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ON A STUDY OF OUTPUT DISTRIBUTION

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INTRODUCTION

In the study of tandem queuing systems, it is important to know the output distributions of these systems. By output distribution (inter-departure time distribution), we mean the distribution of the time period between two successive departures in the steady state.

The purpose of this paper is to investigate these output distributions for some queuing systems.

In the first section of this paper, we consider the outputs of single server queuing systems, including $M/G/1$, $E_l/M/1$ and $E_2/E_2/1$ systems.

In the second and third sections, we investigate tandem queuing systems with two stages and three stages respectively.

Poisson arrival distributions and exponential service time distributions are assumed in these two sections.

In the last fourth section, using these output distributions, we assert that a characteristic of tandem systems is evaluated with satisfactory precision, by means of single server systems approximating these tandem systems. In this section we consider only systems with two stages.

Throughout this paper we use following notations:

λ .... the mean arrival rate of customers

μ .... the mean service rate of a service station

$\rho=\lambda/\mu$ utilization factor

p_n......... the steady state probability that the system is in state n. (Let $p_{(r,s,n)}$ be the steady state probability that the system is in state (r, s, n), so on.)

$p_n^{(+)}$ ... the steady state probability that the system is in state n, regarding the time immediately after the departure of each customer as epoch

$p_n^{(-)}$ ... the steady state probability that the system is in state n, regarding the time immediately before the departure of each customer as epoch

$M_A(\theta)$... the moment generating function of the inter-arrival distribution

$M_S(\theta)$... the moment generating function of the service time distribution

$M_U(\theta)$... the moment generating function of the output distribution

$E(u)$... the expected value of the output distribution

$V(u)$... the variance of the output distribution

C ... the coefficient of variation of the output distribution

In addition, it is known that all systems treated in this paper have unique stationary solution.[1], [10]

Furthermore, let us note the following fact. That is, except for $G/M/1$ system, the relation

$$p_n^{(-)} \propto p_n$$

holds true with all systems that we are going to investigate in this paper. In other words, all instant of time are equivalent, in the sense that departures are equally likely to occur.

1. The Output Distribution from Single Server System

1.1 $M/G/1$ System

We will find the moment generating function (*m.g.f.*), $M_U(\theta)$ of the output distribution from $M(\lambda)/G(\mu)/1$ system.

If there is no customer in the system immediately after the departure of a customer, then the time to the next departure is equal to

(inter-arrival interval)+(service time),

because the distribution of the length of an exponentially distributed variable remains the same if part of the length is chopped off.

On the other hand, if there is at least one customer in the system, then the time interval to the next departure is equal to

(service time).

Table 1.1 shows these situations. From now on, we shall present situations by similar tables, and omit detailed discussions.

Table 1.1

State immediately before a departure	State immediately after a departure	Partioned *m.g.f.*
1	0	$M_A(\theta)\cdot M_S(\theta)$
for $n\geq 2$; n	$n-1$	$M_S(\theta)$

Thus we have the following representation of moment generating function $M_U(\theta)$ of the output distribution.

$$\begin{aligned}(1.1)\qquad M_U(\theta)&=p_0^{(+)}\cdot\{M_A(\theta)\cdot M_S(\theta)\}+\{1-p_0^{(+)}\}\cdot M_S(\theta)\\ &=p_1^{(-)}\cdot\{M_A(\theta)\cdot M_S(\theta)\}+\{1-p_1^{(-)}\}\cdot M_S(\theta)\end{aligned}$$

Differentiating both sides of (1.1) with respect to θ and equating $\theta=0$, we have

$$\frac{1}{\lambda}=p_1^{(-)}\cdot\left(\frac{1}{\lambda}+\frac{1}{\mu}\right)+\left(1-p_1^{(-)}\right)\cdot\frac{1}{\mu},$$

and

$$p_1^{(-)}=p_0^{(+)}=1-\rho.$$

Since $M_A(\theta)$ is given by

$$M_A(\theta)=\frac{\lambda}{\lambda-\theta},$$

substituting these expressions into (1. 1), we obtain the following theorem.

[Theorem 1.1]

The moment generating function $M_U(\theta)$ of the output distribution from $M(\lambda)/G(\mu)/1$ system is given by

$$(1.2) \qquad M_U(\theta)=\frac{\mu-\theta}{\mu}\cdot\frac{\lambda}{\lambda-\theta}\cdot M_S(\theta).$$

(**Corollary 1.1**)

In the case of $M(\lambda)/G(\mu)/1$ svstem, the coefficient of variation of the output distribution is obtained by

$C^2=1-\rho^2(1-C_S^2)$. (C_S denotes the coefficient of variation of the service time distribution.)

Therefore, we have $C=1$ if and only if $C_S=1$.

On the other hand, in the case of $C_S\neq 1$, the value of C is between the value of the coefficient of variation of the arrival distribution and the value of the coefficient of variation of the service time distribution.

We apply the theorem 1.1 to some simple examples.

(**Example 1.1**)

By (1.2), the moment generating function $M_U(\theta)$ of the output distribution from $M(\lambda)/M(\mu)/1$ system is given by

$$M_U(\theta)=\frac{\lambda}{\lambda-\theta}\ .$$

Hence the output distribution coincides with the arrival distribution. (This is a well-known result.[6])

(**Example 1.2**)

In the case of $M(\lambda)/E_k(\mu)/1$ system, where

$$M_A(\theta)=\frac{\lambda}{\lambda-\theta} \quad \text{and} \quad M_S(\theta)=\left(\frac{k\mu}{k\mu-\theta}\right)^k,$$

the moment generating function of the output distribution is given by

$$M_U(\theta)=\left(\frac{\mu-\theta}{\mu}\right)\cdot\left(\frac{\lambda}{\lambda-\theta}\right)\cdot\left(\frac{k\mu}{k\mu-\theta}\right)^k.$$

As to the coefficient of variation of the output distribution, we have

$$C=\sqrt{1-\frac{k-1}{k}\cdot\rho^2}.$$

(Example 1.3)

Since the case of $M(\lambda)/D(\mu)/1$ system is obtained from Example 1.2 as $k\to\infty$, in this case we have

$$M_U(\theta)=\left(\frac{\mu-\theta}{\mu}\right)\left(\frac{\lambda}{\lambda-\theta}\right)\cdot e^{\theta/\mu},$$

$$C=\sqrt{1-\rho^2}.$$

1.2 $E_l/M/1$ System

Similarly to the preceding section, we can find the output distribution from $E_l(\lambda)/M(\mu)/1$ system shown in Fig. 1.1.

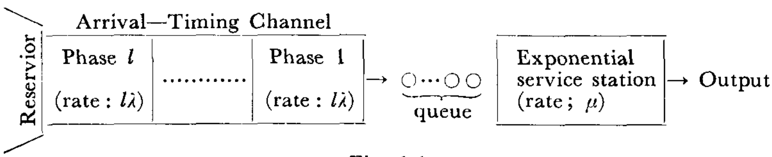

Fig. 1.1

Let us denote a state of the system by (s, n), where s (next customer is in the sth phase in the arrival-timing channel) runs from l to 1, and n is the number of customers in the system (including a customer being served). It is known that the solutions of a system of steady state equations for this system are given[4][9] by

$$p_{s,n}=\rho(1-v)\cdot v^{ln+s-l-1} \qquad (1\leq s\leq l), \tag{1.3}$$

where v is a positive root of

(1.4) $$v^l+v^{l-1}+\cdots+v-l\rho=0\,, \qquad (\rho=\lambda/\mu)\,,$$

which is less than unity. To find the output distribution, we must have

$$p_0=\sum_{s=1}^{l}p_{s,0}\,,\quad p_{11}\,,\quad p_{21}\,,\quad \cdots,\quad p_{l1}$$

and

$$p_2=\sum_{s=1}^{l}p_{s,2}\,,\quad p_3=\sum_{s=1}^{l}p_{s,3}\,,\ \cdots\cdots\,.$$

By (1.3) we can find these values as follows:

$$\begin{aligned}
p_0&=1-\rho\\
p_2&=\rho(1-v^l)\cdot v^l\\
p_3&=\rho(1-v^l)\cdot v^{2l}\\
&\vdots\\
p_n&=\rho(1-v^l)\cdot v^{(n-1)l}
\end{aligned}$$

Referring to the Table 1.2, we obtain the following representation the moment generating function $M_U(\theta)$ of the output distribution.

Table 1.2

State immediately before a departure	State immediately after a departure	$m.g.f.$
(1, 1)	(1, 0)	$\left(\frac{l\lambda}{l\lambda-\theta}\right)^l\cdot\left(\frac{\mu}{\mu-\theta}\right)$
(2, 1)	(2, 0)	$\left(\frac{l\lambda}{l\lambda-\theta}\right)^{l-1}\cdot\left(\frac{\mu}{\mu-\theta}\right)$
$\vdots$		
(l, 1)	(l, 0)	$\left(\frac{l\lambda}{l\lambda-\theta}\right)\left(\frac{\mu}{\mu-\theta}\right)$

Thus we have

$$M_U(\theta)=\left(\frac{\mu}{\mu-\theta}\right)\cdot\left\{\sum_{s=1}^{l}p_{s0}{}^{(+)}\cdot\left(\frac{l\lambda}{l\lambda-\theta}\right)^{l+1-s}+\sum_{n=1}^{\infty}p_n{}^{(+)}\right\}$$

$$=\left(\frac{\mu}{\mu-\theta}\right)\cdot\left\{\sum_{s=1}^{l} p_{s1}{}^{(-)}\cdot\left(\frac{l\lambda}{l\lambda-\theta}\right)^{l+1-s}+\sum_{n=2}^{\infty} p_n{}^{(-)}\right\}.$$

Using the relations

$$p_{s1}{}^{(-)}=\frac{p_{s1}}{1-p_0}, \qquad p_n{}^{(-)}=\frac{p_n}{1-p_0} \quad (\text{for } n\geq 1),$$

we have following theorem.

[Theorem 1.2]

The moment generating function of the output distribution from $E_l(\lambda)/M(\mu)/1$ system is given by

$$(1.5) \quad M_U(\theta)=\left(\frac{\mu}{\mu-\theta}\right)\left[\left(\frac{l\lambda}{l\lambda-\theta}\right)^l\cdot\left\{\frac{1-\left(\left(1-\frac{\theta}{l\lambda}\right)v\right)^l}{1-\left(1-\frac{\theta}{l\lambda}\right)v}\right\}\cdot(1-v)+v^l\right].$$

Note that following corollaries are readily obtained.

(**Corollary 1.2.1**)

In $E_l(\lambda)/M(\mu)/1$ system, we have

$$p_0{}^{(+)}\neq p_0$$

for $l\neq 1$.

(**Proof**)

Considering that

$$p_0{}^{(+)}=p_1{}^{(-)}=\frac{p_1}{1-p_0}$$

and

$$p_0=1-\rho, \qquad p_1=\rho(1-v^l),$$

we have

$$p_0{}^{(+)}=1-v^l.$$

On the other hand we have $v^l<\rho$ for $l>1$, since

$$v^l+v^{l-1}+\cdots+v=l\rho.$$

It follows that

$$p_0{}^{(+)}>1-\rho=p_0.$$

(Corollary 1.2.2)

Concerning the coefficient of variation of the output distribution from $E_l(\lambda)/M(\mu)/1$ system, the relation

$$(1.6) \qquad C=\sqrt{\frac{1}{l}+\frac{2v\cdot(-v^l+\rho)}{l\cdot(1-v)}}$$

is satisfied. From this relation it can be seen that

$$\frac{1}{\sqrt{l}}\leq C\leq 1.$$

(Corollary 1.2.3)

As to $M_U(\theta)$ in (1.5), following relations are satisfied;

$$\lim_{l\to\infty} M_U(\theta)=\left(\frac{\mu}{\mu-\theta}\right)\cdot\left(e^{\theta/\lambda}-e^{2\theta/\lambda}+1\right).$$

1.3 $E_2/E_2/1$ System

Let us consider the output distribution from the system shown in Fig. 1.2.

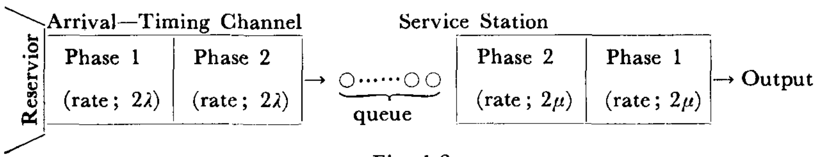

Fig. 1.2

Table 1.3

State immediately before a departure	State immediately after a departure	$m.g.f.$
(1, 1; 1)	(1, 0; 0)	$\left(\frac{2\lambda}{2\lambda-\theta}\right)^2\cdot\left(\frac{2\mu}{2\mu-\theta}\right)^2$
(2, 1; 1)	(2, 0; 0)	$\left(\frac{2\lambda}{2\lambda-\theta}\right)\cdot\left(\frac{2\mu}{2\mu-\theta}\right)^2$
for $n\geq 2$; $(s, 1; n)$	$(s, 2; n-1)$	$\left(\frac{2\mu}{2\mu-\theta}\right)^2$

Note: $(s, 0; 0)$ denotes the state of no customer being in the service station.

Denote each state of the system by $(s, m; n)$, where s is the arrival phase number, m is the service phase number, and n is the number of customers in the system. Similarly to the preceding section we consider the Table 1.3.

Thus we have the moment generating function of the output distribution

$$\text{(1.7)}\qquad M_U(\theta)=p^{(-)}_{(1,1;1)}\cdot\left\{\left(\frac{2\lambda}{2\lambda-\theta}\right)^2\cdot\left(\frac{2\mu}{2\mu-\theta}\right)^2\right\} +p^{(-)}_{(2,1;1)}\cdot\left\{\left(\frac{2\lambda}{2\lambda-\theta}\right)\cdot\left(\frac{2\mu}{2\mu-\theta}\right)^2\right\} +\sum_{s=1}^{2}\sum_{n=2}^{\infty}p^{(1)}_{(s,1;n)}\cdot\left(\frac{2\mu}{2\mu-\theta}\right)^2.$$

It has been shown by Kawamura[9] that

$$p_{(s,m;n)}=\frac{\rho\cdot(\rho-1)}{A\cdot(1-v_3)}\cdot\begin{vmatrix}1 & 1 & 1\\ u_1 & u_2 & 1+\rho\\ U_1^{(s,m;n)} & U_2^{(s,m;n)} & 0\end{vmatrix}$$

where

$$\rho=\lambda/\mu,\qquad A=\rho-u_1,$$
$$U_j^{(s,m;n)}=u_j^{m-1}\cdot v_j^{s-1}\cdot w_j^{n-1},\qquad (j=1,2)$$
$$v_j=\frac{\rho}{1+\rho-u_j},$$
$$w_j=u_j^2,$$
$$u_1=\frac{1+\rho-\sqrt{1+6\rho+\rho^2}}{1},\qquad u_2=\rho,$$
$$u_3=\frac{1+\rho+\sqrt{1+6\rho+\rho^2}}{2},$$

Using the preceding results, we can radily see that

$$p_{(1,1;1)}=\rho\cdot p_{(1,0;0)},$$
$$p_{(2,1;1)}=\frac{-\rho(1-\rho)(1+v_3)}{1-v_3},$$

$$\sum_{s=1}^{2}\sum_{n=1}^{\infty} p_{(s,1;n)}=\frac{1}{2}\rho \ .$$

Nothing that

$$p^{(-)}_{(1,1;1)}=\frac{p_{(1,1;1)}}{\sum_{s=1}^{2}\sum_{n=1}^{\infty} p_{(s,1;n)}}$$

$$p^{(-)}_{(2,1;1)}=\frac{p_{(2,1;1)}}{\sum_{s=1}^{2}\sum_{n=1}^{\infty} p_{(s,1;n)}}$$

$$p^{(-)}_{(s,1;n)}=\frac{p_{(s,1;n)}}{\sum_{s=1}^{2}\sum_{n=1}^{\infty} p_{(s,1;n)}}$$

and considering (1.7), we have following theorem.

[Theorem 1.3]

The moment generating function of the output distribution from $E_2(\lambda)/E_2(\mu)/1$ system is given by

$$(1.8)\qquad M_U(\theta)=\left(\frac{2}{1-v_3}\right)\left(\frac{2\mu}{2\mu-\theta}\right)^2\cdot\left\{(1-\rho)\left(\frac{2\lambda}{2\lambda-\theta}\right)^2 -(1-\rho)(1+v_3)\left(\frac{2\lambda}{2\lambda-\theta}\right)+\frac{1+v_3-2\rho v_3}{2}\right\},$$

where

$$v_3=\frac{-1-\rho-\sqrt{1+6\rho+\rho^2}}{2} \ .$$

By (1.8) we may see that the expectation and the variance of the output distribution is equal to

$$E(u)=\frac{1}{\lambda}\ ,\qquad V(u)=\frac{1}{4\lambda^2}\{3+2\rho-3\rho^2-(1-\rho)\sqrt{1+6\rho+\rho^2}\}\ .$$

Therefore

$$V(u)>\frac{1}{4\lambda^2}\{3+2\rho-3\rho^2-(1-\rho)(1+3\rho)\}=\frac{1}{2\lambda^2}\ .$$

Thus we have,

(Corollary 1.3.1)

The output distribution of $E_2(\lambda)/E_2(\mu)/1$ is different from $E_2(\lambda)$. The coefficient of variation of the output distribution is greater than that of the input distribution.

2. The Output Distribution from a Tandem Type System with Two-Stages

In this section, we consider the output from the system shown in Fig. 2.1[7].

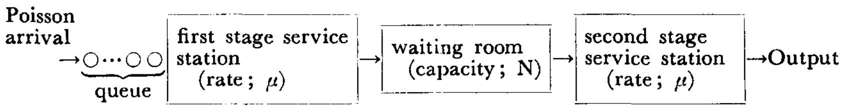

Fig. 2.1

If an arrived customer finds the first service station empty, then he will be served at once.

If he finds the first stage busy, then he will join the queue in front of the first stage station, and customers in the queue will be served in order of arrival. In this case, the length of the queue in front of the first station has no restriction.

After a customer has finished to be served at the first station, he will be served at the second station. If he finds the second station busy, then he joins a queue in front of this station (in a waiting room).

In this case, however, we suppose that the maximum of permissible queue size (the capacity of waiting room) is equal to N. By this restriction, if a customer finished to be served at the first station finds N customers in the waiting room, then he must continue occupying the first station. We call this situation that the first station is blocked.

We suppose that arrivals to the first station have the Poisson distribution with arrival rate λ, and service times at the first stage and the second stage have the exponential distribution with common service rate μ.

All states of this system are listed in Table 2.1.

Table 2.1

State	Queue size	State of the first station	No. of units in the waiting room	State of the second station
(0 0 0)	0	0	0	0
(0 0 1)	0	0	0	1
(0 1 1)	0	0	1	1
⋮				
(0 N 1)	0	0	N	1
(0 N 2)	0	b	N	1
$n\geq 1$; (n 0 0)	$n-1$	1	0	0
(n 0 1)	$n-1$	1	0	1
(n 1 1)	$n-1$	1	1	1
⋮				
(n N 1)	$n-1$	1	N	1
(n N 2)	n	b	N	1

Table 2.2

State immediately before a departure	State immediately after a departure	*m.g.f.*
(0 0 1)	(0 0 0)	$\left(\frac{\lambda}{\lambda-\theta}\right)\cdot\left(\frac{\mu}{\mu-\theta}\right)^2$
$n\geq 1$; (n 0 1)	(n 0 0)	$\left(\frac{\mu}{\mu-\theta}\right)^2$
$n\geq 0$; $r=1, 2, \cdots, N$; (n r 1)	(n, $r-1$, 1)	$\left(\frac{\mu}{\mu-\theta}\right)$
$n\geq 0$; (n N 2)	(n, N, 1)	

The letters 0, 1 and b in Table 2.1 represent that the corresponding station is empty, being in service, and blocked respectively.

The partitioned moment generating function of the output distribution are given in Table 2.2.

From these expressions for partitioned moment generating functions, we can see that the moment generating function of the output distribution is given by

$$M_U(\theta)=p_{001}^{(-)}\cdot\left\{\left(\frac{\lambda}{\lambda-\theta}\right)\left(\frac{\mu}{\mu-\theta}\right)^2\right\}+\{F_{01}^{(-)}-p_{001}^{(-)}\}\cdot\left(\frac{\mu}{\mu-\theta}\right)^2$$
$$+\{F_{11}^{(-)}+F_{21}^{(-)}+\cdots+F_{N1}^{(-)}+F_{N2}^{(-)}\}\cdot\left(\frac{\mu}{\mu-\theta}\right),$$

where

$$F_{rs}^{(-)}=\sum_{n=0}^{\infty}p_{nrs}^{(-)}.$$

Meanwhile, by Makino[7], [11]

$$p_{001}=\rho\cdot p_{000},\qquad F_{00}=1-\rho,\qquad F_{01}=1-\rho-p_{000},$$
$$p_{000}\doteqdot\frac{(N+2)-(N+3)\rho}{(N+2)+(N+1)\rho+N\rho^2+\cdots+2\rho^N+\rho^{N+1}}$$

and

$$p_{001}+\{F_{01}-p_{001}\}+\{F_{11}+F_{21}+\cdots+F_{N1}+F_{N2}\}=1-F_{00}=\rho.$$

Therefore we have the following theorem 2.1 considering that the relations

$$p_{001}^{(-)}=\frac{1}{\rho}\cdot p_{001},\qquad F_{rs}^{(-)}=\frac{1}{\rho}\cdot F_{r,s}.$$

[Theorem 2.1]

The moment generating function of the output distribution from the second stage station of the two-stage tandem system shown in Fig. 2.1 is given by

$$(2.1)\qquad M_U(\theta)=\frac{1}{\rho}\cdot\left(\frac{\mu}{\mu-\theta}\right)\cdot\left[p_{000}\cdot\left\{\rho\left(\frac{\lambda}{\lambda-\theta}\right)\left(\frac{\mu}{\mu-\theta}\right)\right.\right.$$

$$-(1+\rho)\left(\frac{\mu}{\mu-\theta}\right)+1\Big\}$$

$$+\left\{(1-\rho)\left(\frac{\mu}{\mu-\theta}\right)+(2\rho-1)\right\}\Big] .$$

Thus its mean value, variance, and coefficient of variation are given by

$$E(u)=\frac{1}{\lambda} ,$$

$$V(u)=\frac{2}{\lambda\mu}\cdot(2-\rho)+\frac{1}{\lambda^2}\cdot(2\cdot p_{000}-1) ,$$

$$C=\sqrt{1-2\{(1-\rho)^2-p_{000}\}} \tag{2.2}$$

(Corollary 2.1)

In the case where $N=0$, since $p_{000}=\frac{2-3\rho}{2+\rho}$ we have

$$C=\sqrt{1-\frac{2\rho^3}{2+\rho}} , \tag{2.3}$$

and in the case where $N\to\infty$, since $p_{000}\to(1-\rho)^2$ we have

$$C\to 1 .$$

3. The Output Distribution from a Tandem Type System with Three-Stages

We consider the output distribution from the system shown in Fig. 3.1.

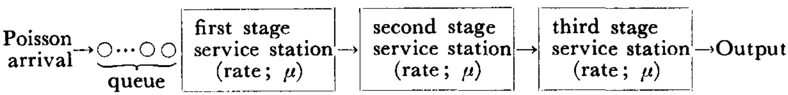

Fig. 3.1

All states of this system are listed in Table 3.1.

The partitioned moment generating functions are shown in Table 3.2.

Steady state probalities are calculated by the usual method of difference equations as follows[11];

Table 3.1

State	Queue size	State of the first station	State of the second station	State of the third station
(0 0 0)	0	0	0	0
(0 0 1)	0	0	0	1
(0 1 0)	0	0	1	0
for $n\geq 1$; (n 0 0)	$n-1$	1	0	0
(0 1 1)	0	0	1	1
(0 0 2)	0	0	b	1
for $n\geq 1$; (n 0 1)	$n-1$	1	0	1
〃 (n 1 0)	$n-1$	1	1	0
〃 (n 2 0)	n	b	1	0
〃 (n 1 1)	$n-1$	1	1	1
〃 (n 0 2)	$n-1$	1	b	1
for $n\geq 0$; (n 2 1)	n	b	1	1
〃 (n 2 2)	n	b	b	1

Putting

$$F_{r,s}=\sum_{n=0}^{\infty} p_{nrs} ,$$

we have

$$F_{01}=F_{00}-p_{000}$$

$$F_{11}=\frac{1}{2}\{3F_{00}-3p_{000}-2p_{001}-p_{010}-p_{002}\}$$

$$F_{02}=\frac{1}{4}\{3F_{00}-3p_{000}-2p_{001}-p_{010}+p_{002}\}$$

$$F_{10}=\frac{1}{4}\{5F_{00}-5p_{000}-2p_{001}+p_{010}-p_{002}\}$$

$$F_{21}=\frac{1}{4}\{3F_{00}-3p_{000}-2p_{001}-2p_{011}-p_{010}-p_{002}\}$$

Table 3.2

State immediately before a departure	State immediately after a departure	Partitioned $m.g.f$
(0 0 1)	(0 0 0)	$\left(\frac{\lambda}{\lambda-\theta}\right)\left(\frac{\mu}{\mu-\theta}\right)^3$
for $n\geq 1$; (n 0 1)	(n 0 0)	$\left(\frac{\mu}{\mu-\theta}\right)^3$
(0 1 1) ⋮ (n 1 1)	(0 1 0) ⋮ (n 1 0)	$\left(\frac{\mu}{\mu-\theta}\right)^2$
(0 2 1) ⋮ (n 2 1)	(0 2 0) ⋮ (n 2 0)	
(0 0 2) ⋮ (n 0 2)	(0 0 1) ⋮ (n 0 1)	$\left(\frac{\mu}{\mu-\theta}\right)$
(0 2 2) ⋮ (n 2 2)	(0 1 1) ⋮ (n 1 1)	

$$F_{20}=\frac{1}{2}\{4F_{00}-4p_{000}-2p_{001}-2p_{010}-p_{011}-p_{002}\}$$

$$F_{22}=\frac{1}{2}\{3F_{00}-3p_{000}-2p_{001}-p_{010}-p_{011}-2p_{002}\}$$

and

$$39\cdot F_{00}=4+(35+18\rho)p_{000}+9p_{010}+9p_{002}+6p_{011}$$

$$p_{001}=\rho\cdot p_{000}$$

$$p_{002}=\rho(1+\rho)\cdot p_{000}-p_{010}$$

$$p_{011}=\rho(1+\rho)^2\cdot p_{000}-(1+\rho)\cdot p_{010}$$

$$p_{100}=2(1+\rho)\cdot p_{010}-\rho(1+\rho)^2\cdot p_{000}$$

$$p_{101}=-\rho(2+3\rho+3\rho^2+\rho^3)\cdot p_{000}-2(1+\rho)^2\cdot p_{010}$$

Considering that the relation

$$p_{nrs}^{(-)}=\frac{1}{\rho}\cdot p_{nrs}$$

holds, we have the following expression for the moment generating function of the output distribution

$$(3.1)\qquad M_U(\theta)=\frac{1}{\rho}\left\{p_{001}\cdot\left(\frac{\lambda}{\lambda-\theta}\right)\cdot\left(\frac{\mu}{\mu-\theta}\right)^3+(F_{01}-p_{001})\left(\frac{\mu}{\mu-\theta}\right)^3+(F_{11}+F_{21})\left(\frac{\mu}{\mu-\theta}\right)^2+(F_{02}+F_{22})\left(\frac{\mu}{\mu-\theta}\right)\right\}.$$

Its coefficient of variation is given by

$$(3.2)\qquad C=\sqrt{1-2\left(1+\frac{\rho}{3}\right)\{(1-\rho)^3-p_{000}\}-\frac{2}{3}\rho^4}$$

On the other hand, by simple calculation we can show that

$$(3.3)\qquad p_{000}<\frac{1-\rho}{1+2\rho+3\rho^2+\left(4+\frac{3}{4}\right)\rho^3+\left(5+\frac{5}{18}\right)\rho^4},$$

and we can conclude that

$$C<\sqrt{1-\frac{2\rho^3}{2+\rho}}$$

Note that the right hand side of the above inequality is the coefficient of variation of the output distribution from two-stage tandem system (with $N=0$).

Therefore we have the following;

[Theorem 3.1]

The coefficient of variation of the output distribution from three-stage tandem system shown in Fig. 3.1 is smaller than that from two-stage tandem system (with $N=0$) shown in Fig. 2.1.

4. Similar Systems

4.1 General Consideration

In the study of tandem queues with blocking effect, it is difficult to find the distributions of queue sizes and waiting times, since it is impossible to regard its stations os independent and treat them separately.

By this reason, it may be natural to require that all states should be put together and be regarded as an equivalent, at least approximately, service station. In this paper we restrict ourselves to the consideration of two-stage tandem system (Fig. 2.1).

In the case that the capacity N of the waiting room is equal to zero, it is known that

[the mean number L of customers in the system]
$=4\rho(2-\rho^2)/(2+\rho)(2-3\rho)$.

For $N\geq 1$, however, the values of L is not yet obtained.

In section 1, we have shown the moment generating function of the output distribution from $M(\lambda)/G(\mu')/1$ system to be

$$M_U(\theta)=\left(\frac{\mu'-\theta}{\mu'}\right)\cdot\left(\frac{\lambda}{\lambda-\theta}\right)\cdot M_S(\theta)\,, \tag{4.1}$$

where $M_S(\theta)$ is the moment generating function of the service time distribution.

On the other hand, it has been shown in section 2, that the moment generating function of the output distribution from two-stage $M(\lambda)/M(\mu)/1$ system (with the waiting room of capacity N) is given by

$$\begin{aligned}M_U(\theta)=\frac{1}{\rho}\cdot\frac{\mu}{\mu-\theta}\Big[p_{000}\cdot\Big\{\rho\Big(\frac{\lambda}{\lambda-\theta}\Big)\Big(\frac{\mu}{\mu-\theta}\Big)-(1+\rho)\cdot\Big(\frac{\mu}{\mu-\theta}\Big)+1\Big\}\\+\Big\{(1-\rho)\cdot\Big(\frac{\mu}{\mu-\theta}\Big)+(2\rho-1)\Big\}\Big]\,.\end{aligned} \tag{4.2}$$

Regarding (4.1) and (4.2) to be equal, we have

$$\begin{aligned}M_S(\theta)=\frac{1}{\rho}\Big(\frac{\mu'}{\mu'-\theta}\Big)\Big(\frac{\mu}{\mu-\theta}\Big)\Big(\frac{\lambda-\theta}{\lambda}\Big)\\\cdot\Big[p_{000}\cdot\Big\{\rho\Big(\frac{\lambda}{\lambda-\theta}\Big)\cdot\Big(\frac{\mu}{\mu-\theta}\Big)-(1+\rho)\Big(\frac{\mu}{\mu-\theta}\Big)+1\Big\}\\+\Big\{(1-\rho)\cdot\Big(\frac{\mu}{\mu-\theta}\Big)+(2\rho-1)\Big\}\Big]\,.\end{aligned} \tag{4.3}$$

Our approach to the tandem system is through the substitution of a single server system for the original system. In other words, instead of the tandem system with service rates μ, Poisson arrival, and waiting room capacity N, we consider $M(\lambda)/G(\mu')/1$ system with the service distribution determined by (4.3).

As to the mean number of customers in $M(\lambda)/G(\mu')/1$ system the following results have been obtained.

$$F(z)\equiv\sum_{j=0}^{\infty}p_jz^j=(1-\rho')(-z)\cdot S^*[\lambda(1-z)]/\{S^*[\lambda(1-z)]-z\}\ . \quad (\rho'\equiv\lambda/\mu')$$

Where p_j denotes the steady state probability that the number of customers in the system is j, and

$$S^*(\theta)=\int_0^\infty e^{-\theta t}dS(t)$$

is the Laplase transform of the service distribution function $S(t)$.

Nothing that

$$M_S(-\theta)=S^*(\theta)\ ,$$

we have

$$\text{(4.4)} \qquad F(z)=(1-\rho')(1-z)\cdot M_S[-\lambda(1-z)]/\{M_S[-\lambda(1-z)]-z\}\ .$$

Substituting (4.3) in (4.4), we obtain that

$$\text{(4.5)} \qquad F(z)=(1-\rho')\cdot M_S[-\lambda(1-z)] \cdot\frac{\{1+\rho'(1-z)\}\cdot\{1+\rho(1-z)\}^2}{[(1-z)\cdot p_{000}+4\rho(1-z)+z-z\{\rho^2(1-z)+\rho'(1+\rho(1-z))^2\}]}$$

Since we want to consider $M/G/1$ system instead of the two-stage system, it is preferable to compare the queue size of both systems.

Let $F_q(z)$ be the generating function of queue size.

Then we have

$$\text{(4.6)} \qquad F_q(z)=(1-\rho')\cdot\left(1-\frac{1}{z}\right)+\frac{1}{z}\cdot F(z)\ .$$

Hence the mean queue size L'_q in $M(\lambda)/G(\mu')/1$ is given by

$$\text{(4.7)} \qquad L'_q=\frac{d}{dz}F_q(z)\Big|_{z=1}=\frac{p_{000}+\rho'^2-(1-\rho)^2}{1-\rho'}\ .$$

Therefore our problem becomes to find out the value of

$$(4.8) \qquad \rho' \equiv \lambda/\mu'$$

4.2 Calculation of ρ'

Table 4.1 gives all states of a two-stage system (with the waiting room of capacity N), and corresponding mean passage time.

By the mean passage time, we mean the mean passage time of the first customer through the system after the arrival of a customer (we observe the system at the epoch immediately before an arrival). For example, if the state immediately before an arrival is $(0, r, 1)$, then the mean time to the first departure from the first stage station (the time spent in the first station) is equal to $1/\mu$.

Table 4.1

State	Queue size	State of the 1st station	No. in the waiting room	State of the 2nd station	Mean passage time to the first station
(0 0 0)	0	0	0	0	$1/\mu$
(0 0 1)	0	0	0	1	$1/\mu$
⋮					
$(0, N-1, 1)$	0	0	$N-1$	1	$1/\mu$
$(0\ N\ 1)$	0	0	N	1	$\frac{3}{2}\cdot\frac{1}{\mu}$
for $n \geq 1$; $(n\ 0\ 0)$	$n-1$	1	0	0	[8] mean: $\frac{N+3}{N+2}\cdot\frac{1}{\mu}$
〃 $(n\ 0\ 1)$	$n-1$	1	0	1	
〃 ⋮					
〃 $(n\ N\ 1)$	$n-1$	1	N	1	
for $n \geq 0$; $(n\ N\ 2)$	n	b	N	1	

Denote the mean service rate when the system is regarded as a single server system by μ', then

$$(4.9) \qquad \mu' = \frac{1}{(p_{000}+p_{001}+\cdots+p_{0,N-1,1})\cdot\dfrac{1}{\mu}+p_{0N1}\cdot\left(\dfrac{3}{2}\cdot\dfrac{1}{\mu}\right) + \{1-(p_{000}+p_{001}+\cdots+p_{0N1})\}\left(\dfrac{N+3}{N+2}\cdot\dfrac{1}{\mu}\right)}$$

it follows that

$$\begin{aligned}\rho' &= \frac{\lambda}{\mu'}\\ &= \Big[(p_{000}+p_{001}+\cdots+p_{0,N-1,1})+\frac{3}{2}\cdot p_{0N1}\\ &\quad +\{1-(p_{000}+p_{001}+\cdots+p_{0N1})\}\cdot\frac{N+3}{N+2}\Big]\cdot\rho\ .\end{aligned}$$

It is clear that

$$1-\rho > p_{000}+p_{001}+\cdots+p_{0N1} > 1-\frac{N+3}{N+2}\cdot\rho\ .$$

The values of p_{000} and p_{0N1} can be precisely evaluated as necessary. However, for the large values of N, we may suppose that

$$(4.10) \qquad \rho' \doteqdot \left\{1+\frac{(N+3)}{(N+2)^2}\cdot\rho\right\}\cdot\rho\ .$$

4.3 Approximate Solutions

Let us compare the mean queue size L'_q of $M/G/1$ system obtained in the preceding paragraph, with the mean queue size L_q of the original two-stage system. At first we calculate L_q.

By [7],

$$L=[\sum_{r,s} dF_{r,s}(z)/dz]_{z=1}\ . \qquad (F_{r,s}(z)\equiv\sum_{n=0}^{\infty} p_{n,r,s}z^{n+r+s})\ .$$

Since we have obtained the expression of $F_{r,s}(z)$, the mean queue size L_q is given by

$$\begin{aligned}(4.11) \quad L_q &= L-\{F_{00}(1)+2\cdot F_{01}(1)+3\cdot F_{11}(1)+3\cdot F_{12}(1)\}+(p_{000}+p_{001}+p_{011})\\ &= L-\{(\rho+3)\cdot p_{000}+(7\rho-3)\}\ .\end{aligned}$$

where

$$(4.12)\qquad L=\frac{(\rho^2+4\rho-1)\cdot p_{000}+(10\rho^2-12\rho+1)}{4\rho-3}\quad [7]\ .$$

If we substitute $M(\lambda)/G(\mu')/1$ system for this two-stage system, then we have

$$(4.13)\qquad \rho'=\left[(p_{000}+p_{001})+\frac{3}{2}p_{011}+\frac{4}{3}\{1-(p_{000}+p_{001}+p_{011})\}\right]\cdot\rho$$

$$=\left\{\left(\frac{11}{6}-\frac{2}{3}\rho\right)\cdot-\left(\frac{5}{6}+\frac{2}{3}\rho\right)\cdot p_{000}\right\}\cdot\rho\ ,$$

using the relations (see [7]

$$p_{001}=\rho\cdot p_{000}$$
$$p_{011}=3-4\rho-(3+2\rho)\cdot p_{000}$$

On the other hand, we can see that

$$(4.14)\qquad p_{000}\doteqdot\frac{(N+2)-(N+3)\rho}{(N+2)+(N+1)\rho+N\rho^2+\cdots+2\rho^N+\rho^{N+1}}\quad [11]\ ,$$

where N is the capacity of the waiting room. In the present case of $N=1$, we have

$$p_{000}\doteqdot\frac{3-4\rho}{3+2\rho+\rho^2}\ .$$

It follows that

$$(4.15)\qquad \rho'=\frac{(18+18\rho+19\rho^2-4\rho^3)\cdot\rho}{6\cdot(3+2\rho+\rho^2)}\ .$$

Table 4.2 gives the values of

$$(4.16)\qquad L'_q=\frac{p_{000}+\rho'^2-(1-\rho)^2}{1-\rho'}$$

and the values of L_q for various values of ρ.

The numerical values of L_q in Table 4.2 is calculated from (4.11), (4.12) and (4.14) by

$$(4.17) \qquad L_q = L - \{(\rho+3)\cdot p_{000} + (7\rho-3)\}$$
$$\fallingdotseq \frac{3\rho^2+5\rho-8}{3+2\rho+\rho^2} + \frac{8-21\rho+18\rho^2}{3-4\rho} .$$

In this relation we used approximate values for p_{000}, so that the values of L_q given in the Table are approximate.

Table 4.2

ρ	ρ'	p_{000}	Mean queue size of approximate single server system L'_q	Mean queue size of original two-stage system L_q
0	0	1	0	0
0. 1	0. 1038	0. 8100	0. 01	0. 01
0. 2	0. 2164	0. 6395	0. 06	0. 05
0. 3	0. 3388	0. 4878	0. 17	0. 16
0. 4	0. 4711	0. 3535	0. 41	0. 38
0. 5	0. 6127	0. 2353	0. 93	0. 88
0. 6	0. 7626	0. 1316	2. 33	2. 27
0. 7	0. 9194	0. 0409	9. 89	9. 98
3/4	1	0	∞	∞

Table 4. 2 shows the fairly good agreement of the value of L_q and L'_q. When N becomes large, the agreement is expected to become more satisfactory.

Table 4. 3 gives the approximate mean queue sizes L'_q for some values of N.

Note: For $N \to \infty$, L'_q is calculated by $L'_q = \frac{\rho}{1-\rho} - \rho$.

Remark

Only two-stage systems are considered in this section. However, the concept of similar system is applied to three-stage systems in the similar way.

Table 4.3

ρ	Mean queue size of approximate single server system; L'_q							
	$N=1$	$N=2$	$N=3$	$N=4$	$N=5$	$N=10$	$N=20$	$N=\infty$
0	0	0	0	0	0	0	0	0
0.1	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01
0.2	0.06	0.06	0.06	0.05	0.05	0.05	0.05	0.05
0.3	0.17	0.16	0.15	0.15	0.14	0.13	0.13	0.13
0.4	0.41	0.37	0.35	0.33	0.32	0.29	0.27	0.27
0.5	0.93	0.81	0.74	0.67	0.64	0.57	0.52	0.50
0.6	2.33	1.80	1.56	1.35	1.27	1.09	0.95	0.90
0.7	9.89	4.94	3.80	3.06	2.75	2.16	1.77	1.63
(3/4)	∞	11.63	7.03	5.18	4.44	3.22	2.49	2.25
(4/5)		∞	20.10	11.13	8.47	5.16	3.65	3.20
(5/6)			∞	29.06	16.62	7.70	4.90	4.17
(6/7)				∞	41.04	11.11	6.24	5.14
(7/8)					∞	15.88	7.91	6.88
0.9						35.03	10.86	8.10
(12/13)						∞	16.54	11.08
(22/23)							∞	21.05
1.0								∞

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