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ON THE STABILITY OF VEHICULAR TRAFFIC FLOW—A PHENOMENOLOGICAL VIEWPOINT

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ABSTRACT

An attempt is made to explain the cause of instability in vehicular traffic flow by assuming that the equilibrium flow of a system is related to the concentration of vehicles in the system by a parabolic curve. Measurements of tunnel and freeway traffic are the basis for this assumption. It is found that the flow for densities less than that for maximum flow, *i.e.*, the unsaturated flow regime, is completely stable. On the other hand the saturated flow regime is unstable to certain perturbations but will support finite oscillations. Finite oscillations between the two regimes can be either stable or unstable, but in particular oscillations around maximum flow are unstable.

INTRODUCTION

The general behavior of traffic flow as a function of concentration is well known, at least qualitatively, for many systems. Examples of such systems are single lane flow in tunnels for which there is good quantitative data [1] and multi-lane flow on freeways where the situation is more difficult [2]. This behavior of the steady state flow is shown schematically in Fig. 1 where the abscissa is the number of vehicles in the syssem at one time and the ordinate is the flow in vehicles per unit time. The solid line represents the gross or averaged flow and the circles represent data points. The average flow increases linearly at first with the number of vehicles in the system but as the number of vehicles increases the flow reaches a maximum and then drops off and finally breaks up. The scatter in the data points, of course, is because the flow and the number of vehicles are both statistical quantities and subject to fluctuations. Large fluctuations indicate the onset of instability and it is this effect that we want to investigate here.

We will use an extremely simple model. Fig. 2, which is an idealization of Fig. 1, shows the flow as a function of the number of vehicles in the system assuming that there are no fluctuations; we do not supply the reasons for such behavior but take this result as our starting point. For simplicity we take the flow curve to be a parabola following Helly [3] who worked with the discrete case. We assume that this curve determines the dynamic behavior of the system and thus have restricted our considerations to the slowly varying case. We further assume the flow out of the system is equal to this flow. Finally, we assume that the flow into the system is independent of the number of vehicles in the system—at least up to the point where the flow out is zero when the model breaks down.

If we let $N(t)$ be the number of vehicles in the system at time t it then follows that

$$\frac{dN}{dt}=q(t)-\gamma N(2M-N),\ 0\leq N(t)<2M, \tag{1}$$

where

$q(t)$ = flow into system, vehicles per unit time,
γ = constant,
M = constant, the number for which the system has maximum flom out.

In writing down (1) we have assumed that the flow out is immediately affected by any change in the input flow. As is well known a similar physically unrealizable condition obtains for the heat equation. Such a change in the flow out is patently unrealizable for a well defined bottleneck downstream of the input. However, if the bottleneck is ill defined or distributed as for example can be the case in multilane flow on a more or less homogeneous highway, it seems reasonable that there should be a regime where (1) is a valid approximation to a physical system. Remarks akin to these can be made about the validity of the assumption that the flow in is an independent variable. Unfortunately because of the present state of the theory of vehicular traffic flow no quantitative limits can be put on the regime of validity; the plausibility of the model must rest on the results which can be obtained from it. Two results which will be obtained which argue in favor of plausibility are (1) the instability of the flow to a small change in the input on the retrograde side of the flow curve, which would thus cause the scatter in measured data, and (2) the impossibility of maintaining a flow at maximum, which is not documented but is noised about.

2. PRELIMINARY ANALYSIS

For mathematical convenience we set (see the dashed axes in Fig. 2)

$$\begin{aligned} \eta &= (N-M)/M \\ \omega &= q/\gamma M^2 \\ \tau &= \gamma M^2 t \end{aligned} \tag{2}$$

so that becomes

$$\frac{d\eta}{d\tau} = \omega(\tau) - 1 + \eta^2, \quad -1 \leq \eta < 1. \tag{3}$$

The normalized flow out is $[1-\eta^2]$ and thus for steady state flow a necessary condition is $0 \leq \omega \leq 1$. The time variable τ has a unit such that if the system starts at time $\tau=0$ with no vehicles in it, $\eta=-1$, vehicles enter at the maximum steady steady state rate, $\omega=1$, and no vehicles are allowed to leave, then at time $\tau=1$ we find that $\eta=0$, *i.e.*, the number of vehicles in the system is such that the flow is a maximum.

The steady state solutions of (3) are obtained by setting $\omega(\tau)=\theta$, a constant, and $d\eta/d\tau=0$. Then

$$\eta(\tau)=\pm\sqrt{(1-\theta)}, \text{ for } 0\leq\theta\leq1. \tag{4}$$

The negative sign is for the unsaturated flow case and the positive sign for the saturated flow. Of course, these solutions were purposely built into the model.

Since we are interested in time varying solutions of (3) let us examine this equation from the mathematical viewpoint briefly. It is a Riccatti equation and is discussed in many places, *e.g.*, Ince [4], Forsythe [5], Watson [6]. The substitution $\eta=-v'/v$ linearizes it,

$$v''+[\omega(\tau)-1]v=0. \tag{5}$$

Ideally, analytic solutions to (3) or (5) for noiselike $\omega(\tau)$ are desired. What is available, however, is the following: For ω a constant the solution of (5) is trivial. For ω a periodic function (5) is Hill's equation and for the special case $\omega=\alpha+\beta\cos p\tau$ it is Mathieu's equation, [4,5] Whittaker and Watson [7], Stoker [8], McLachlin [9]. The periodic cases are difficult enough; apparently nothing has been done for more noiselike ω. Thus there is little use for the existing literature.

3. CONSTANT INPUT

Let us set $\omega(\tau)=\theta$, a constant; we can then find η straightforwardly by solving (5). It is convenient to define

$$\mu=+\sqrt{|1-\theta|},\ 0\leq\theta. \tag{6}$$

Case I: $0\leq\theta<1$.

$$\eta(\tau)=-\mu+2\mu\left[1+\frac{\mu-\eta_0}{\mu+\eta_0}e^{2\mu\tau}\right]^{-1},\ \eta(0)=\eta_0,\ \tau\geq0. \tag{7}$$

In the range of regularity of η it is a monotone function of τ. For

$\eta_o<0$, $\eta(\tau)$ increases or decreases, as the case may be, smoothly to $\eta(\infty)=-\mu$, a point in the unsaturated steady state flow region in Fig. 2. For $0<\eta_o<\mu$ the same is true, *i.e.*, starting at the point η_o, η decreases to $\eta(\infty)=-\mu$ while the flow increases to a maximum and then decreases toward its steady state value. On the other hand, for $\eta_o>\mu$ the flow decreases and η builds up to $\eta=1$ so that the flow stops and the model breaks down. Hence, points representing flows which are to left of the maximum in Fig. 2 are stable to step function changes in the input while points on the retrograde side of the curve are unstable to step function changes because either the flow goes to the unsaturated condition or the flow breaks down no matter how small the change. As has already been remarked in the Introduction this behavior gives us an important plausibility argument for the validity of the model. Its importance is seen when the measurement of the statistics involved is considered. The measurement must take place during a finite length of time; if a step function change in input occurs during this time the instability would cause considerable scatter in the numerical results.

Case II: $\theta>1$.

Clearly this case is transient

$$\eta(\tau)=\eta_o-\frac{\mu^2+\eta_o^2}{\eta_o-\mu\cot\mu\tau},\ \eta(0)=\eta_o,\ \tau\leq 0. \tag{8}$$

which is valid for $0\leq\tau<\tau_o$ where $\eta(\tau_o)=1$.

There remains the transition case $\theta=1$; one finds that

$$\eta(\tau)=\eta_o/(1-\eta_o\tau),\ \eta(0)=\eta_o,\ \tau\geq 0. \tag{9}$$

The solution is stable for $\eta_o\leq 0$ and otherwise holds only for $0\leq\tau<\tau_o$ where $\eta(\tau_o)=1$.

4. SMALL OSCILLATIONS

Let us next consider the more noiselike case of an oscillatory input; put

$$\omega(\tau)=\alpha+\beta\cos p\tau,\ \alpha>0,\ |\beta|\leq\alpha. \tag{10}$$

Since α is the average flow in the only case of interest is $\alpha\leq 1$ and so we set

$$1>k^2=1-\alpha\geq 0. \tag{11}$$

The differential equation (3) then becomes

$$\eta'=-k^2+\beta\cos p\tau+\eta^2. \tag{12}$$

As already noticed the known asymptotic results for the Mathieu equation, which is the equation (12) goes into when linearized by the substitution $\eta=-v'/v$, do not appear to be useful to us. We therefore restrict the examination to the case $|\beta|$ small and obtain a few terms of a perturbation expansion of η in powers of β. We set

$$\eta(\tau)=y_0(\tau)+\beta y_1(\tau)+\beta^2 y_2(\tau)+\ldots, \tag{13}$$

substitute it into (12) and set the coefficients of β to zero; thus

$$\begin{aligned} &y_0'-y_0^2=-k^2,\\ &y_1'-2y_0y_1=\cos p\tau,\\ &y_2'-2y_0y_2=y_1^2,\\ &\ldots\ldots \end{aligned} \tag{14}$$

The analysis can be simplified by choosing the initial conditions judiciously. Let us set $y_0(0)=k$; the solution to the first of equation (14) is then

$$y_0(\tau)=k,\ -1<k<+1. \tag{15}$$

With the above substitution the second of equation (14) has the general solutioa

$$y_1(\tau)=-\frac{1}{\sqrt{(4k^2+p^2)}}\cos(p\tau+\delta_1)+\alpha_1 e^{2k\tau},\ \tan\delta_1=p/2k. \tag{16}$$

For k negative the last term goes to zero as τ increases regardless of the initial canditions; for $k\geq 0$ the initial conditions can be chosen such that $\alpha_1=0$ to preserve the oscillatory character of the solution. We can proceed in a like manner for higher powers of β and the results, except for $k=0$, are the same; that is, for k less than zero the solution is unconditionally asymptotically oscillatory, for k greater than zero the initial conditions may be chosen so that the solution is oscillatory. For the case $k=0$ the solution (16), with $\alpha_1=0$, reduces to

$$y_1(\tau)=\frac{1}{p}\sin p\tau\,. \tag{17}$$

Substituting this into the third of equations (14) and solving we find

$$y_2(\tau)=\frac{\tau}{2p^2}-\frac{1}{4p^3}\sin 2p\tau+\alpha_2\,, \tag{18}$$

which is unstable for any choice of α_2.

Summing up the results of this section we have found that (1) the flow is unconditionally stable to small oscillations in the input flow for flows represented by points to the left of the maximum in Fig. 2; (2) for input flows equal to the maximum flow in Fig. 2 the flow is unconditionally unstable to small oscillations, as had been mentioned in the Introduction; and (3) for flows represented by points on the retrograde side of Fig. 2 the flow is conditionally stable to small oscillations.

5. FINITE OSCILLATIONS

It is possible to investigate the stability under finite oscillations in the input by using the results for step function inputs, section 3. Suppose the input flow is the rectangular wave shon in Fig. 3. For a time r the input is θ_r and for a time f it is θ_f; without loss of generality we demand $\theta_r>\theta_f$. Then for stable oscillations during the time r, η will rise to a peak value η and during the following time f, η will fall to its minimum value $\underline{\eta}$ and so on. Clearly

$$\eta>\underline{\eta}\text{ since }\theta_r>\theta_f\,; \tag{19}$$

in Fig. 3 the zero of the η axis is purposely not indicated. For stability

$\underline{\eta}$ and $\bar{\eta}$ must exist and, conversely, if they exist there is stability. The average value of the input flow is

$$\text{ave } \omega(\tau)=\frac{r\theta_r+r\theta_f}{r+f}\leq 1\,, \tag{20}$$

and clearly must be restricted as indicated.

It is convenient to define

$$\mu_r=\sqrt{|1-\theta_r|}\,,\ \mu_f=\sqrt{|1-\theta_f|}\,. \tag{21}$$

As before there are two cases:

Cases I: $0\leq\theta_f<\theta_r<1$.

By the use of (7) it is found that

$$\underline{\eta}=\frac{-\mu_f^2+p_f\bar{\eta}}{p_f-\bar{\eta}}\,, \tag{22}$$

$$\bar{\eta}=\frac{-\mu_r^2+p_r\underline{\eta}}{p_f-\underline{\eta}}\,, \tag{23}$$

where we have set

$$p_f=\mu_f \coth f\mu_f\,,\ p_r=\mu_r \coth r\mu_r \tag{24}$$

for convenience.

Case II: $0\leq\theta_f<1<\theta_r$

From (7) and (8) it is found that

$$\underline{\eta}=\frac{-\mu_f^2+p_f\bar{\eta}}{p_f-\bar{\eta}}\,, \tag{25}$$

$$\bar{\eta}=\frac{\mu_r^2+p_r\underline{\eta}}{p_r-\underline{\eta}}\,, \tag{26}$$

where for this case we have set

$$p_f=\mu_f \coth f_f\,,\ p_r=\mu_r \cot r\mu_r\,. \tag{27}$$

It is instructive to plot these equations for some typical θ's before analyzing the equations. Fig. 4 shows these equations superimposed on one another. On the abscissa the values are plotted for p_f and each of the p_r's and the p's are used as the indepenent variables; the values of $\underline{\eta}$ and $\bar{\eta}$ are plotted on the ordinate. A typical stable oscillation for Case I with $\theta_f=1/2$ and $\theta_r=0.96<1$ has $\bar{\eta}=-1/4$, $\underline{\eta}=-1/2$ with p_f indicated by ▼ and p_r by ●. A companion stable oscillation has $\bar{\eta}=1/2$, $\underline{\eta}=1/4$ with the same values for p_f and p_r which are now indicated by ■ and △, respectively. It is seen generally from this figure that only two kinds of stable oscillations exist for this case: $-\mu_f<\underline{\eta}<\bar{\eta}<-\mu_r<0$ or $0<\mu_r<\underline{\eta}<\bar{\eta}<\mu_f$. For Case II, $\theta_f=1/2$, $\theta_r=1.04<1$, a typical stable oscillation has $\bar{\eta}=-1/4$, $\underline{\eta}=-1/2$ with p_f still the value indicated by ▼ but p_r now indicated by □; the companion oscillation $\bar{\eta}=1/2$, $\underline{\eta}=1/4$ again exists. A type of stable oscillation not allowed under the previous case has $\bar{\eta}=1/4$, $\underline{\eta}=-1/2$ with p_f indicated by ○ and $p_r(<0)$ indicated by ×, the companion oscillation has $\bar{\eta}=1/2$, $\underline{\eta}=-1/4$. Generally it is clear from the figure that a necessary condition for stable oscillations is $-\mu_f<\underline{\eta}<\bar{\eta}<\mu_f$.

Let us now obtain the general conditions for stability; we will only analyze the more interesting case, Case II, for which $\theta_r>1$. By combining (25) and (26) we obtain the following quadratic equation for $\underline{\eta}$:

$$\underline{\eta}^2(p_r+p_f)+\underline{\eta}(\mu_r^2+\mu_f^2)-(\mu_f^2p_r-\mu_r^2p_f)=0. \tag{28}$$

The quadratic equation $\bar{\eta}$ satisfies is the same except that the sign of the middle term is negative. It is expedient to introduce the definitions

$$p=p_r+p_f,\ \Delta\theta=\theta_r-\theta_f=\mu_r^2+\mu_f^2, \tag{29}$$

where the last is from (21). The solutions of (28) are then

$$\underline{\eta}=-(\Delta\theta/2p)^2\pm\sqrt{R} \tag{30}$$

where

$$R=(\Delta\theta/2p)^2+1-\theta_fp_r/p-\theta_rp_f/p. \tag{31}$$

One can readily show by multiplying (25) and (26) through by their respective denominators and then eliminating the terms $\bar{\eta}\cdot\underline{\eta}$ between these equations that for any solution $\underline{\eta}$

$$\bar{\eta}=\underline{\eta}+\Delta\theta/p\,. \tag{32}$$

By (29) and (19) necessarily

$$p>0\,. \tag{33}$$

From Fig. 4, as already mentioned, or by analysis, it is necessary that

$$-\mu_f<\underline{\eta}<\bar{\eta}<\mu_f\,. \tag{34}$$

Thus the necessary condition for stable oscillations is

$$-\mu_f+(\Delta\theta/2p)<\pm\sqrt{R}<\mu_f-(\Delta\theta/2p)\,, \tag{35}$$

where the left limit must obviously be non-positive and the right limit non-negative.

First let us look at the requirement that R be non-negative. We can define ν such that

$$p_f=\nu p\,,\ p_r=(1-\nu)p,\ \nu>0\,, \tag{36}$$

where the last is from (33) and (27). A convenient mixed notation for R as a function of ν is

$$R(\nu)=(\Delta\theta/2p)^2+\mu_f^2-\nu\Delta\theta\,. \tag{37}$$

For R to be positive

$$0<\nu(p)<A(p) \tag{38}$$

where

$$A(p)=[\Delta\theta/2p)^2+\mu_f^2]/\Delta\theta\,. \tag{39}$$

For p large, *i.e.*, r and f small, it can be shown that (39) implies (20). Next from the signs of the limits in (35) it is seen that

$$p\geq\Delta\theta/2\mu_f>0\,, \tag{40}$$

which further restricts the range of p. Also, (40) can be found directly by the instructive method of letting f be infinite, which means that $p_f=\mu_f$, setting $\underline{\eta}=-\mu_f$, $\bar{\eta}=+\mu_f$, and then solving for p. Finally, under the assumption that the previous conditions, (38) and (40), are satisfied, the remaining inequalities in (35) give only

$$\nu p=p_f>\mu_f, \qquad (41)$$

which adds nothing that was not already known; however, it is reassuring. The canstants (39), (40), (41) all have the point $p=\Delta\theta/2\mu_f$ in common on their boundaries.

In Fig. 5 the area defined by the above constraints has been sketched for the case considered before, namely $\theta_f=1/2$, $\theta_r=1.04$. The parameter p is some sort of a generalized frequency because of the behavior of its components p_r and p_f. To show the precise relation between r and f and the stability criterion three of the curves $r+f=$ constant have also been plotted in Fig. 5. Clearly the qualitative aspects of the stability criterion will not change with the values of θ_r and θ_f.

The results for Case I, $\theta_r<1$, are not particularly interesting. Briefly, oscillations of the variety $\underline{\eta}<0<\bar{\eta}$ are not allowed; on the other hand for any $\underline{\eta}$, $\bar{\eta}$ such that $-\mu_f<\underline{\eta}<\bar{\eta}<-\mu_r$ or $\mu_r<\underline{\eta}<\bar{\eta}<\mu_f$ there exist compatible chocies of p_r and p_f.

6. CONCLUSIONS

The model we have used in investigating the stability of vehicular traffic flow, which is all contained in equation (1) in its physical form, appears to have a range of validity. The evidence for this statement is all qualitative and it is not clear at this time how to obtain quantitative validation. Some of the qualitative evidence has already been given in the Introduction, *i.e.*, the derivation of the equation and some results. The same effect found using the step function can be illustrated in another way: Suppose these is a steady state flow with $\eta=\eta_o$ which is due to the constant input θ_o; let μ_o be defined in the usual manner.

Now let us perturb the input by a small amount ε for a short time δ. By section 3 at the end of time δ, $\eta(\delta)=\eta_0=\eta_0+\varepsilon\delta$; thus, for $\eta_0=-\mu_0$ then $\eta(\infty)=-\mu_0$, *i.e.*, stability, while for $\eta_0=+\mu_0$ then $\eta(\infty)=-\mu_0$ for ε negative and $\eta(\infty)$ does not exist for ε positive, *i.e.*, instability. On the other hand we have seen in section 5 that there may be large, stable oscillations on the retrograde side of the flow curve. In fact the only oscillations which are unstable are some of those for which the low point in the number of vehicles in the system is to the left of the maximum of the flow curve and the high point to the right, as was derived in section 5. So, essentially we have found that the model shows completely stable flow when the flow is constrained to be in the unsaturated region, which certainly checks with observed traffic behavior; on the other hand, when the flow is partly or wholly in the saturated region the stability of the flow is dependent on the type of disturbance, which is not an unreasonable conclusion to reach from observing real traffic flow. The next problem is to find out how to handle stochastic inputs and then find the statistics that the model would predict for such inputs.

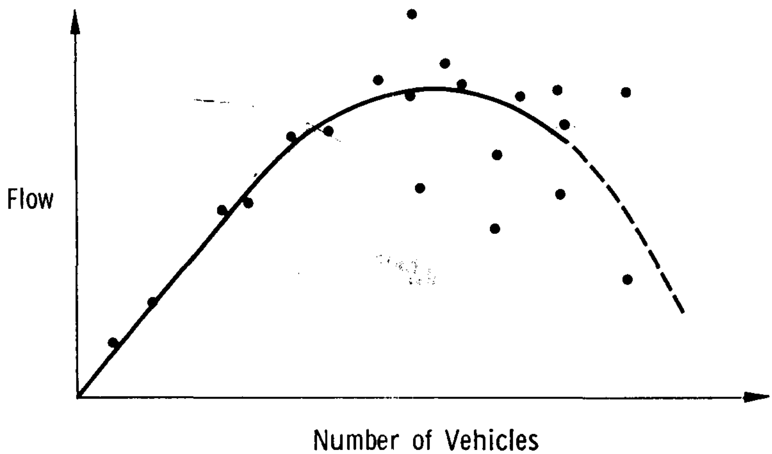

Fig. 1. Schematic representation of the flow vs. the number of vehicles in a system showing scatter of data points about average.

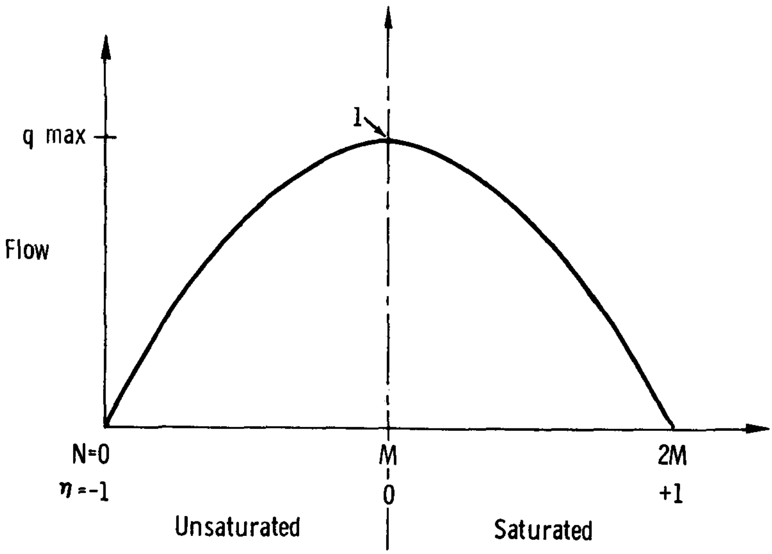

Fig. 2. Idealization showing flow as a parabolic function of number, N, of vehicles in system. Normalized flow vs. normalized number, η, is shown by dashed axes, see equations (2).

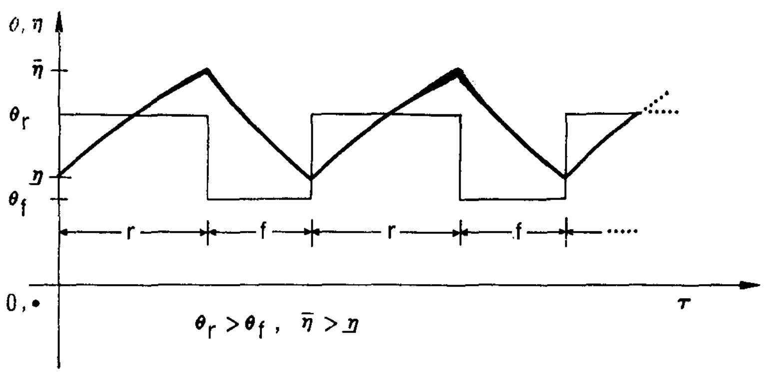

Fig. 3. Schematic drawing of rectangular wave input into system, $\omega(\tau)$, and resultant steady state variation of number in system, $\eta(\tau)$. Here, max $\eta(\tau)$ $=\bar{\eta}$, min $\eta(\tau)=\underline{\eta}$, and the zero of the η axis is not shown.

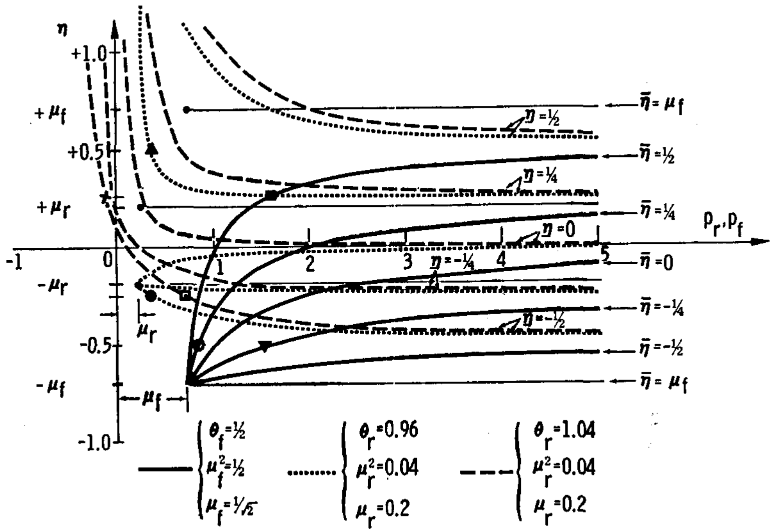

Fig. 4. Surerimposed curves with the values for the variables $\underline{\eta}$ and $\bar{\eta}$ given on the ordinate and the values for p_r and p_f on the abscissa. Three families are shown: the dependent variable $\underline{\eta}$ as a function of the independent variable p_f for the parameter $\theta_f=1/2$ and for various initial values $\bar{\eta}$, the variable $\bar{\eta}$ as a function of p_r for $\theta=0.96$ and various initial $\underline{\eta}$, and the variable $\bar{\eta}$ as a function of p_r for $\theta_r=1.04$ and various initial $\underline{\eta}$.

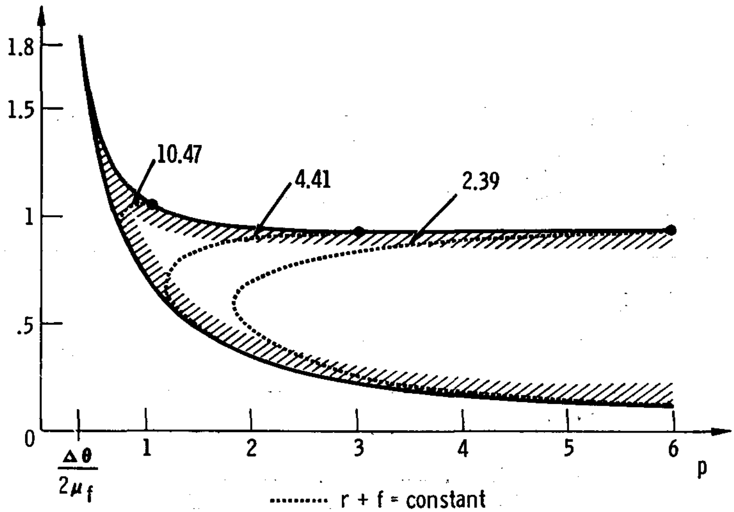

Fig. 5. The region bounded by hatched lines gives ν, p for which stable oscillations exist for the case $\theta_f=1/2$, $\theta_r=1.04$. Constant period curves are indicated by the dashed lines with periods (in units of τ) as shown. On such curves r decreases monotonely from its value on the upper curves, $\nu=A(p)$, and goes to zero as the curve asymptotically approaches the lower curve, $\nu p=\mu_f$.

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