is decreased.

When $\Delta^*<0$, we should calculate the first term of (15) before coming to the conclusion of decreasing the frequency of services. However in the similar way, if

$$\Delta^{*\prime}\equiv\Phi^*(u_n)-\frac{c}{ab}<0$$

then, decreasing the numbers of transportation from n to $n-1$ will be adequate.

The problems of policy making for determining the traffic capacity have been studied. Some simple criterions are obtained which may be useful for practical application.

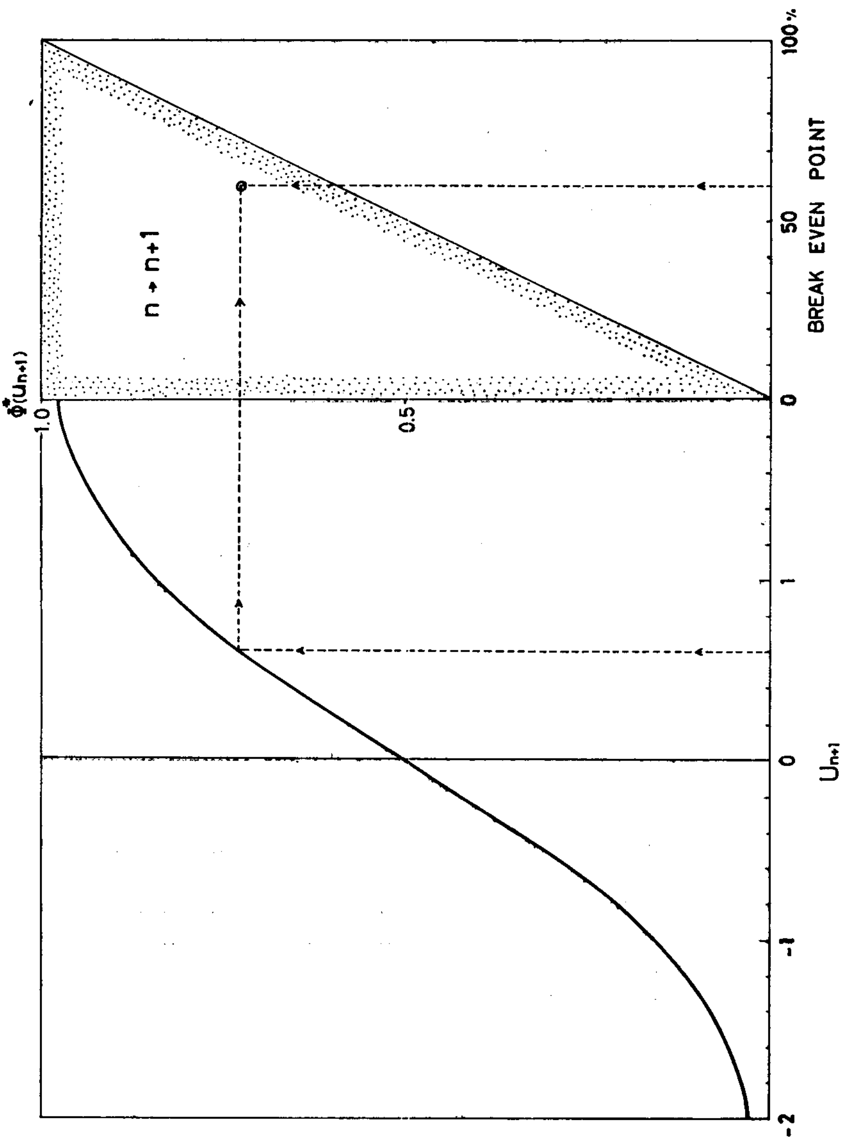

Figure 1.