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SOME RESULTS ON ROAD TRAFFIC DISTRIBUTION

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§ 1. Introduction

In a paper [1] by Leo Breiman it is shown that, under weak assumptions, the number of cars in an arbitrary interval on the space will be asymptotically Poisson distributed as the time tends to infinity. After that, under the same assumptions as those of Breiman, the above result is generalized by T. Thedéen [2] as follows: the cars on the space as the time tends to infinity will be distributed according to a Poisson process.

In this paper it will be shown that under the similar assumptions as above the number of cars on the time-axis will be distributed according to a Poisson process as the time tends to infinity (Theorem 1).

Furthermore, it is shown that that the number of cars in the time-interval $(0, t]$ is asymptotically linear in t as $t \to \infty$ (Theorem 2). Also the relation between the spatial and time density of the velocity is shown in the theorem 3.

§ 2. Assumptions and Notations

At $t=0$, let there be a set of starting points X_1, $X_2, \cdots$ on the negative spatial axis which are obtained as observations of a stochastic point process. Concerning this process $\{X_n,\ n=1, 2, 3, \cdots\}$ we assume the following three conditions.

(a) A spatial density σ exists with probability one,

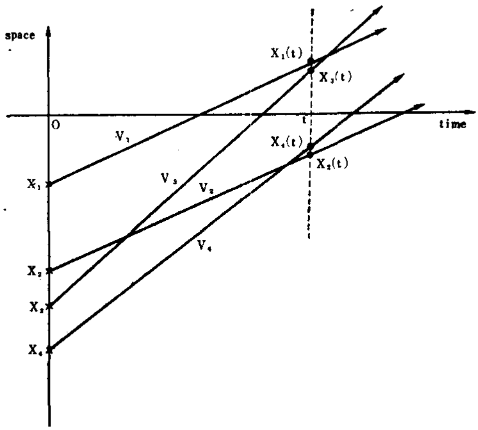

Fig. 1.

i. e.,

$$\lim_{x\to\infty} \{\text{no. of } X_k \text{ in } (0, -x]\}/x=\sigma,$$

with probability one.

(b) For any finite interval I on the spatial axis, the expected number of X_k in I can be bounded above by M, where M depends only on the length of I.

Associated with X_k is V_k, its velocity. The V_k are assumed independent each other and of the $X_1, X_2, \cdots$, with common distribution $G(v)=P\ (V_k<v)$, $G(v)=0\ (v\leqq 0)$.

Further,

(c) $G(v)=\int_0^v g(u)du$, where $g(u)$ is almost everywhere continuous (with respect to Lebesgue measure), bounded on every finite interval and $g(v)=0\ (v>v_0)$ for some positive number v_0. Then the expectation of V_k exists and finite, say $E(V)$.

Denote by $X_k(t)$ the position of the k th car after time t, then we have $X_k(t)=X_k+tV_k$. Let T_k be the time at that $X_k(t)=0$ and $N(t_1, t_2)$ be the number of T_k in the time interval $(t_1, t_2]$, where $t_1<t_2$ and define

$N_v(t)$=no. of k such that $v \leqq V_k < v+\Delta v$ and $T_k \in (0, t]$.

3. Theorems

Theorem 1. Under (a), (b), (c) above for fixed T, j,

$$\text{(1)} \qquad \lim_{t\to\infty} P(N(t, t+T)=j) = \frac{\lambda^j}{j!} e^{-\lambda}$$

where $\lambda = \sigma TE(V)$. Further, let $(t_1, t_1+T_1]$, $(t_2, t_2+T_2]$, $\cdots$, $(t_n, t_n+T_n]$ be n disjoint but otherwise arbitrary time-intervals, then for fixed T_1, $T_2, \cdots, T_n$; $j_1, j_2, \cdots j_n$,

$$\text{(2)} \quad \lim_{t_1,\cdots\cdots,t_n\to\infty} P(N(t_\nu, t_\nu+T_\nu)=j_\nu,\ \nu=1, 2, \cdots, n) = \prod_{\nu=1}^{n} \left(\frac{\lambda_\nu^{j_\nu}}{j_\nu!}\right) e^{-\lambda_\nu}$$

where $\lambda_\nu = \sigma T_\nu E(V)$.

Proof. The proof of the theorem can be done in exactly the same way as in [1] and [2].

We will note that, for fixed $X_1, X_2, \cdots$, the T_k are independent. For sake of completeness of the proof it will be sufficient only to show that

$$\text{(3)} \qquad \lim_{t\to\infty} \sum_{k=1}^{\infty} P(t \leqq T_k \leqq t+T \mid X_1, X_2, \cdots) = \lambda$$

and

$$\text{(4)} \qquad \lim_{t\to\infty} \sup_k P(t \leqq T_k \leqq t+T \mid X_1, X_2, \cdots) = 0$$

for fixed T.

Proof of (3).

$$\begin{aligned} S_t &= \sum_{k=1}^{\infty} P(t \leqq T_k \leqq t+T \mid X_1, X_2, \cdots) \\ &= \sum_k P\left(-\frac{X_k}{t+T} \leqq V_k \leqq -\frac{X_k}{t} \,\middle|\, X_1, X_2, \cdots\right) \\ &= \sum_k \left[G\left(-\frac{X_k}{t}\right) - G\left(-\frac{X_k}{t+T}\right)\right]. \end{aligned}$$

Now let $M_t(y)$=no. of X_k in $(0, -ty]$ $(y\geqq 0)$ and place $m_t(y)=M_t(y)/t$. Then using the assumption (a),

$$m_t(y)\to \sigma y \quad (t\to\infty)$$

with probability one (and the same exceptionl set for all y). Then S_t may be rewritten as follows.

$$\begin{aligned}
S_t&=\int_0^\infty\left[G(y)-G\left(\frac{t}{t+T}y\right)\right]dM_t(y)\\
&=\int_0^\infty\frac{G(y)-G\left(\frac{t}{t+T}y\right)}{\frac{y}{t}}\cdot y\,dm_t(y)\\
&=\int_0^\infty\left(\frac{1}{\frac{y}{t}}\int_{\frac{t}{t+T}y}^{y}g(u)du\right)y\,dm_t(y)\\
&=\int_0^\infty\left(t\int_{\frac{t}{t+T}}^{1}g(yz)dz\right)y\,dm_t(y).
\end{aligned}$$

If $g(v)$ is continuous, then as $t\to\infty$,

$$t\int_{\frac{t}{t+T}}^{1}g(yz)dz\to\begin{cases}g(y)T & \text{unif. in } y\\ 0 & (y>v_0+\varepsilon),\end{cases}$$

where ε is any small positive number.

Because, for any $\varepsilon>0$ there exists a number δ indenpendent of y such that $|g(yz)-g(y)|<\varepsilon$ for $|z-1|<\delta$. For t sufficiently large,

$$\begin{aligned}
&\left|t\int_{\frac{t}{t+T}}^{1}g(yz)dz-g(y)T\right|\\
&=\left|t\int_{\frac{t}{t+T}}^{1}g(yz)-(t+T)\int_{\frac{t}{t+T}}^{1}g(y)dz\right|\\
&\leqq t\int_{\frac{t}{t+T}}^{1}|g(yz)-|dz+\frac{T^2}{t+T}g(y)\\
&\leqq\frac{tT}{t+T}\varepsilon+\frac{T^2}{t+T}\max g(y)<\varepsilon'.
\end{aligned}$$

Therefore,

$$(5) \qquad \lim_{t\to\infty} S_t = \sigma T \int_0^\infty y g(y) dy = \sigma T E(V) = \lambda \text{ (say)}.$$

If $g(v)$ is not continuous, then using the assumption (c) and the same method as in [1] we have the result (5).

Proof of (4). Using (c),

$$\begin{aligned} &P(t \leqq T_k \leqq t+T \mid X_1, X_2, \cdots\cdot) \\ &= P\left(t \leqq -\frac{X_k}{V_k} \leqq t+T \mid X_1, X_2, \cdots\cdot\right) \\ &= P\left(-\frac{X_k}{t+T} \leqq -\frac{X_k}{t} \,\middle|\, X_1, X_2, \cdots\cdot\right) \\ &= \int_{-\frac{X_k}{t+T}}^{-\frac{X_k}{t}} g(u) du \leqq \frac{T}{t+T} v_0 \max g(u). \end{aligned}$$

Therefore, for fixed T,

$$\begin{aligned} &\lim_{t\to\infty} \sup_k P(t \leqq T_k \leqq t+T \mid X_1, X_2, \cdots\cdot) \\ &\leqq \lim_{t\to\infty} \frac{T}{t+T} v_0 \max g(v) = 0. \end{aligned}$$

In the following we denote $N(0,t)$ by $N(t)$.

Theorem 2. Under (a), (b) and (c),

$$(6) \qquad \lim_{t\to\infty} \frac{N(t)}{t} = \sigma E(V)$$

in probability.

Proof. First we consider the case where $X_1, X_2, \cdots\cdot$ are fixed. Define the following random variables,

$$Y_k(t) = \varphi\left(V_k + \frac{X_k}{t}\right),$$

where $\varphi(x) = 1$ $(x \geqq 0)$, $= 0$ $(x < 0)$.

Then

$$(7) \qquad N(t)=\sum_{k=1}^{\infty} Y_k(t)$$

and for fixed t the $Y_k(t)$ are independent random variables, however, they have not a common distribution. In fact,

$$P(Y_k(t)=0|X_1, X_2, \cdots)=P\left(V_k<-\frac{X_k}{t}\middle|X_1, X_2, \cdots\right)$$
$$=G\left(-\frac{X_k}{t}\right)=q_k \to 1 \ (k\to\infty)$$

and put $p_k=1-q_k$.

According to the assumption (c), the summation on the right-hand side of (7) is finite.

Now if we define

$$k(t)=\min\left\{k \,\middle|\, -\frac{X_k}{t}\leqq v_0, \quad -\frac{X_{k+1}}{t}>v_0\right\},$$

then

$$N(t)=\sum_{k=1}^{k(t)} Y_k(t)$$

$$(8) \qquad =\frac{k(t)}{t}\cdot\frac{\sum_{k=1}^{k(t)} Y_k(t)}{k(t)}.$$

For the application of the strong law of large numbers we need to examine some properties.

$k(t)$=no. of cars in the interval $(0, -tv_0]$ on the spatial axis.

$$=\text{no. of } \frac{X_k}{t} \text{ in } (0, -v_0]$$
$$=\text{no. of } -\frac{X_k}{t} \text{ in } (0, v_0]$$
$$=\int_0^{v_0} dM_t(y)=\int_0^{v_0} t\,dm_t(y).$$

Therefore,

$$(9)\qquad \lim_{t\to\infty}\frac{k(t)}{t}=\lim_{t\to\infty}\int_0^{v_0}dm_t(y)=\int_0^{v_0}\sigma dy=\sigma dy=\sigma v_0\,.$$

Furthermore,

$$E(Y_k(t)|X_1, X_2, \cdots)\leqq 1 \quad\text{and}\quad V_{ar}(Y_k(t)|X_1, X_2, \cdots)=p_kq_k<1\,,$$

then

$$(10)\qquad \sum_{k=1}^{\infty}\frac{V_{ar}(Y_k(t)|X_1, X_2, \cdots)}{k^2}\leqq\sum_{k=1}^{\infty}\frac{1}{k^2}<\infty\,.$$

And

$$\begin{aligned}E(N(t)|X_1, X_2, \cdots)&=E(\sum_{k=1}^{k(t)}Y_k(t)|X_1, X_2, \cdots)\\&=\sum_{k=1}^{k(t)}P(Y_k(t)=1|X_1, X_2, \cdots)\\&=\sum_{k=1}^{k(t)}\left[1-G\left(-\frac{X_k}{t}\right)\right]\\&=\int_0^{v_0}[1-G(y)]dM_t(y)=t\int_0^{v_0}[1-G(y)]dm_t(y)\,,\end{aligned}$$

then from (9),

$$\begin{aligned}\lim_{t\to\infty}E\left(\frac{N(t)}{k(t)}\middle|X_1, X_2, \cdots\right)&=\lim_{t\to\infty}\frac{t}{k(t)}\int_0^{v_0}[1-G(y)]dm_t(y)\\(11)\qquad&=\frac{1}{\sigma v_0}\int_0^{v_0}[1-G(y)]\sigma dy=\frac{1}{v_0}E(V)\,.\end{aligned}$$

By (8), (10), (11) and the strong law of large numbers, for fixed X_1, $X_2, \cdots$,

$$(12)\qquad \lim_{t\to\infty}\frac{N(t)}{t}=\sigma E(V)$$

with probability one. According to the assumption (a) $P(N(t)/t\leqq x)\to 0$ $(x<\sigma E(V))$ or 1 (elsewhere), then $N(t)/t\to\sigma E(V)$ in probability, i.e., the theorem is true.

Corollary. Under (a), (b) and (c),

$$\lim_{t\to\infty} E\left(\frac{N(t)}{t}\right)=\sigma E(V).$$

Proof. With the proof of the above and the Lebesgue bounded convergence theorem,

$$\begin{aligned}\lim_{t\to\infty} E\left(\frac{N(t)}{t}\right)&=\lim_{t\to\infty} E\left\{E\left(\frac{N(t)}{t}\middle| X_1,\ X_2,\cdots\right)\right\}\\&=\lim_{t\to\infty} E\left(\int_0^{v_0}[1-G(y)]\,dm_t(y)\right)\\&=E\left(\int_0^{v_0}[1-G(y)]\sigma dy\right)=\sigma E(V).\end{aligned}$$

Theorem 3. Under (a), (b) and (c),

$$\lim_{t\to\infty}\frac{N_v(t)}{N(t)}=\frac{1}{E(V)}\int_v^{v+\varDelta v} ug(u)du \tag{13}$$

in probability.

Proof. First we consider the case where $X_1, X_2, \cdots$ are fixed. Define the following random variables,

$$Z_k(t)=\varphi\left(V_k+\frac{X_k}{t}\right)\varphi(V_k-v)\,\varphi(v+\varDelta v-V_k).$$

Then

$$N_v(t)=\sum_{k=1}^{k(t)} Z_k(t),$$

where $k(t)$ is defined in the proof of theorem 2.

For the application of the strong law of large numbers to the following expression,

$$\frac{N_v(t)}{t}=\frac{k(t)}{t}\cdot\frac{\sum_{k=1}^{k(t)} Z_k(t)}{k(t)}, \tag{14}$$

we will examine some properties as follows.

Clearly,

(15) $$\sum_{k=1}^{\infty}\frac{V_{ar}(Z_k(t)|X_1, X_2, \cdots)}{k^2} \leqq \sum_{k=1}^{\infty}\frac{1}{k^2} < \infty.$$

since $0 \leqq Z_k(t) \leqq 1$ for all k.

And also

$$E(N_v(t)|X_1, X_2, \cdots) = \sum_k P(Z_k(t)=1|X_1, X_2, \cdots)$$
$$= \int_v^{v+\Delta v} g(u)du \int_0^u dM_t(y)$$
$$= t\int_v^{v+\Delta v} g(u)du \int_0^u dm_t(y).$$

Therefore, using (9),

$$\lim_{t\to\infty} E\left(\frac{N_v(t)}{N(t)}\middle| X_1, X_2, \cdots\right)$$
$$= \lim_{t\to\infty}\frac{t}{k(t)} \cdot \lim_{t\to\infty}\int_v^{v+\Delta v} g(u)du \int_0^u dm_t(y)$$
$$= \frac{1}{v_0}\int_v^{v+\Delta v} ug(u)du.$$

From (14), (15), (16) and the strong law of large numbers,

$$\lim_{t\to\infty}\frac{N_v(t)}{t} = \sigma\int_v^{v+\Delta v} ug(u)du. \tag{17}$$

The above result (17) is true for each fixed values of $X_1, X_2, \cdots$ again with the possible exception of a set of probability zero, i.e., it is true with probability one.

With the result of the preceding theorem and (17), we have

$$\lim_{t\to\infty}\frac{N_v(t)}{N(t)} = \frac{1}{E(V)}\int_v^{v+\Delta v} ug(u)du$$

in probability.

REFERENCES

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[2] Thedéen, T., "A Note on the Poisson Tendency in Traffic Distribution," *Ann. Math. Stat.*, **35** (1964), 1823—1829.