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A METHOD FOR COUNTING THE NUMBER OF FEASIBLE SUBSETS OF A PARTIALLY ORDERED FINITE SET

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ABSTRACT

This paper gives a method for counting the number of feasible subsets of a finite set $P=\{1,2,\cdots,n\}$ with a partial ordering $<$ expressing the precedence relations among the elements, where a subset $S=\{i_1, i_2, \cdots, i_\nu\}$ of P is said to be feasible with respect to P if $i_\mu \varepsilon S$ and $i_\lambda < i_\mu$ imply $i_\lambda \varepsilon S$.

The method is as follows;

Step 1: Assign a 2×2 index matrix to each arrow in the diagram which corresponds to the set P.

Step 2: Reduce the diagram to a single arrow by applications of rules for series, parallel, etc.

Then we can calculate the number from the last index matrix of the reduced arrow.

1. INTRODUCTION

Let us consider a finite set $P=\{1, 2, \cdots, n\}$ with a partial ordering $<$expressing the precedence relations among the elements.

A subset $S=\{i_1, i_2, \cdots, i_\nu\}$ of P is said to be feasible with respect to P if $i_\mu \varepsilon S$ and $i_\lambda < i_\mu$ imply $i_\lambda \varepsilon S$.

The problem to count the number of feasible subsets of a partially ordered finite set, arises whenever sequencing problems with precedence constraints, *e.g.*, a line-balancing problem, are treated. This paper gives a method for counting the number of feasible subsets.

2. METHOD

It is convenient to introduce some concepts associated with the partial ordering.

Connected elements: Two elements i and j of P are said to be connected if there exist elements $k_1, k_2, \cdots, k_\nu$ of P such that i is comparable to k_1, k_1 is comparable to $k_2, \cdots, k_\nu$ is comparable to j.

Disjoint subsets: Two subsets P_1 and P_2 with no common elements are said to be disjoint if there exist no connected elements (i, j) such that $i \varepsilon P_1$ and $j \varepsilon P_2$.

Direct predecessor or successor: If $i<j$ and there is no third element k such that $i<k<j$, it is said that i is a direct predecessor of j or that j is a direct successor of i.

Let $[P]$ be the number of feasible subsets of a set P.

THEOREM 1. If P is empty, $[P]=1$.

THEOREM 2. If P consists of a single element, $[P]=2$.

THEOREM 3. If P is divisible to two or more subsets $P_1, P_2, \cdots, P_\nu$, which are disjoint each other, $[P]=\prod_{\alpha=1}^{\nu}[P_\alpha]$.

When P consists of two or more elements, let us represent P in a convenient diagram $D(P)$ as follows; node i corresponds to element i and an arrow is drawn from node i to node j if i is a direct predecessor of j.

[Procedure A]

Step 1: Assign the index matrix $\begin{pmatrix}1&0\\1&1\end{pmatrix}$ to each arrow in $D(P)$.

Step 2: Apply the following rules which are illustrated in Fig. 1, successively until $D(P)$ is reduced to a signal arrow.

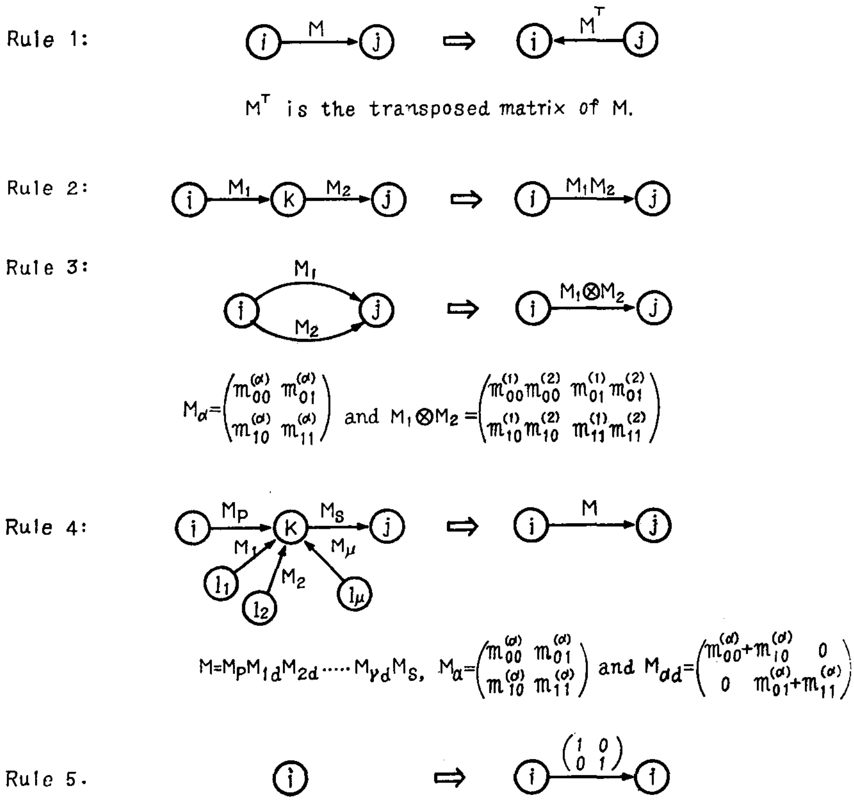


Fig. 1. Illustration of Rules.

Rule 1: When we change the direction of an arrow, transpose its index matrix.

Rule 2: Reduce two sequent arrows to an arrow whose index matrix is the product of their index matrices.

Rule 3: Reduce two parallel arrows to an arrow whose index matrix is the element-wise product of their index matrices.

Rule 4: When there exist an arrow from i to k with the index matrix M_p, an arrow from k to j with the index matrix M_s, and arrows from l_α to k with the index matrices $M_\alpha=\begin{pmatrix} m_{00}^{(\alpha)} & m_{01}^{(\alpha)} \\ m_{10}^{(\alpha)} & m_{11}^{(\alpha)} \end{pmatrix}$ $(\alpha=1, 2, \cdots, \nu)$, reduce them to an arrow from i to j whose index matrix is $M=M_pM_{1d}M_{2d}\cdots M_{\nu d}M_s$, where $M_{\alpha d}=\begin{pmatrix} m_{00}^{(\alpha)}+m_{10}^{(\alpha)} & 0 \\ 0 & m_{01}^{(\alpha)}+m_{11}^{(\alpha)} \end{pmatrix}$

Rule 5: If necessary, we make divide a node into two nodes and an arrow between them whose index matrix is $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Then connect the arrows connected to the old node to either of the new nodes.

THEOREM 4. If $D(P)$ is reduced to a single arrow whose index matrix is $\begin{pmatrix} m_{00} & m_{01} \\ m_{10} & m_{11} \end{pmatrix}$, then $[P]=m_{00}+m_{01}+m_{10}+m_{11}$.

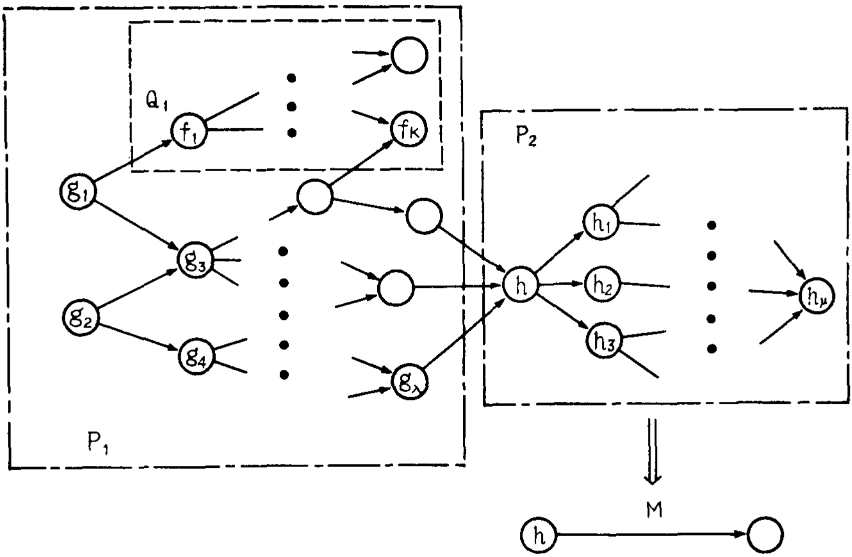


Fig. 2. Division of P into P_1 and P_2.

If $D(P)$ is not reducible to a single arrow, divide P into two subsets P_1 and P_2 satisfying the conditions of THEOREM 5, and calculate $[P_1]$, $[Q_1]$ and the index matrix of $D(P_2)$.

THEOREM 5. If P is divisible into two subsets P_1 and P_2 such that (1) only one element h is the direct successor in P_2 of any element of P_1, (2) there exist no successors of h in P_1, and (3) by applications of Rules, $D(P_2)$ is reducible to a single arrow from h with the index matrix $\begin{pmatrix} m_{00} & m_{01} \\ m_{10} & m_{11} \end{pmatrix}$, then $[P]=[P_1]\ (m_{00}+m_{01})+[Q_1]\ (m_{10}+m_{11})$.

Here Q_1 is the set consisting of all the elements in P_1 which don't precede h and having the partial ordering between them. (See Fig. 2.)

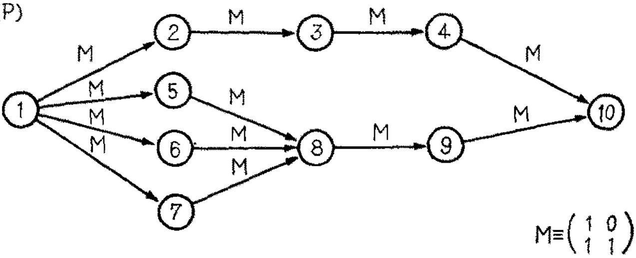
The proof of these theorems is shown in §4.

If we cannot apply THEOREMS 1-4 for counting $[P_1]$ or $[Q_1]$, continue to divide P_1 or Q_1.

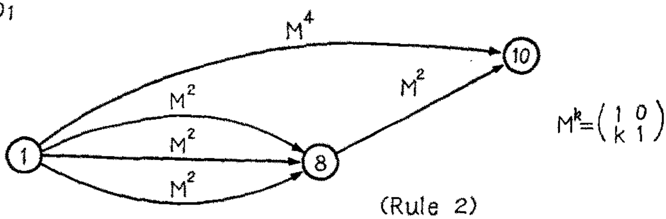
3. EXAMPLES

Example 1:

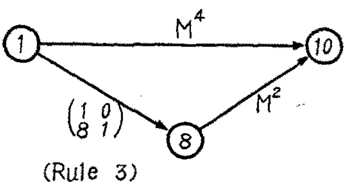
D(P)



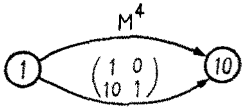
D1



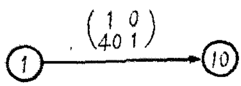
D2



D3



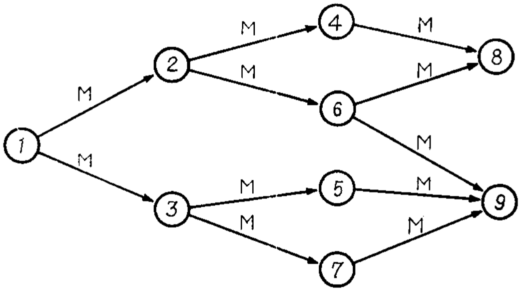
D4



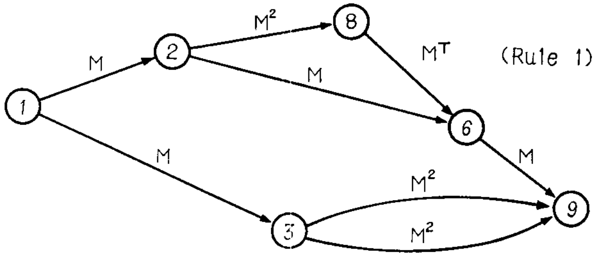
Hence $(P)=1+0+40+1=42$.

Example 2:

D(P)

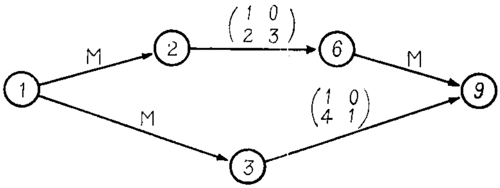


D_1

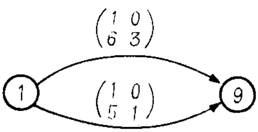


(Rule 2)

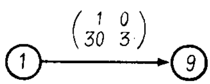
D_2



D_3

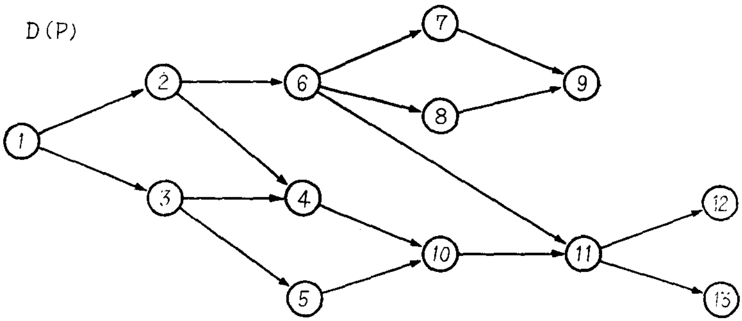


D_4

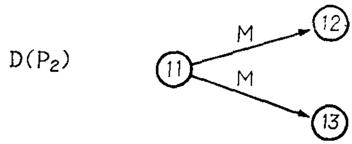


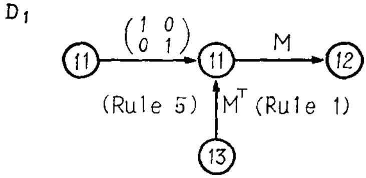
Hence $[P]=1+0+30+3=34$.

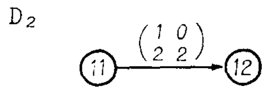
Example 3:

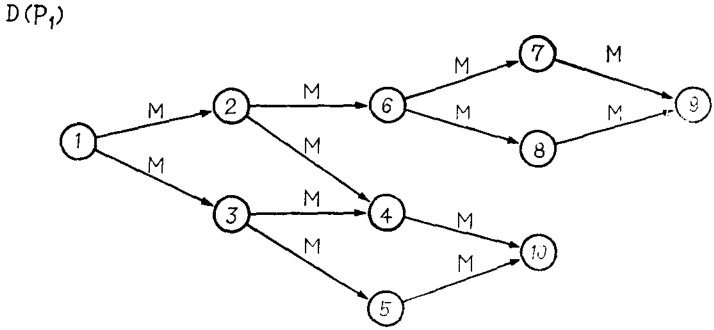


Put h=11 and divide P into P_1 and P_2



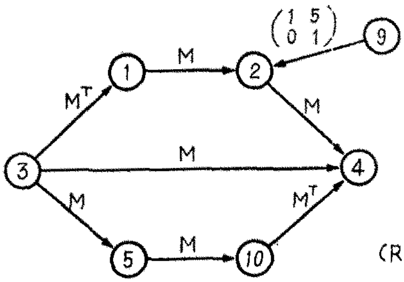






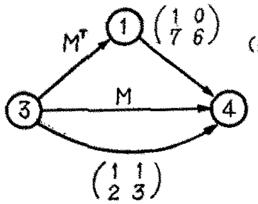
Example 3 (continued):

D_1



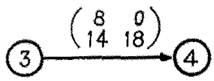
(Rules 1, 2 and 3)

D_2



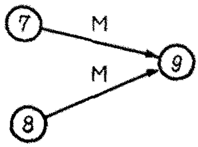
(Rule 4)

D_3

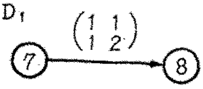


$[P_1]=8+0+14+18=40$

$D(Q_1)$



D_1



$[Q_1]=1+1+1+2=5$

Hence $[P]=40\times 1+5\times 4=60$.

4. MATHEMATICAL JUSTIFICATION

THEOREMS 1-3 can be easily verified.

Before we prove THEOREM 4, let us consider what each element of the index matrix means.

Now it is convenient to represent a subset S of P in $(x_1, x_2, \cdots, x_n)$ where $x_i=0$ if $i \notin S$ and $x_i=1$ if $i \in S$. Obviously $(0, 0, \cdots, 0)$ is an empty set and $(1, 1, \cdots, 1)$ is the whole set P. We define $\phi(x_{i_1}, x_{i_2}, \cdots, x_{i_\nu})$ as follows;

$$\phi(x_{i_1}, x_{i_2}, \cdots, x_{i_\nu})=\begin{cases} 1 & \text{if } (x_{i_1}, x_{i_2}, \cdots, x_{i_\nu}) \text{ is a feasible subset,} \\ 0 & \text{if not so,} \end{cases}$$

when the set $\{i_1, i_2, \cdots, i_\nu\}$ is considered as if the whole set.

Then $\sum_{x_1}\sum_{x_2}\cdots\sum_{x_n}\phi(x_1, x_2, \cdots, x_n)$ gives the number of feasible subsets in P.

LEMMA 1. Let $F=\{f_1, f_2, \cdots, f_\kappa\}$ be a subset which separates two subsets $G=\{g_1, g_2, \cdots, g_\lambda\}$ and $H=\{h_1, h_2, \cdots, h_\mu\}$, as shown Fig. 3. Then

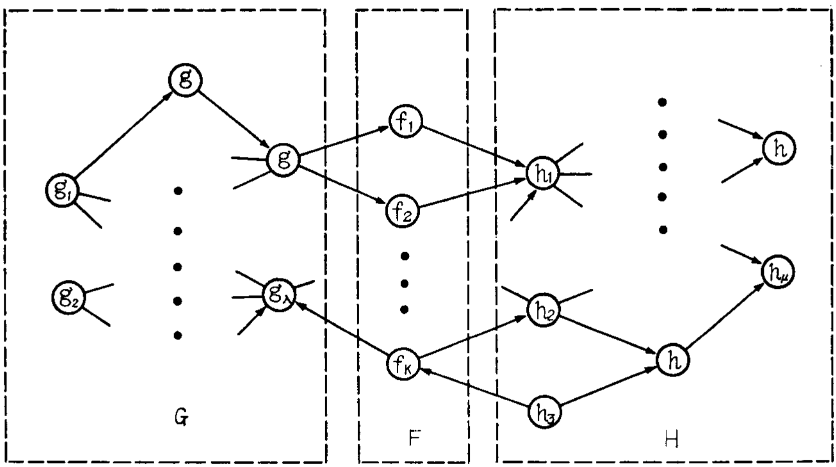


Fig. 3. Relation of F, G and H in LEMMA 1.

$$\phi(x_{f_1}, x_{f_2}, \cdots, x_{f_\kappa}, x_{g_1}, \cdots, x_{g_\lambda}, x_{h_1}, \cdots, x_{h_\mu})$$
$$=\phi(x_{f_1}, x_{f_2}, \cdots, x_{f_\kappa}, x_{g_1}, \cdots, x_{g_\lambda})$$
$$\times\phi(x_{f_1}, x_{f_2}, \cdots, x_{f_\kappa}, x_{h_1}, \cdots, x_{h_\mu}).$$

Proof. If a subset $(x_1, x_2, \cdots, x_n)$ is feasible w.r.t. P, a subset $(x_{i_1}, x_{i_2},$

$\cdots, x_{i_\nu})$ is also feasible w.r.t. $Q=\{i_1, i_2, \cdots, i_\nu\}$. So, if $\phi(x_{f_1}, \cdots, x_{f_\kappa}, x_{g_1}, \cdots, x_{g_\lambda}, x_{h_1}, \cdots, x_{h_\mu})=1$, $\phi(x_{f_1}, \cdots, x_{f_\kappa}, x_{g_1}, \cdots, x_g)=\phi(x_{f_1}, \cdots, x_{f_\kappa}, x_{h_1}, \cdots, x_{h_\mu})=1$.

If $\phi(x_{f_1}, \cdots, x_{f_\kappa}, x_{g_1}, \cdots, x_{g_\lambda}, x_{h_1}, \cdots, x_{h_\mu})=0$, there exists a pair (i, j) such that $i, j \varepsilon F \cup G \cup H$, $i<j$, $x_i=0$, $x_j=1$.

If $i, j \varepsilon F \cup G$, then $\phi(x_{f_1}, \cdots, x_{f_\kappa}, x_{g_1}, \cdots, x_{g_\lambda})=0$.

If $i, j \varepsilon F \cup H$, then $\phi(x_{f_1}, \cdots, x_{f_\kappa}, x_{h_1}, \cdots, x_{h_\mu})=0$.

If $i \varepsilon G$ and $j \varepsilon H$(or i H and $j \varepsilon G$), then there exists an element $k \varepsilon F$ such that $i<k<j$, because F separates G and H. Hence, $\phi(x_{f_1}, \cdots, x_{f_\kappa}, x_{g_1}, \cdots, x_{g_\lambda})=0$ if $x_k=1$ (or $x_k=0$), or $\phi(x_{f_1}, \cdots, x_{f_\kappa}, x_{h_1}, \cdots, x_{h_\mu})=0$ if $x_k=0$ (or $x_k=1$).

Consequently, $\phi(x_{f_1}, \cdots, x_{f_\kappa}, x_{g_1}, \cdots, x_{g_\lambda}, x_{h_1}, \cdots, x_{h_\mu})=\phi(x_{f_1}, \cdots, x_{f_\kappa}, x_{g_1}, \cdots, x_{g_\lambda})\times\phi(x_{f_1}, \cdots, x_{f_\kappa}, x_{h_1}, \cdots, x_{h_\mu})$.

LEMMA 2. Let $\begin{pmatrix} m_{00} & m_{01} \\ m_{10} & m_{11} \end{pmatrix}$ be the index matrix of an arrow from i to j, and let $K=\{k_1, k_2, \cdots, k_\nu\}$ be the elements absorbed in the arrow when some of the Rules are applied. Then

$$(*)\ m_{ab}=\sum_K \phi(x_i=a, x_{k_1}, x_{k_2}, \cdots, x_{k_\nu}, x_j=b) \quad (a=0,1\,;\ b=0,1).$$

Here $\sum_K$ represents $\sum_{x_{k_1}}\sum_{x_{k_2}}\cdots\sum_{x_{k_\nu}}$, and

$\sum_K \phi(x_i=a, x_{k_1}, x_{k_2}, \cdots, x_{k_\nu}, x_j=b)=\phi(x_i=a,\ x_j=b)$ if K is empty.

Proof. We shall prove it inductively.

If i is a direct predecessor of j, $\phi(0,0)=\phi(1,0)=\phi(1,1)=1$ and $\phi(0,1)=0$. The relation (*) is true for the elemental matrix $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ which is assigned to every arrow at first.

Next we shall show that if the relation (*) is true now, then it is so after we apply one of the rules.

For Rule 1: Let $(m_{ab}^{(T)})$ be the index matrix of the arrow from j to i whose direction is changed.

$$m_{ab}^{(T)}=m_{ba}$$
$$=\sum\phi(x_i=b, x_{k_1}, x_{k_2}, \cdots, x_{k_\nu}, x_j=a)$$

The relation (*) is true since there is no absorption of elements.

For Rule 2: Let $G=\{g_1, g_2, \cdots, g_\lambda\}$ be the elements absorbed in an arrow from i to k whose index matrix is $M_1=(m_{ab}^{(1)})$, and let $H=\{h_1, h_2, \cdots, h_\mu\}$ be the elements absorbed in an arrow from k to j whose index matrix $M_2=(m_{ab}^{(2)})$.

Then the element m_{ab} of the index matrix M of the reduced arrow from i to j is as follows;

$$\begin{aligned} m_{ab} &= \sum_k m_{ax_k}^{(1)} m_{x_k b}^{(2)} \\ &= \sum_k \sum_G \phi(x_i=a, x_{g_1}, \cdots, x_{g_\lambda}, x_k) \times \sum_H \phi(x_k, x_{h_1}, \cdots, x_{h_\mu}, x_j=b) \\ &= \sum_k \sum_G \sum_H \phi(x_i=a, x_{g_1}, \cdots, x_{g_\lambda}, x_k, x_{h_1}, \cdots, x_{h_\mu}, x_j=b) \end{aligned}$$

The relation (*) is true, since the elements absorbed in the reduced arrow are $\{g_1, \cdots, g_\lambda, k, h_1, \cdots, h_\mu\}$.

For Rule 3. Let $G=\{g_1, g_2, \cdots, g_\lambda\}$ be the elements absorbed in an arrow from i to j whose index matrix $M_1=(m_{ab}^{(1)})$, and let $H=\{h_1, h_2, \cdots, h_\mu\}$ be the elements absorbed in another arrow from i to j whose index matrix $M_2=(m_{ab}^{(2)})$.

Then the elements m_{ab} of the index matrix M of the reduced arrow from i to j is as follows;

$$\begin{aligned} m_{ab} &= m_{ab}^{(1)} m_{ab}^{(2)} \\ &= \sum_G \phi(x_i=a, x_{g_1}, \cdots, x_{g_\lambda}, x_j=b) \times \sum_H \phi(x_i=a, x_{h_1}, \cdots, x_{h_\mu}, x_j=b) \\ &= \sum_G \sum_H \phi(x_i=a, x_{g_1}, \cdots, x_{g_\lambda}, x_{h_1}, \cdots, x_{h_\mu}, x_j=b) \end{aligned}$$

(from LEMMA 1)

The relation (*) is true.

For Rule 4: Denote F_α, G and H as follows;

$F_\alpha=\{f_1^{(\alpha)}, f_2^{(\alpha)}, \cdots, f_{k(\alpha)}^{(\alpha)}\}$: the elements absorbed in the arrow from l_α to k whose index matrix is $M_\alpha=(m_{ab}^{(\alpha)})$,

$G=\{g_1, g_2, \cdots, g_\lambda\}$: the elements absorbed in the arrow from i to k whose index matrix is $M_p=(m_{ab}^{(p)})$, and

$H=\{h_1, h_2, \cdots, h_\mu\}$: the elements absorbed in the arrow from k to j whose index matrix is $M_s=(m_{ab}^{(s)})$.

Let $M=(m_{ab})$ be the index matrix of the reduced arrow from i to j.

$$m_{ab}=\sum_k m_{ax_k}^{(b)}\{\prod_{\alpha=1}^{\nu}(m_{0x_k}^{(\alpha)}+m_{1x_k}^{(\alpha)})\}m_{x_k b}^{(s)}$$

$$=\sum\sum\phi(x_i=a,x_{g_1},\cdots,x_{g_\lambda},x_k)$$

$$\times\prod_\alpha\{\sum_{l\alpha}\sum_{F\alpha}\phi(x_{l_\alpha},x_{f_1^{(\alpha)}},\cdots,x_{f_{k(\alpha)}^{(\alpha)}},x_k)\}\times\sum_H\phi(x_k,x_{h_1},\cdots,x_{h_\mu},x_j=b)$$

$$=\sum_k\sum_G\phi(x_i=a,x_{g_1},\cdots,x_{g_\lambda},x_k)$$

$$\times\sum_L\sum_{UF\alpha}\phi(x_k,x_{l_1},\cdots,x_{l_\nu},x_{f_1^{(1)}},\cdots,x_{f_{k(1)}^{(1)}},x_{f_1^{(2)}},\cdots,x_{f_{k(\nu)}^{(\nu)}})$$

$$\times\sum_H\phi(x_k,x_{h_1},\cdots,x_{h_\mu},x_j=b)$$

$$=\sum_k\sum_G\sum_H\sum_L\sum_{UF\alpha}\phi(x_i=a,x_{g_1},\cdots,x_{g_\lambda},x_k,x_{l_1},\cdots,x_{l_\nu},$$

$$x_{f_1^{(1)}},\cdots,x_{f_{k(\nu)}^{(\nu)}},x_{h_1},\cdots,x_{h_\mu},x_j=b)$$

where $L=\{l_1,l_2,\cdots,l_\nu\}$ and $UF_\alpha=F_1UF_2U\cdots UF_\nu$.
The relation (*) is true.

For Rule 5: It is obvious that we may divide a node into two nodes and an arrow between them whose index matrix is $\begin{pmatrix}1&0\\0&1\end{pmatrix}$, because feasible subsets neither increase nor decrease in number by adding an element i' to P such that $\phi(x_1,\cdots,x_{i-1},x_i=a,x_{i'}=b,x_{i+1},\cdots,x_n)$

$$=\begin{cases}\phi(x_1,\cdots,x_{i-1},x_i=a,x_{i+1},\cdots,x_n) & \text{if } a=b\\ 0 & \text{if } a\neq b.\end{cases}$$

Consequently this LEMMA is true after rules are applied successively. The proof is accomplished.

For example, that the index matrix of the arrow from 3 to 9 in Example 2-D2 is $\begin{pmatrix}1&0\\4&1\end{pmatrix}$ implies that when $\{3,5,7,9\}$ is considered as if the whole subset, the number of feasible sub sets which contain neither 3 nor 9, contain 9 but not 3, contain 3 but not 9, or contain both 3 and 9 is 1, 0, 4 or 1 respectively.

Proof of THEOREM 4.

The fact that $D(P)$ is reduced to a single arrow from i to j means that all the elements but i and j are absorbed in the arrow. Hence $\sum_i\sum_j m_{xixj}=\sum_P\phi(x_1,x_2,\cdots,x_n)=[P]$, which completes the proof.

Before we prove THEOREM 5, we shall present LEMMA 3, easily verified.

LEMMA 3. If h has no successor in $G=\{g_1, g_2, \cdots, g_\lambda\}$, then

$$\phi(x_{g1}, x_{g2}, \cdots, x_{g\lambda}, x_h=0)=\phi(x_{g1}, x_{g2}, \cdots, x_{g\lambda}).$$

If f has no predecessor in G, then

$$\phi(x_f=1, x_{g1}, x_{g2}, \cdots, x_{g\lambda})=\phi(x_{g1}, x_{g2}, \cdots, x_{g\lambda}).$$

Proof of THEORAM 5.

Let the elements of P_1, Q_1 or P_2 be $\{f_1, f_2, \cdots, f_\kappa, g_1, \cdots, g_\lambda\}$, $\{f_1, f_2, \cdots, f_\kappa\}$ or $\{h, h_1, h_2, \cdots, h_\mu\}$ respectively.

$$\begin{aligned}[P]&=\sum_{P_1}\sum_{P_2}\phi(x_{f1}, \cdots, x_{f\kappa}, x_{g1}, \cdots, x_{g\lambda}, x_h, x_{h1}, \cdots, x_{h\mu})\\ &=\sum_{P_1}\phi(x_{f1}, x_{f2}, \cdots, x_{f\kappa}, x_{g1}, \cdots, x_{g\lambda}, x_h=0)\\ &\times\sum_{P_2-h}\phi(x_h=0, x_{h1}, \cdots, x_{h\mu})\\ &+\sum_{Q_1}\phi(x_{f1}, x_{f2}, \cdots, x_{f\kappa}, x_{g1}=x_{g2}=\cdots=x_{g\lambda}=x_h=1)\\ &\times\sum_{P_2-h}\phi(x_h=1, x_{h1}, x_{h2}, \cdots, x_{h\mu})\end{aligned}$$

From LEMMA 3,

$$\begin{aligned}&\phi(x_{f1}, x_{f2}, \cdots, x_{f\kappa}, x_{g1}, \cdots, x_{g\lambda}, x_h=0)\\ &\quad=\phi(x_{f1}, x_{f2}, \cdots, x_{f\kappa}, x_{g1}, \cdots, x_{g\lambda}), \text{ and}\\ &\phi(x_{f1}, x_{f2}, \cdots, x_{f\kappa}, x_{g1}=x_{g2}=\cdots=x_{g\lambda}=x_h=1)\\ &\quad=\phi(x_{f1}, x_{f2}, \cdots, x_{f\kappa}).\end{aligned}$$

Consequently

$$[P]=[P_1]\ (m_{00}+m_{01})+[Q_1]\ (m_{10}+m_{11}),$$

which completes the proof.

5. FINAL REMARKS

If $D(P)$ is a series-parallel diagram, e.g. Example 1, the number can be counted more easily by Procedure B and THEOREM 4B.

[Procedure B]

Step 1: Assign the index number 1 to each arrow in $D(P)$.

Step 2: Apply the following Rules, which are illustrated in Fig. 4, until $D(P)$ is reduced to a single arrow.

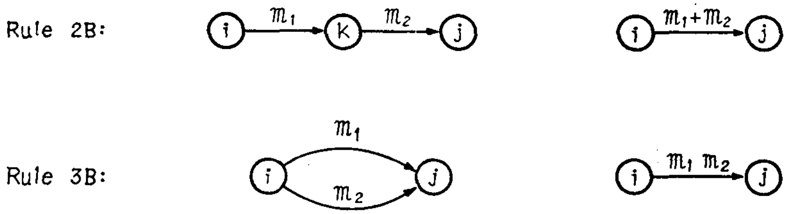


Fig. 4. Illustration of Rules (B).

Rule 2B: Reduce two sequent arrows to an arrow whose index number is the sum of their index numbers.

Rule 3B: Reduce two parallel arrows to an arrow whose index number is the product of their index numbers.

THEOREM 4B. If $D(P)$ is reduced to a single arrow whose index number is m by Procedure B, then $[P]=m+2$.

The index number in Procedure B corresponds to m_{10} of the index matrix in Procedure A. When $D(P)$ is series-parallel diagram, by applications of Rule 2 or 3, the index matrix always retains the form of $\begin{pmatrix} 1 & 0 \\ m & 1 \end{pmatrix}$.

Surely

$$\begin{pmatrix} 1 & 0 \\ m_1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ m_2 & 1 \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ m_1+m_2 & 1 \end{pmatrix} \qquad \text{(Rule 2)}$$

and

$$\begin{pmatrix} 1 & 0 \\ m_1 & 1 \end{pmatrix}\otimes\begin{pmatrix} 1 & 0 \\ m_2 & 1 \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ m_1\times m_2 & 1 \end{pmatrix}. \qquad \text{(Rule 3)}$$

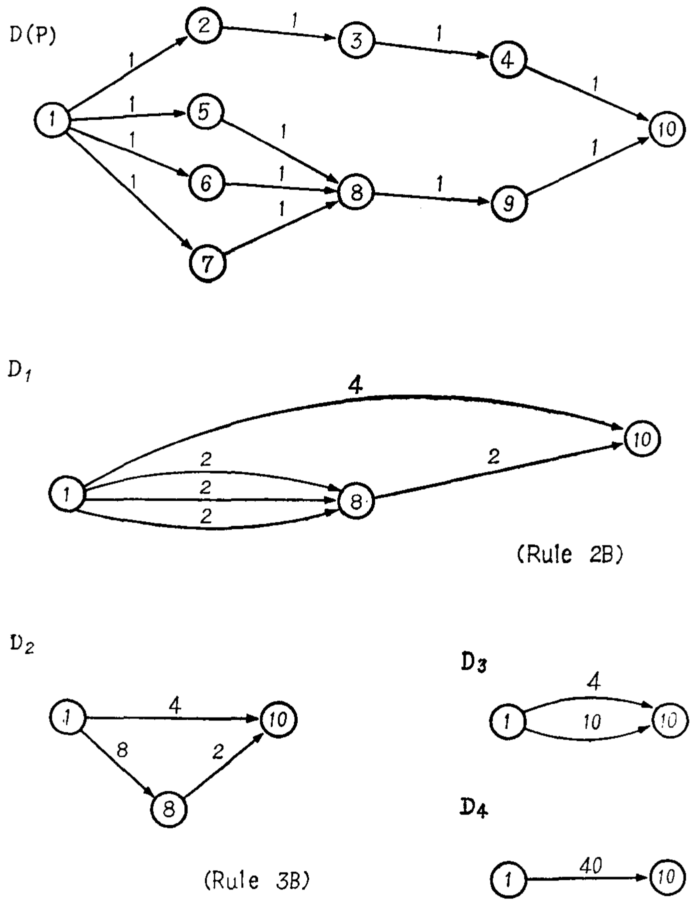
Hence it is enough to calculate only m_{10} of the index matrix. Note that $D(P)$ is always reducible to a single arrow by applications of Rule 2B or 3B in this case.

Held, Karp and Shareshian[2)] introduced the concepts of basic complement and component of P, and proved that

(1) $[P]$ is equal to the sum of the numbers of feasible subsets of the basic complements and that

(2) $[P]$ is equal to the product of the numbers of feasible subsets

Example 1B:



Hence [P]=40+2=42.

of the components of *P*. (This corresponds to THEOREM 3 in this paper.)

They presented a method for counting the number by applying these theorems and a procedure in a special case when $D(P)$ is a tree.

However Procedure A is applicable in much wider case. In many practical cases, $D(P)$ is reducible to a single arrow by Procedure A, and necessity of separating P rarely occurs.

In a line-balancing problem, the number of feasible subsets gives the total number of storage locations and a rough measure of the time required for the calculation by the computer. So we can see whether it is practically possible to apply an exhaustive procedure like Jackson's[1] or Gutjahr's[3].

REFERENCES

[1] Jackson, J.R., "A Computing Procedure for a Line Balancing," Man. Sci. Vol. 2 (1956) 261—271.

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[3] Gutjahr, A.L. and Nemhauser, G.L., "An Algorithm for the Line Balancing Problem," Man. Sci. Vol. 11 (1964) 308—315.