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TREE ALGORITHM FOR SOLVING RESOURCE ALLOCATION PROBLEM

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§ 0. PREFACE

A new algorithm is proposed for solving *resource allocation problem* (§ 1). It is basically dual method (I) in linear programming theory. But in view of network topology the idea of tree or forest plays essential role in our algorithm, as in (II). E. Balas and P.L. Ivanescu, K. Eisemann and J.L. Lourie have treated the same problem. The me hods are basically primal method (IV), (V), (VI).

§ 1. RESOURCE ALLOCATION PROBLEM

Definition

Let us call the following problem *resource allocation without slack*

(1·1) $$\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} \rightarrow \text{Min.},$$

(1·2) $$\sum_{j=1}^{n} x_{ij}=a_i \quad (i=1\sim m),$$

(1·3) $$\sum_{i=1}^{m} r_{ij}x_{ij}=b_j \quad (j=1\sim n),$$

(1·4) $$x_{ij}\geqq 0 \quad (i=1\sim m,\ j=1\sim n).$$
$$(a_i,\ b_j,\ c_{ij}\geqq 0,\ r_{ij}>0).$$

For example each of m resources is to be allocated to produce n products. Let a_i denote the available amount of i th resource $(i=1\sim m)$, and b_j the demand of j th product $(j=1\sim n)$. Unit amount of i th resource produces r_{ij} of jth product at a cost of c_{ij}. Find allocation program minimizing total cost.

Instead of (1.2) we have often inequality constraints

$$(1.2)' \qquad \sum_{j=1}^{n} x_{ij} \leqq a_i \quad (i=1\sim m).$$

Let problem (1·1) (1·2)' (1·3) (1·4) be called *resource allocation with slack*, which we treat in §5.

Network Representation

We represent resource allocation without slack (1·1)~(1·4) in network figure as Fig. 1 (a). It shows that the resource allocation problem is the same as transportation problem except that when a commodity flows from node i to node j its amount is multiplied by r_{ij}. Let the coefficient r_{ij} be called *trancemitance*. Of course each *source* may not connect with every *sink* directly, so in general we have a network shown in Fig. 1 (b).

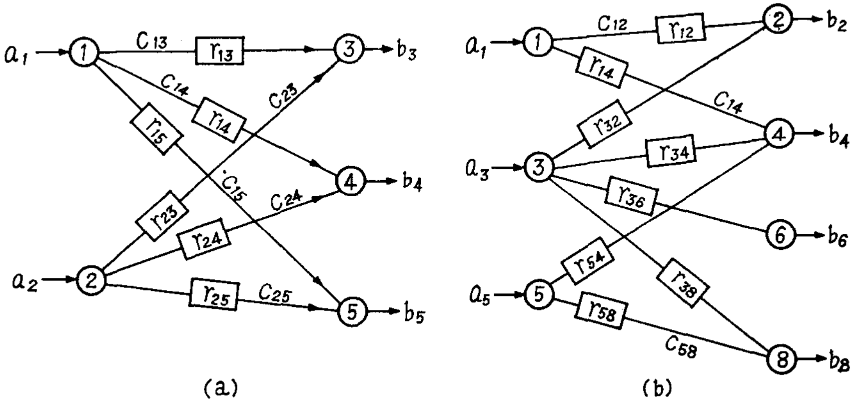

Fig. 1. Network representation ot resource allocation

Dual Problem

Resource allocation (without slack) in general network (Fig. 1 (b)) is formulated as follows:

$$\sum_{(i,j)\in A} c_{ij}x_{ij} \to \text{Min.}, \tag{1·5}$$

$$a_i = \sum_{\beta} x_{i\beta}\,,\ i\in S, \tag{1·6}$$

$$\sum_{\alpha} r_{\alpha k}x_{\alpha k} = b_k\,,\ k\in T, \tag{1·7}$$

$$x_{ij} \geqq 0\,,\ (i,j)\in A, \tag{1·8}$$

where S denotes the set of *source nodes*, T the set of *sink nodes*, and A the set of *arcs*. Dual problem of above is:

$$\sum_{k\in T} b_k u_k - \sum_{i\in S} a_i u_i \to \text{Max.} \tag{1·9}$$

$$c_{ij} + u_i - r_{ij}u_j \geqq 0\,,\ (i,j)\in A. \tag{1·10}$$

Let us call u_i *node potential* of node i and

$$\zeta_{ij} = c_{ij} + u_i - r_{ij}u_j \tag{1·11}$$

imputed cost of arc (i,j).

§2. GENERAL ALGORITHM BASED ON DUAL METHOD

We reinterpret dual method (I) as follows. Let us consider a primal linear program

$$c'x \to \text{Min.},\ Ax=b\,,\ x\geqq 0 \tag{2·1}$$

and its dual program

$$b'y \to \text{Max.},\ c\geqq A'y\,. \tag{2·2}$$

Select linear independent column vectors $P_1, \cdots, P_m$ of A, such that y determined by

$$y'P_i = c_i \quad (i=1\sim m) \tag{2·3}$$

is dual feasible (where $c_1, \cdots, c_m$ are components of c corresponding $P_1, \cdots, P_m$ respectively).

If some of $x_1, \cdots, x_m$ determined by

$$(2\cdot4)\qquad x_1P_1+\cdots+x_mP_m=b$$

are negative, then select one of them, say x_1. Delete the equation for $i=1$ from (2·3) to get a *relaxed system of equation*

$$(2\cdot5)\qquad \bar{y}'P_i=c_i \quad (i=2\sim m).$$

The solution of (2·5) has one parameter θ to be arbitrarily chosen, so we denote it

$$(2\cdot6)\qquad \bar{y}_i=y_i+\theta\eta_i \quad (i=1\sim m)$$

where y_i $(i=1\sim m)$ is the solution of (2·3).

Then we can see easily that η (whose components are η_i $(i=1\sim m)$) is orthogonal to P_i $(i=2\sim m)$, that is

$$(2\cdot7)\qquad \eta'P_i=0 \quad (i=2\sim m).$$

Further from (2·4) we have

$$(2\cdot8)\qquad x_1\eta'P_1=\eta'b.$$

The objective function of dual program for $\bar{y}$ is

$$(2\cdot9)\qquad b'\bar{y}=b'y+\theta b'\eta=b'y+\theta x_1\eta'P_1.$$

Let us select

$$(2\cdot10)\qquad \begin{cases}\theta<0 & \text{if } \eta'P_1>0,\\ \theta>0 & \text{if } \eta'P_1<0\end{cases}$$

then we can increase the objective function of dual program (because $x_1<0$), and $\bar{y}$ is also dual feasible for small* (absolute) value of θ.
(*If a column vector P_k of A which does not belong to $[P_1,\cdots,P_m]$ satisfy $y'P_k=c_k$, that is degenerate occurs, then for any non zero θ, $\bar{y}$ breaks feasibility. However we can avoid the case by purterbation procedure (III)).

Consideirng as above, we have the following general algorithm;

Step 1: Select liner independent base vectors $P_1,\cdots,P_m$ such that y determined by (2·3) is dual feasible.

Step 2: If $x_1, \cdots, x_m$ determined by (2·4) are all non-negative, then these are optimum. If some of them are negative, then select one of them, say x_1.

Step 3: Delete equation for $i=1$ from (2·3) to get (2·5), then (2·5) relaxes the constraints by one parameter. In (2·5) the solution can move freely (by one parameter) and the direction is η (see (2·6)).

Step 4: Move the solution from y in the direction (according to rule (2·10)) and stop not until feasibility of dual program is broken somewhere, to get new better solution y, and a new base vector P_k. Return to Step 2 with new basis $[P_2 \cdots P_m P_k]$.

§ 3. PRELIMINARY CONCEPTS AND GENERAL DESCRIPTION OF TREE ALGORITHM

0-tree, 1-tree, forest and path characteristic

On a given network (such as Fig. 1 (b) or Fig. 2), let us call a sub network, which contain no loops, *0-tree* and a sub network which contain only one loop, *1-tree*. Further let a sub network be called *forest* if components of the sub network are 0-trees or 1-trees and every nodes of the given network is contained in the sub network.

In Fig. 2, for example, $T_1=\{(1,2), (1,4), (1,6), (3,2), (3,6), (5,6)\}$ is a 1-tree and $T_2=\{(7,8), (7,10), (9,8), (9,12)\}$ is a 0-tree, and the set of all solid arcs $\{T_1, T_2\}$ is a forest whose components are T_1 and T_2.

A *path characteristic* is the product of trancemitances of arcs with same direction as the path and reciprocals of trancemitances of arcs with opposite direction on the path. In Fig. 2 characteristic of path $(5,6,3,2)=2\times(1/3)\times4=8/3$. *Loop characteristic* is defined same way. In Fig. 2 characteristic of loop $(1,2,3,6,1)=3\times(1/4)\times3\times(1/1)=9/4$.

Several Propertier

Property I: Flows on a 1-tree which satisfy conservation constraints are uniquely determined if and only if the loop characteristic is not unity.

In Fig. 2, T_1 has non unity loop characteristic and flows of T_1 which satisfy conservation constraints are as follows; $x_{12}=-4.6$, $x_{14}=7.5$, $x_{16}=6.1$, $x_{32}=6.7$, $x_{36}=1.3$, $x_{56}=10$.

Property II: Node potentials on a 1-tree which yield zero imputed cost to all arcs on the 1-tree are uniquely determined if and only if the loop characteristic is not unity.

In Fig. 2, node potentials on T_1 which yield zero imputed cost to all arcs on T_1 are as follows; $u_1=-0.1$, $u_2=0.3$, $u_3=-0.8$, $u_4=1.2$, $u_5=-2.2$, $u_6=0.9$.

Property III: Node potentials on a 0-tree which yield zero imputed cost to all arcs on the 0-tree has one parameter θ to be arbitrarily chosen.

In Fig. 2, node potentials on T_2 which yield zero imputed cost

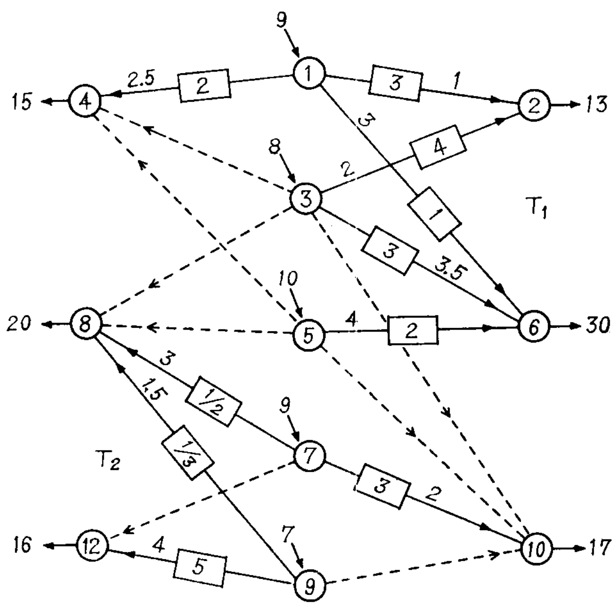

Fig. 2. 0-Tree, 1-Tree, Forest

on T_2 are as follows; $u_7=\theta$, $u_8=6+2\theta$, $u_9=1/2+2\theta/3$, $u_{10}=2/3+\theta/3$, $u_{12}=9/10+2\theta/15$.

Gain of 0-tree

Consider a 0-tree Z. The increasing amount of the function $\sum_{i\in T\cap Z} b_i u_i - \sum_{i\in S\cap Z} a_i u_i$ (a part of dual objective function) by raising up node potentials of Z, with imputed costs of arcs on Z to be invariant, play an important role, and let the amount be called *gain* of the 0-tree Z. More precisely, we call the increasing amount (above mentioned) per unit increasing amount of u_i $(i\in Z)$ *gain of Z to node i*. For example in Fig. 2, gain of T_2 to node 7 is $20\times 2+17\times 1/3+16\times 2/15-9\times 1-7\times 2/3=512/15$.

General Description of Tree Algorithm

(i) Initial Treatment

Choosing appropriate node potentials, construct an initial forest F composed of 1-trees such that every arc of F has zero imputed cost, every remaining arc has positive imputed cost and the characteristics of loops contained in F are not unity.*

Let flows on arcs which are not contained in F be zero, then determine flows on arcs in F so that conservation constraints are satisfied (see Property I). (F is corresponds to initial basis in general dual method).

(ii) Iterative Process

If flows on F are all non negative, then optimal solution attains. Otherwise select one of arcs with negative flows and delete it out of F, to get F'. Then one component of F' becomes a 0-tree which we denote Z.

Raise (or lower according to rule (2·10)) the node potentials of Z with imputed cost of arcs on Z to be invariant (see Property III), until for the first time an imputed cost of an arc which is not contained in

* We assume such initial forest exist for convenience to illustrate but detailed algorithm in § 4 (or § 5) is valid in the case lacking in the assumption.

F become zero.

Adding the arc with newly zero imputed cost to F' we get a new forest.

Repeat above procedures.

§ 4. TREE ALGORITHM FOR RESOURCE ALLOCARION WITHOUT SLACK

We state tree algorithm for resource allocation without slack (1·1)~(1·4) through an example shown in Table 1.

Table 1

c_{ij}

	1	5	6	7
1	1	6	7	10
2	5	3	17	3
3	14	11	2	15

r_{ij}

	4	5	6	7	a_i
1	1	2	3	1	9
2	2	1	4	3	15
3	3	2	5	1	10
b_j	13	15	30	20	

In table-from such as Table 1 or Table 2, a row number corresponds to a source node, a column number to a sink node and an entry to an arc.

(i) Determination of Initial Forest

Step 1: Select a sink, say sink 7. Set

$$u_7=0,\ u_1=-c_{17}\quad u_4=\text{Min.}_i\ ((u_i+c_{i4})/r_{i4})$$
$$u_2=-c_{27}\quad u_5=\text{Min.}_i\ ((u_i+c_{i5})/r_{i5})$$
$$u_3=-c_{37}\quad u_6=\text{Min.}_i\ ((u_i+c_{i6})/r_{i6}),$$

and compute $\zeta_{ij}=c_{ij}+u_i-r_{ij}u_j$. Square entries with zoro imputed costs (Table 2). (When more than one zeros in a column, except column 7, select one with highest row number of these zeroes—see treatment on degenerate case). Then squared entries construct a 0-tree.

Step 2: Set $\eta_7=1$ and η_i to be reciprocal of characteristic of the path on the 0-tree from node 7 to node i for each i, and compute $\xi_{ij}=\eta_i-r_{ij}\eta_j$ for all entries—of course ξ_{ij} on the 0-tree are zero. ζ_{ij}/ξ_{ij} is shown in each entry without square in Table 2.

Step 3: If for all arcs (i,j) $\xi_{ij}=0$ then let initial forest F to be the 0-tree and go to Step 4. If some $\xi_{ij}>0$ then select θ to be negative and determine a magnitude of θ to be the maximum value which satisfy

$$(4\cdot1) \qquad \zeta_{ij}+\theta\xi_{ij}\geqq 0$$

for all entries (i,j). In our example $\theta=-4/5$* for which arc (2,5) satisfies (4·1) with equality sign.

Add the arc to the 0-tree to get initial forest F which is itself 1-tree and does not contain loops with unity-characteristics. (Of course more than one such arcs exist we select one with highest arc number).

Step 4: Compute new node potentials $u_i'=u_i+\theta\eta_i$ and imputed costs $\zeta_{ij}'=\zeta_{ij}+\theta\xi_{ij}$ (Table 3 (i)). Set flows on arcs not contained in F to be zero, and determine flows on F to satisfy conservation constraints** (Table 3 (i)).

(ii) Iterative Process

Step 1: If all flows on F are non negative, optimum solutions attain. Otherwise select one arc (m,n) of arcs with negative flows on F, and delete it from F to get F', then further to get a 0-tree Z. (In Fig. 3 (i), $(m,n)=(1,7)$, $Z=\{(1,4)\}$. In Fig. 3 (ii), $(m,n)=(3,7)$, $Z=\{(3,4), (3,5), (3,6), (1,4), (2,5), (2,7)\}$).

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* Of course in our example we can select θ to be positive and set $\theta=17/2$. Generally if $\xi_{ij}\geqq 0$ for all arcs then we must set θ to be negative and if $\xi_{ij}\leqq 0$ for all arcs then we must set θ to be positive, otherwise we can arbitrarily select sign of θ.

** When F is 0-tree such flows may not exist. In this case feasible solution does not exist.

Step 2:
In case $m \in Z$, $n \notin Z$:
Raise up potentials of Z with imputed cost on Z to be invariant.

(4·2) The gain of Z to node $m = -x_{mn}$.

In case $n \in Z$, $m \notin Z$:
Lower potentials of Z with imputed cost on Z to be invariant.

(4·3) The gain of Z to node $n = -r_{mn}x_{mn}$.

In case both m and $n \in Z$:
Raise up (Lower) potentials of Z with imputed cost on Z to be invariant if the gain of Z to node m is positive (negative)

(4·4) The gain of Z to node $m = (\eta_n - 1)x_{mn}$.

(The gain does not vanish because F does not contain loop with unity-characteristic). (In Fig. 3 (i), $m=1 \in Z = \{(1,4)\}$, so set $\eta_1 = 1$, $\eta_4 = 1$ other $\eta_i = 0$. In Table 3 (ii), both $m=3$, $n=7 \in Z$, so set $\eta_3=1$, $\eta_4=1/3$, $\eta_5=1/2$, $\eta_6=1/5$, $\eta_2=1/2$, $\eta_7=1/6$, and the gain of Z to node $3=13/3+15/2+30/5+20/6-9/3-15/2-10=2/3$.)

Step 3: Continue to raise ($\theta>0$) or lower ($\theta<0$) the potentials until for the first time an imputed cost of an arc $\in Z$ (complement of Z) becomes newly zero. That is, the amount to be raised

(4·5) $|\theta| = \text{Min.}\ (\zeta_{ij}/|\xi_{ij}| : \xi_{ij}<0,\ (i,j) \in Z)$

(Replacing $\xi_{ij}>0$ instead of $\xi_{ij}<0$ in (4·5) we get the amount to be lowered)
Let (k, l) denote a minimizing arc in (4·5).* In Fig. 3 (i), $\theta=31/5$, $(k, l)=(3, 4)$.

Step 4: Compute new node potentials u_i' on Z by

$$u_i' = u_i + \theta\eta_i \quad (i \in Z)$$

* If there are more than one such arcs then let (k, l) be an arc with the highest arc number.—See treatment on degenerate case.

and new imputed costs ζ_{ij}' on $\bar{Z}$ by

$$\zeta_{ij}'=\zeta_{ij}+\theta\xi_{ij} \quad ((i,j)\in\bar{Z})$$

Step 5: Add $(k, 1)$ to F', we get a new forest and compute flows satisfying conservation constraints on the new forest. Go to Step 1.

Table 2

	u_i / η_i	4	5	6	7
		-6 / 1	-2 / $\frac{1}{2}$	$-\frac{13}{5}$ / $\frac{1}{5}$	0 / 1
1	-10 / 1	(1) [ ]	(2) 0 / 0	(3) $\frac{24}{5}$ / $\frac{2}{5}$	(1) [ ]
2	-3 / 3	(2) 14 / 1	(1) 2 / $\frac{5}{2}$	(4) $\frac{122}{5}$ / $\frac{11}{5}$	(3) [ ]
3	-15 / 1	(3) 17 / -2	(2) [ ]	(5) [ ]	(1) [ ]

$\theta=-\frac{4}{5}$

Table 3

(i)

	u_i / η_i	4	5	6	7	
		$-\frac{34}{5}$ / 1	$-\frac{12}{5}$ / 0	$-\frac{69}{25}$ / 0	$-\frac{4}{5}$ / 0	
1	$-\frac{54}{5}$ / 1	(1) [13]	(2) 0 / 1	(3) $\frac{112}{25}$ / 1	(1) [-4]	9
2	$-\frac{27}{5}$ / 0	(2) $\frac{66}{5}$ / -2	(1) [7]	(4) $\frac{566}{25}$ / 0	(3) [8]	15
3	$-\frac{79}{5}$ / 0	(3) $\frac{93}{5}$ / -3	(2) [4]	(5) [6]	(1) [0]	10
		13	15	30	20	

$\theta=\frac{31}{5}$

Table 3

(ii) $\theta = \frac{24}{5}$

	u_i / η_i	4	5	6	7	
		$-\frac{3}{5}$; $\frac{1}{3}$	$-\frac{12}{5}$; $\frac{1}{2}$	$-\frac{69}{25}$; $\frac{1}{5}$	$-\frac{4}{5}$; $\frac{1}{6}$	
1	$-\frac{23}{5}$; $\frac{1}{3}$	1; 9	2; $\frac{31}{5}$; $-\frac{2}{3}$	3; $\frac{267}{25}$; $-\frac{4}{15}$	1; $\frac{31}{5}$; $\frac{1}{6}$	9
2	$-\frac{27}{5}$; $\frac{1}{2}$	2; $\frac{4}{5}$; $-\frac{1}{6}$	1; $\frac{121}{15}$	4; $\frac{566}{25}$; $-\frac{3}{10}$	3; $\frac{104}{15}$	15
3	$-\frac{75}{5}$; 1	3; $\frac{4}{3}$	2; $\frac{52}{15}$	5; 6	1; $-\frac{4}{5}$	10
		13	15	30	20	

(iii) $\theta = 4$

	u_i / η_i	4	5	6	7	
		1; $\frac{1}{4}$	0; $\frac{1}{2}$	$-\frac{4}{5}$; $\frac{1}{5}$	0; $\frac{1}{6}$	
1	-3; $\frac{1}{4}$	1; 9	2; 3; $\frac{-3}{4}$	3; $\frac{47}{5}$; $-\frac{7}{20}$	1; 7; $\frac{1}{12}$	9
2	-3; $\frac{1}{2}$	2; 4	1; $\frac{13}{3}$	4; $\frac{106}{5}$; $-\frac{3}{10}$	3; $\frac{20}{3}$	15
3	-11; 1	3; $-\frac{4}{3}$	2; $\frac{16}{3}$	5; 6	1; 4; $\frac{5}{6}$	10
		13	15	30	20	

(iv)

	u_i	4	5	6	7	
		2	2	-1	$\frac{2}{3}$	
1	-2	1; $\frac{77}{9}$	2; $\frac{4}{9}$	3; 8	1; $\frac{22}{3}$	9
2	-1	2; $\frac{20}{9}$	1; $\frac{55}{9}$	4; 20	3; $\frac{20}{3}$	15
3	-7	3; 1	2; 4	5; 6	1; $\frac{22}{3}$	10
		13	15	30	20	

Optimum Solution

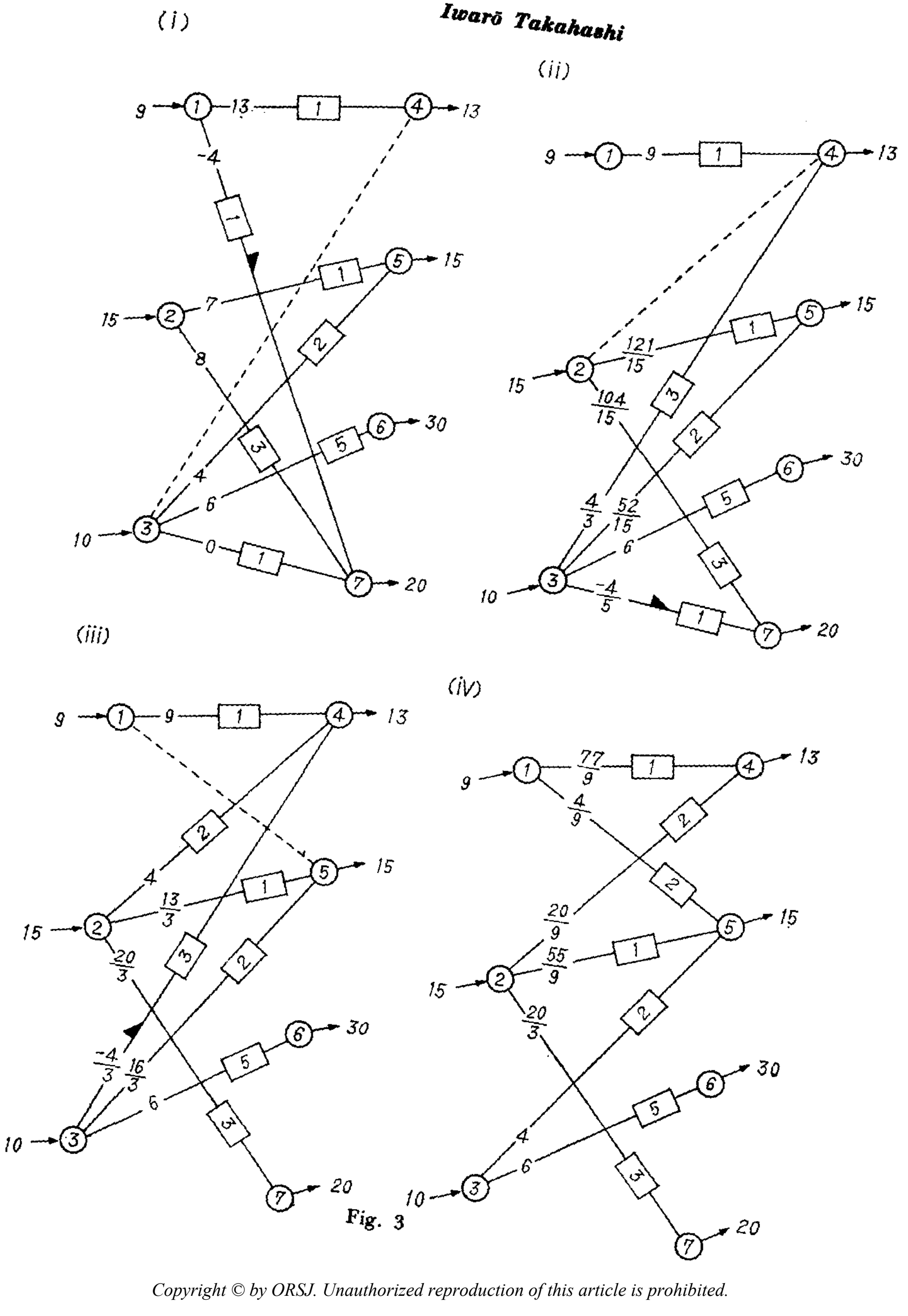


Fig. 3

Treatment on Degenerate Case (III)

We can imagine, unit cost of an arc (i,j) is $c_{ij}+\varepsilon^{i+j}$ where $\varepsilon>0$ is arbitrary small number. Thus even if for some values ρ_{ij}

$$c_{ij}+\rho_{ij}=c_{jl}+\rho_{kl}$$

we have

$$c_{ij}+\varepsilon^{i+j}+\rho_{ij}<c_{ij}+\rho_{kl}+\varepsilon^{k+l}$$

for $i+j>k+l$.

This is the reason why we select an arc with highest row (or column) number when tie occur in minimum imputed costs.

§ 5. TREE ALGORITHM FOR RESOURCE ALLOCATION WITH SLACK

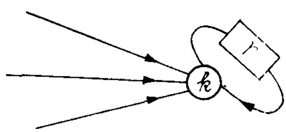
Arbitrary Amount of Demand

Consider the case that we are free from the constraints on demands at some sink nodes in resource allocation without slack. For example constraint at sink node k is

$$\sum_i r_{ik}x_{ik}=y_k$$

and y_k is an arbitrary amount.

Now imagine that node k has a loop (consisting of only one arc) with characteristic smaller than unity.



(0 < r < 1)

This imaginery loop absorbs the discrepancy of conservation constraint at node k and plays the same role as arbitrary amount of demand. And we can treat it as though it be an actual loop considered in § 4.

Resource Allocation with Slack

Consider problem (1·1) (1·2)′ (1·3) (1·4) and insert slacks $x_{i,n+1}$ $(i=1\sim m)$ to (1·2′) to obtain

$$\sum_{j=1}^{n+1} x_{ij}=a_i \quad (i=1\sim m). \tag{5·1}$$

Let y_{n+1} denote total of slacks, that is

$$\sum_{i=1}^{m} x_{i,n+1}=y_{n+1}. \tag{5·2}$$

The problem (1·1) (1·2)′ (1·3) (1·4) is equivalent to (5·1) (5·2) (5·3) (5·4) (5·5) with y_{n+1} arbitrary.

$$\sum_{i=1}^{m} r_{ij}x_{ij}=b_j \quad (j=1\sim n), \tag{5·3}$$

$$x_{ij}\geqq 0 \quad (i=1\sim m,\ j=1\sim n+1), \tag{5·4}$$

$$\sum c_{ij}x_{ij} \to \text{Min}. \tag{5·5}$$

For example resource allocation with slack shown in Table 4 is equivalent to a network shown in Fig. 4. And the algorithm for solving it is completely same as in §4 except that we should not delete the imaginary loop out of basis forest. Determination of initial forest is simpler than that of without slack type.

Table 4

c_{ij}

	3	4	5
1	4	6	7
2	5	4	17

r_{ij}

	3	4	5	a_i
1	1	2	3	≦10
2	2	1	4	≦15
b_j	13	15	30	

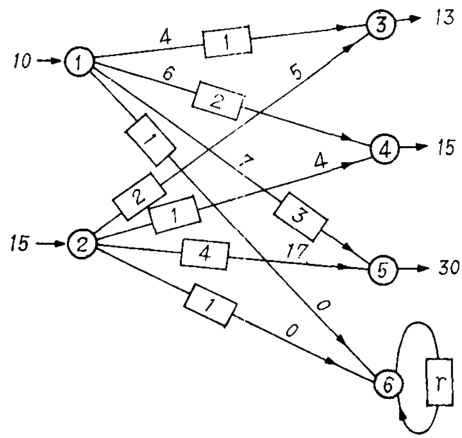


Fig. 4

Numerical Example

We trace tree algorithm through an example shown in Table 5. We insert slacks and set necessary informations in Table 6, where node 0 is slack node. Iterative procedures are same as that of without-slack type, so we have only to explain initial treatment.

Set

$$u_0=0,$$
$$u_i=0 \quad \text{for every source node } i,$$
$$u_j=\text{Min. } (c_{ij}/r_{ij}) \quad \text{for every sink node } j$$

and compute

$$\zeta_{ij}=c_{ij}+u_i-r_{ij}u_i \quad \text{for every arc (entry).}$$

And square entry with zero imputed cost. (Of course if more than one zeroes in a column, square the arc with highest arc number, except slack column). Then squared entry and arc (0, 0) construct initial forest (Table 6 (i), Fig. 5 (i)).

Table 5

c_{ij}

	6	7	8	9	10	11
1	4	6	7	10	8	9
2	5	3		3	4	1
3	14	11	2			2
4		10	1	8		
5	3	4		8	3	12

r_{ij}

	6	7	8	9	10	11	a_i
1	1	2	3	1	2	3	9
2	2	1		3	2	1	8
3	3	2	3			2	10
4		2	4	3			9
5	2	3		1	3	4	7
b_j	13	15	30	20	17	16	

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Table 6

(i) $\theta = 2$

u_i \ η_i	0	2	4	6	8	10	12	
	0	$\frac{3}{2}$ / $\frac{1}{2}$	$\frac{4}{3}$ / $\frac{1}{3}$	$\frac{1}{4}$	1	1 / $\frac{1}{3}$	1	
1: 0	1: 9	1: $\frac{5}{2}$ / $-\frac{1}{2}$	2: $\frac{10}{3}$ / $-\frac{2}{3}$	3: $\frac{25}{4}$	1: 9	2: 6 / $-\frac{2}{3}$	3: 6	9
3: 0	1: $\frac{4}{3}$	2: 2 / -1	1: $\frac{5}{3}$ / $-\frac{1}{3}$		3: $\frac{20}{3}$	2: 2 / $-\frac{2}{3}$	1: 0	8
5: 0	1: 2	3: $\frac{19}{2}$ / $-\frac{3}{2}$	2: $\frac{25}{3}$ / $-\frac{2}{3}$	3: $\frac{5}{4}$			2:	10
7: 0	1: $\frac{3}{2}$		2: $\frac{22}{3}$ / $-\frac{2}{3}$	4: $\frac{15}{2}$	3: 5			9
9: 0 / 1	1: $-\frac{61}{6}$	2: $\frac{13}{2}$	3: 5		1: 7 / 1	3:	4: 8 / 1	7
		13	15	30	20	17	16	

(ii) $\theta = 3$

u_i \ η_i	0	2	4	6	8	10	12	
	0	$\frac{5}{2}$ / $\frac{1}{2}$	2 / $\frac{1}{3}$	$\frac{1}{4}$	1 / $\frac{1}{3}$	$\frac{5}{3}$ / $\frac{1}{3}$	1	
1: 0	1: 9	1: $\frac{3}{2}$ / $-\frac{1}{2}$	2: 2 / $-\frac{2}{3}$	3: $\frac{25}{4}$	1: 9 / $-\frac{1}{3}$	2: $\frac{14}{3}$ / $-\frac{2}{3}$	3: 6	9
3: 0 / 1	1: $-\frac{53}{6}$	2: $\frac{61}{6}$	1: 1 / $\frac{2}{3}$		3: $\frac{20}{3}$	2: $\frac{2}{3}$ / $\frac{1}{3}$	1: 0 / 1	8
5: 0	1: 2	3: $\frac{13}{2}$ / $-\frac{3}{2}$	2: 7 / $-\frac{2}{3}$	3: $\frac{5}{4}$			2: 8	10
7: 0	1: $\frac{3}{2}$		2: 6 / $-\frac{2}{3}$	4: $\frac{15}{2}$	3: 5 / -1			9
9: 2 / 1	2 / 1	2: $-\frac{11}{3}$	3: 5		1: 9 / $\frac{2}{3}$	3: $\frac{17}{3}$	4: 10 / 1	7
		13	15	30	20	17	16	

Table 6

(iii)

		0	2	4	6	8	10	12	
		0	4; 3/4	3; 1/2	1/4	2; 1/2	8/3; 1/2	1	
1	0; 1	1: **-17/4**	1: 0; 1/4	2: **53/4**	3: 25/4; 1	1: 8; 1/2	2: 8/3; 0	3: 6; 1	9
3	3; 3/2	1: 3; 3/2	2: **4/3**	1: 3; 1	▨	3: **20/3**	2: 5/3; 1/2	1: 3; 3/2	8
5	0	1: **2**	3: 2; -9/4	2: 5; -1	3: 5/4	▨	▨	2: **8**	10
7	0	1: **3/2**	▨	2: 4; -1	4: **15/2**	3: 2; -3/2	▨	▨	9
9	5; 3/2	1: 5; 3/2	2: **31/6**	3: **-23/6**	▨	1: 11; 1	3: **17/3**	4: 13; 3/2	7
			13	15	30	20	17	16	

$\theta = \frac{8}{9}$

)

		0	2	4	6	8	10	12	
		0	14/3	31/9	1/4	22/9	28/9	1	
1	8/9	1: 8/9; -2	1: 2/9; -2	2: **9**	3: 257/36; -2	1: 76/9; -2	2: 8/3; -2	3: 62/9; -2	9
3	13/3	1: 13/3	2: **4/3**	1: 35/9; 1	▨	3: **20/3**	2: 19/9	1: 13/3	8
5	0	1: **1/9**	3: **17/9**	2: 37/9; 2	3: 5/4	▨	▨	2: **8**	10
7	0	1: **3/2**	▨	2: 28/9; 2	4: **15/2**	3: 2/3	▨	▨	9
9	19/3	1: 19/3	2: **7/3**	3: **-1**	▨	1: 107/9	3: **17/3**	4: 43/3	7
			13	15	30	20	17	16	

$\theta = \frac{2}{9}$

Table 6

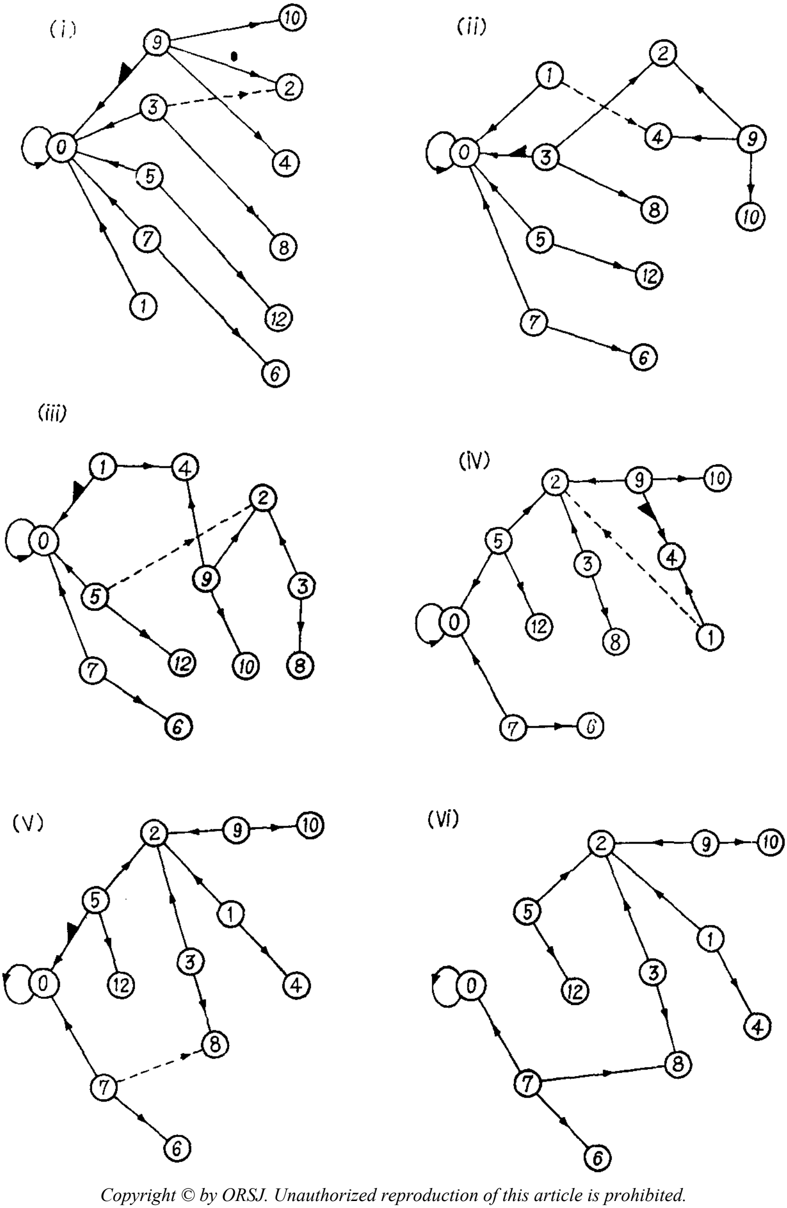
(V) $\theta = 1$

		0	2	4	6	8	10	12	
		0	$\frac{14}{3}$; $\frac{1}{3}$	$\frac{29}{9}$; $\frac{1}{6}$	$\frac{1}{4}$	$\frac{22}{9}$; $\frac{2}{9}$	$\frac{28}{9}$; $\frac{2}{9}$	1 ; $\frac{1}{2}$	
1	$\frac{4}{9}$; $\frac{1}{3}$	1 $\frac{4}{9}$; $\frac{1}{3}$	[1] $\frac{3}{2}$	[2] $\frac{15}{2}$	3 $\frac{241}{36}$; $\frac{1}{3}$	1 8 ; $\frac{1}{9}$	2 $\frac{20}{9}$; $-\frac{1}{9}$	3 $\frac{58}{9}$; $-\frac{7}{6}$	9
3	$\frac{13}{3}$; $\frac{2}{3}$	1 $\frac{13}{3}$; $\frac{2}{3}$	[2] $\frac{4}{3}$	1 $\frac{37}{9}$; $\frac{1}{2}$	▨	[3] $\frac{20}{3}$	2 $\frac{19}{9}$; $\frac{2}{9}$	1 $\frac{13}{3}$; $\frac{1}{6}$	8
5	0 ; 1	[1] $-\frac{1}{18}$	[3] $\frac{37}{18}$	2 $\frac{41}{9}$; $\frac{2}{3}$	3 $\frac{5}{4}$; 1	▨	▨	[2] 8	10
7	0	[1] $\frac{3}{2}$	▨	2 $\frac{32}{9}$; $-\frac{1}{3}$	[4] $\frac{15}{2}$	3 $\frac{2}{3}$; $-\frac{2}{3}$	▨	▨	9
9	$\frac{19}{3}$; $\frac{2}{3}$	1 $\frac{19}{3}$; $\frac{2}{3}$	[2] $\frac{4}{3}$	3 $\frac{2}{3}$; $\frac{1}{6}$	▨	1 $\frac{107}{9}$; $\frac{4}{9}$	[3] $\frac{17}{3}$	4 $\frac{43}{3}$; $-\frac{4}{3}$	7
			13	15	30	20	17	16	

(VI)

		0	2	4	6	8	10	12	
	u_i	0	5	$\frac{61}{18}$	$\frac{1}{4}$	$\frac{8}{3}$	$\frac{10}{3}$	$\frac{3}{2}$	
1	$\frac{7}{9}$	$\frac{7}{9}$	[1] $\frac{3}{2}$	[2] $\frac{15}{2}$	$\frac{253}{36}$	$\frac{73}{9}$	$\frac{19}{9}$	$\frac{95}{18}$	9
3	5	5	[2] $\frac{17}{12}$	$\frac{83}{18}$	▨	[3] $\frac{79}{12}$	$\frac{7}{3}$	$\frac{25}{6}$	8
5	1	1	[3] 2	$\frac{47}{9}$	$\frac{9}{4}$	▨	▨	[2] 8	10
7	0	[1] $\frac{17}{12}$	▨	$\frac{29}{9}$	[4] $\frac{15}{2}$	[3] $\frac{1}{12}$	▨	▨	9
9	7	7	[2] $\frac{4}{3}$	$\frac{5}{6}$	▨	$\frac{37}{3}$	[3] $\frac{17}{3}$	13	7
			13	15	30	20	17	16	

Optimal!


(i)
(ii)
(iii)
(iV)
(V)
(Vi)