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ON A CLASS OF OPTIMAL STOPPING RULE PROBLEMS

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§ 1. INTRODUCTION

One of the typical decision processes, to which we are confronted, in our daily activities, may be to decide whether to accept or refuse a proposal from several kinds of them, at the instant it is offered. Consider, for example, a situation where we have three kinds of traffic facilities in a town: street car, bus and taxi-cab. We are waiting for one of them to attend, perhaps, a weekly meeting. At the instant when one of them comes along the street, we have tc make up our mind whether to take it or pass it over and wait for the one which will come afterwards.

Such kind of decisions can only be made *rationally* on some definite criteria. In our example, we may suppose that we should have to pay penalty, if we would be late, and on the other hand, the fee is different for each kind of the traffic facilities. In this circumstance, the criterion may be the total amount of the fee and the penalty.

The refusal against a proposal may be done only in the hope that we would have another proposal, in the future, which is expected to be more favourable than the current one either in sense of the kinds of the proposals or the opportunities they would be made.

Moreover, in order to estimate the figure of merit associated with a particular decision in advance, we must have some information about the figure of merit to accept the proposals which would be made in future.

In this article, we would like to investigate how we can decide optimally whether to accept or refuse a proposal at the instant when it is offered to us, assuming that there are N kinds of them, each of which is made stochastically from time to time, independent to the other kinds, with the known law of '*arrival*' distribution. Suppose that it is admitted to accept a proposal only once. The figure of merit to accept the i-th kind of proposal made at a particular instant is assumed to be known as a function of the time.

The problem is formulated into a system of non-linear integral equations by the method of dynamic programming. The existence and the uniqueness of the solution of the system is proved. Iterative methods for numerical solution are examined with the estimation of the associated error.

§ 2. FORMULATION OF THE PROBLEM

Assume now, that there are N kinds of proposals, each of which is made to us stochastically from time to time with a known '*arrival*' distribution.

Let

(2.1) $$\varphi^j(t) \qquad t \geqq 0, \qquad j=1, \cdots, N$$

be the probability density function that a proposal of the kind 'j' will follow a proposal of the same kind after t time units. We assume that $\varphi^j(t)$ is independent to the other kinds of proposals, what we have replied to the proposals made in the past and the times at which they were made. In other words, it depends only on the kind of the proposal and the time lapse after the latest proposal.

We postulate that $\varphi^j(t)$ is uniformly bounded:

(2.2) $$\varphi^j(t) < M < \infty \qquad t \geqq 0, \qquad j=1, \cdots, N$$

And the *a posteriori* probability given that we have not been offered by proposals of the j-th kind during θ time units after the latest proposal of the same kind, that we would not be offered by a proposal of the same kind until $\theta+\tau$ time units after the latest proposal of the j-th kind is assumed to be defined for any $\theta>0$ and $\tau>0$ by the fraction of the integrals,

(2.3)
$$\frac{\int_{\theta+\tau}^{\infty}\varphi^j(t)dt}{\int_{\theta}^{\infty}\varphi^j(t)dt} \qquad j=1,\cdots,N$$

which is continuous as to θ. Denote by Ω the class of distribution functions satisfying the above conditions.

Denote by

(2.4)
$$g^j(t), \qquad j=1,\cdots,N, \qquad t_0\leqq t<\infty$$

the figure of merit that we will obtain, if we accept, at time t, the proposal of the j-th kind made at that time. We assume that this function is known in advance and belongs to the class Γ, defined below:

$$g^j(t)\epsilon\Gamma, \qquad j=1,\cdots N .$$

Definition 1

The set of uniformly bounded functions $g(t)$ defined on (t_0, ∞) which are continuous except on the set of points $\{t_n\}$, $(n=1,\cdots)$ will be called the class Γ, where the set $\{t_n\}$ is assumed to be fixed for all g in Γ and to have no finite limit points (Fig. 1).

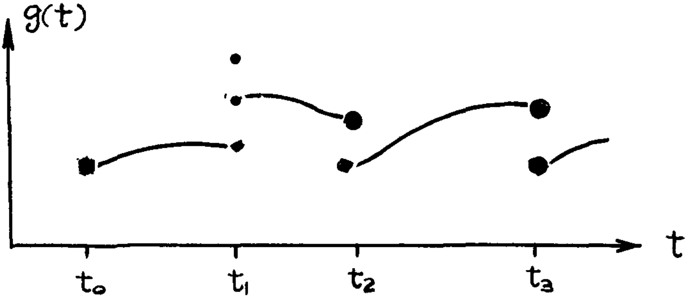

Fig. 1

Let us define the metric on Γ.

Definition 2

The distance between two elements of Γ, $g(t)$ and $g'(t)$ will be defined as follows.

(2.5) $$\rho(g(t),\ g'(t)) = \sup_{t_0 \leqq t < \infty} |g(t) - g'(t)|\ .$$

Next, we introduce a discount factor

(2.6) $$\exp(-\gamma\tau), \qquad \gamma > 0, \qquad \gamma : \text{const.}$$

which will be multiplied by the figure of merit, in order to evaluate it at the instant τ time units prior to the instant at which it will be obtained. Remark that, with this factor, the figure of merit $g(t+\tau_1+\tau_2)$ which is expected to be obtained at time $t+\tau_1+\tau_2$ (τ_1, $\tau_2 > 0$) will be evaluated at time t as

(2.7) $$\exp(-\gamma(\tau_1+\tau_2))\cdot g(t+\tau_1+\tau_2),$$

while the *re-evaluation* at time t of the figure of merit evaluated at time $t+\tau_1$,

$$\exp(-\gamma\tau_2)\cdot g(t+\tau_1+\tau_2)$$

will be

$$\exp(-\gamma\tau_1)\cdot \exp(-\gamma\tau_2)\cdot g(t+\tau_1+\tau_2)$$
$$= \exp(-\gamma(\tau_1+\tau_2))\cdot g(t+\tau_1+\tau_2)$$

which is consistent with the evaluation (2.7).

Let us consider the situation where we are admitted to accept only one proposal, only once; thus all the decisions before the acceptance of a proposal are refusals. Using different words, we are considering a class of *optimal stopping rule* problems——problems of stopping to look over proposals.

The state of a decision maker at time t may be described by the time duration

(2.8) $$\theta^k \geqq 0, \qquad k = 1, \cdots, N$$

from the time when the latest proposal of the k-th kind was made.

Hence, our problem is to decide, at the time when one or several proposals are made, which one to accept or to refuse all of them, related to θ^k, $k=1, \cdots, N$, so as to maximize the expected figure of merit evaluated at that time.

We apply the method of *dynamic programming* [2][1] to formulate the

1) Numbers in brackets [ ] refer to the references cited at the end of the paper.

problem. In order to do so, we define an unknown function as follows.

(2.9) $f(t;\ \theta^1,\cdots,\theta^N)$: the maximum expected figure of merit evaluated at time t, related to the time duration $\theta^k \geqslant 0$ from tne latest proposal of the kind 'k', $k=1,\cdots,N$.

With this, the *principle of optimality* leads to a non-linear integral equation

(2.10) $$f(t;\ \theta^1,\cdots,\theta^N)=\max\begin{bmatrix} g^j(t),\ j\epsilon J \\ h(t;\ \theta^1,\cdots,\theta^N) \end{bmatrix} \quad t\geqq 0,\quad \theta^k\geqq 0\,,$$

where J is the set of all the superfices j for which $\theta^j=0$ and the function $h(t;\ \theta^1,\cdots,\theta^N)$ is defined as follows.

(2.11) $h(t;\ \theta^1,\cdots,\theta^N)$: the maximum expected figure of merit evaluated at time t after refusing all the proposals made up to ($\leqq t$) the time t.

Let us evaluate the function $h(t;\ \theta^1,\cdots,\theta^N)$ in terms of $f(t;\ \theta^1,\cdots,\theta^N)$ and $\varphi^i(t)$, $i=1,\cdots,N$. If we have refused all the proposals made up to ($\leqq$) time t, the next moment at which we would be urged to make decision will be the time at which a proposal would be made for the first time after t, regardless of its kind.

On the other hand, the probability that a proposal of the i-th kind will be made in a small interval $(t+\tau,\ t+\tau+\varDelta\tau)$ as the first proposal after time t is given by

(2.12) $$\left(\prod_{\substack{l=1\\ l\neq i}}^{N}\frac{\int_{\tau+\theta^l}^{\infty}\varphi^l(t)dt}{\int_{\theta^l}^{\infty}\varphi^l(t)dt}\right)\cdot\frac{\varphi^i(\tau+\theta^i)}{\int_{\theta^i}^{\infty}\varphi^i(t)dt}\cdot\varDelta\tau+\mathrm{o}(\varDelta\tau)$$

$$=\left(\prod_{l=1}^{N}\frac{\int_{\tau+\theta^l}^{\infty}\varphi^l(t)dt}{\int_{\theta^l}^{\infty}\varphi^l(t)dt}\right)\cdot\frac{\varphi^i(\tau+\theta^i)}{\int_{\tau+\theta^i}^{\infty}\varphi^i(t)dt}\cdot\varDelta\tau+\mathrm{o}(\varDelta\tau)\ ,$$

where $\mathrm{o}(\varDelta\tau)$ denotes the quantity for which

$$\lim_{\varDelta\tau\to 0}\ \frac{\mathrm{o}(\varDelta\tau)}{\varDelta\tau}=0$$

holds.

Remark here, also, that the probability that more than two proposals would be made in this small interval of length $\Delta\tau$ would be so small as the order $o(\Delta\tau)$ that we may even neglect in reality as in the subsequent discussions.

If we follow the optimal decisions at time $t+\tau$ and thereafter, the maximum expected figure of merit evaluated at time $t+\tau$ is, by definition,

$$f(t+\tau\,;\ \theta^1+\tau,\cdots,\theta^i=0,\cdots,\theta^N+\tau)\,. \tag{2.13}$$

Thus, according to the *principle of optimality*, the function $h(t\,;\ \theta^1,\cdots,\theta^N)$ is evaluated as follows.

$$\begin{aligned} &h(t\,;\ \theta^1,\cdots,\theta^N) \\ &=\sum_{i=1}^{N}\int_{0+}^{\infty} e^{-r\tau}f(t+\tau,\theta^1+\tau,\cdots,\theta^i=0,\cdots,\theta^N+\tau) \\ &\times\left(\prod_{l=1}^{N}\frac{\int_{\tau+\theta^l}^{\infty}\varphi^l(t)dt}{\int_{\theta^l}^{\infty}\varphi^l(t)dt}\right)\cdot\frac{\varphi^i(\tau+\theta^i)}{\int_{\theta^i+\tau}^{\infty}\varphi^i(t)dt}\,d\tau \end{aligned} \tag{2.14}$$

so that it might be, at least in some cases, appropriate to denote it in the form

$$h(t\,;\ \theta^1,\cdots,\theta^N\,;\ f)\,. \tag{2.15}$$

Thus, our problem reduces to the non-linear integral equation

$$f(t\,;\theta^1,\cdots,\theta^N)=\max\begin{bmatrix}g^j(t), & j\epsilon J \\ h(t\,;\theta^1,\cdots,\theta^N\,;\ f)\end{bmatrix} \tag{2.10, bis}$$

$$t\geqq 0,\ \theta^k\geqq 0\,.$$

And the optimal decisions at time t when we are offered a single proposal of the j-th kind are:

(2.16)

(i) accept the proposal of the j-th kind, if $g^j(t)>h(t\,;\ \theta^1,\cdots,\theta^N)$,

(ii) refuse the proposal of the j-th kind, if $g^j(t)<h(t\,;\ \theta^1,\cdots,\theta^N)$,

(iii) either accept of refuse the proposal of the j-th kind, if $g^j(t)=h(t\,;\ \theta^1,\cdots,\theta^N)$.

In case, we are offered more than two proposals at a time, we only have to take the superfix j for which the figure of merit $g^j(t)$ is the greatest among them.

Besides the basic formulation in (2.10), we may formulate the problem in another form. Although this new formulation is essentially equivalent to (2.10), it has more convenient appearance for numerical treatments especially in the case when the '*arrival*' distribution is of *Poisson* type, which will be considered later in examples. This formulation is based on the fact that we may neglect, in reality, the case of two proposals made at a moment, as remarked above.

We define again, unknown functions which are slightly different from that defined in (2.9) as follows

(2.17) $f^j(t;\ \theta^1,\cdots,\theta^{j-1},\theta^{j+1},\cdots,\theta^N)$: the maximum expected figure of merit, evaluated at time t, when a single proposal of the j-th kind is made, related to the time duration θ^k after the latest proposal of the k-th kind.

And again, by the *principle of optimality*, we are lead to a system of non-linear integral equations

(2.18)
$$f^j(t;\ \theta^1,\cdots,\theta^{j-1},\theta^{j+1},\cdots,\theta^N)$$
$$=\max\begin{bmatrix} g^j(t) \\ h(t;\ \theta^1,\cdots,\theta^j=0,\ \cdots,\theta^N;\ \boldsymbol{f}(t)) \end{bmatrix} \quad j=1,2,\cdots,N,$$

where,

$$\boldsymbol{f}(t)=(f^1(t),\cdots,f^N(t))$$

and

(2.19)
$$h(t;\ \theta^1,\cdots,\theta^j=0,\cdots,\theta^N;\ \boldsymbol{f})$$
$$=\sum_{i=1}^{N}\int_{0+}^{\infty} e^{-r\tau} f^j(t+\tau;\ \theta^1+\tau,\cdots,\theta^{i-1}+\tau,\theta^{i+1}+\tau,\cdots,\theta^N+\tau)$$
$$\times\left(\prod_{l=1}^{N}\frac{\int_{\tau+\theta^l}^{\infty}\varphi^l(t)dt}{\int_{\theta^l}^{\infty}\varphi^l(t)dt}\right)\cdot\frac{\varphi^i(\tau+\theta^i)}{\int_{\theta^i+\tau}^{\infty}\varphi^i(t)dt}\,d\tau$$

as in (2.14). Indeed it may be readily remarked that the relation below holds between the functions in (2.9) and (2.17).

(2.20)
$$f(t;\ \theta^1,\cdots,\theta^N)=\max_{j\in J}(\{f^j(t;\ \theta^1,\cdots,\theta^{j-1},\theta^{j+1},\cdots,\theta^N)\}),$$

where J is the set of j for which

(2.21) $\theta^j=0.$

Also, the optimal decisions against the j-th proposal offered at time t are:

(i) accept the proposal of the j-th kind, if $g^j(t)<h(t;\ \theta^1,\cdots,\theta^j=0,\cdots,\theta^N)$,

(2.22) (ii) refuse the proposal of the j-th kind, if $g^j(t)>h(t;\ \theta^1,\cdots,\theta^j=0,\cdots,\theta^N)$,

(iii) either accept or refuse the proposal of the i-th kind, if $g^j(t)=h(t;\ \theta^1,\cdots,\theta^j=0,\cdots,\theta^N)$.

§ 3. EXISTENCE AND UNIQUENESS OF THE SOLUTION

Let us now examine the existence of the solution of the functional equations (2.10) and (2.18). In order to do so, we first define several metric spaces on which the functional equations are to be defined.

Definition 3

The metric space Ψ_{ab} is the set of all the functions $\psi(t:\boldsymbol{\theta})$ defined on the region,

(3.1) $a<t<b$ (b may be finite or infinite)

and

$$\Theta:\ \boldsymbol{\theta}=(\theta^1,\cdots,\theta^N)\geqq 0\ i.e.,\ \theta^j\geqq 0,\ j=1,\cdots,N,$$

which are continuous as to t and $\boldsymbol{\theta}$ and uniformly bounded. The distance between two functions ψ and ψ' in Ψ_{ab} is given by the formula

(3.2) $$\rho(\psi,\ \psi')=\sup_{t\in(a,\ b)}|\psi(t)-\psi'(t)|\,.$$

Definition 4

The metric spase Ψ_a is the set of all the functions $\psi(a,\ \boldsymbol{\theta})$ defined on the region,

$$\Theta:\ \boldsymbol{\theta}=(\theta^1,\cdots,\theta^N)\geqq 0\quad i.e.,\quad \theta^j\geqq 0,\ i=1,\cdots,N,$$

(3.3) a: a fixed parametre

which are continuous as to $\boldsymbol{\theta}$ and uniformly bounded. The distance between two functions ψ_a and ψ'_a in Ψ_a is given by the formula

$$(3.4) \qquad \rho(\psi, \psi')=\sup_{\boldsymbol{\theta}\epsilon\Theta} |\psi(a, \boldsymbol{\theta})-\psi'(a, \boldsymbol{\theta})| .$$

It may be well known that Ψ_{ab} and Ψ_a are complete as to the metrics (3.2) and (3.4) respectively.

Definition 5

The metric space Ψ is the set of all the functions $\psi(t, \boldsymbol{\theta})$ defined on the product set $T_0\times\Theta$, where

$$(3.5) \qquad \begin{aligned} &T_0: \ t\geqq t_0 \\ &\Theta: \ \boldsymbol{\theta}=(\theta^1,\cdots,\theta^N)\geqq 0, \ \ i.e., \ \ \theta^j\geqq 0, j=1,\cdots,N, \end{aligned}$$

which are uniformly bounded and continuous as to $\boldsymbol{\theta}$, and belong to Γ if $\boldsymbol{\theta}$ is fixed. The distance between two functions ψ and ψ' in Ψ is given by

$$(3.6) \qquad \rho(\psi, \psi')=\sup_{\substack{t\epsilon T_0 \\ \boldsymbol{\theta}\epsilon\Theta}}|\psi(t, \theta)-\psi'(t, \theta)| .$$

For this metric space Ψ, we have

Theorem 1

The metric space Ψ is complete as to the metric (3.6).

Proof

Denote by $\{\psi_n(t, \boldsymbol{\theta})\}$ a fundamental sequence in Ψ, *i.e.*, for any $\varepsilon>0$, there exists a positive integer N_ε for which the relations

$$(3.7) \qquad \rho(\psi_p, \psi_q)<\varepsilon$$

hold for all the integers

$$(3.8) \qquad p, q>N_\varepsilon .$$

The parts of the functions $\psi_n(t, \boldsymbol{\theta})$ defined on the interval (t_i, t_{i+1}) belong to $\Psi_{t_i, t_{i+1}}$ and constitute a fundamental sequence in it, as to the metric (3.2). Moreover, $\psi_n(t_i, \boldsymbol{\theta})$, $n=1, 2,\cdots$ is a fundamental sequence in Ψ_{t_i}, as to the metric (3.4). Hence, each of these sequences converges to a limit function with repect to either the distances (3.2) or (3.4):

$$(3.9) \qquad \begin{aligned} &\psi_n(t, \boldsymbol{\theta})\longrightarrow\psi(t, \boldsymbol{\theta}) , \quad t_i<t<t_{i+1} , \quad \boldsymbol{\theta}\epsilon\Theta \\ &\psi_n(t_i, \boldsymbol{\theta})\longrightarrow\psi(t_i, \boldsymbol{\theta}) , \quad \boldsymbol{\theta}\epsilon\Theta \end{aligned}$$

thus the fundamental sequence $\psi_n(t, \boldsymbol{\theta})$, $t_0\leqq t$, $\boldsymbol{\theta}\epsilon\Theta$ converges to the limit function which is constructed by the limit functions given in (3.9) with respect to the distance (3.6). Hence, Ψ is complete as to the metric (3.6).

Q. E. D.

Corollary

Γ is complete as to the metric (2.5).

Let us now define metric on the product space of N complete metric spaces.

(3.10) $$\boldsymbol{\Psi}_N=\Psi^1\times\Psi^2\times\cdots\times\Psi^N .$$

Definition

The distance between two elements of $\boldsymbol{\Psi}_N$

(3.11) $$\boldsymbol{\psi}=(\psi^1, \psi^2,\cdots, \psi^N)\epsilon\boldsymbol{\Psi}_N$$ $$\boldsymbol{\psi}'=(\psi'^1, \psi'^2,\cdots, \psi'^N)\epsilon\boldsymbol{\Psi}_N$$

is defined by

(3.12) $$\rho(\boldsymbol{\psi}, \boldsymbol{\psi}')=\max_{1\leqslant j\leqslant N} \quad \rho(\psi^j, \psi'^j) .$$

It may be obvious that

Theorem 2

Ψ_N is complete as to the metric (3.12).

Let us recall, here, three theorems on *contraction operator* defined on a complete metric space.

Theorem 3

If a mapping T defined on a complete metric space (Π, ρ) into itself is associated with a constant L (which is called the Lipschitz constant) less ($<$) than unity such that

(3.13) $$\rho(T(\pi), T(\pi'))\leqq L\rho(\pi, \pi')$$

for any two elements $\pi, \pi'\epsilon\Pi$, then the operator T is called a *contraction operator* and the sequence defined by the recurrence relation

(3.14) $$\pi_{k+1}=T(\pi_k)$$

with an arbitrary π_0 converges to a limit π, which is the unique fixed point of the operator T *i.e.*,

(3.15) $$\pi=T(\pi) .$$

Corollary

$$\rho(\pi, \pi_n) \leqq \frac{L^n}{1-L}\rho(\pi_1, \pi_0) \tag{3.16}$$

The proof of this theorem and the corollary may be found in any of the standard textbooks on functional analysis (*e.g.*, [4]) and will not be given here.

Theorem 4

If we have an operator system $\boldsymbol{T}$ mapping $\boldsymbol{\Psi}_N$ into itself of the form

$$\begin{aligned} &T^1(\psi^1, \psi^2, \cdots, \psi^N) \epsilon \Psi^1 \\ &T^i(\psi^1, \psi^2, \cdots, \psi^N) \epsilon \Psi^i \\ &T^N(\psi^1, \psi^2, \cdots, \psi^N) \epsilon \Psi^N \end{aligned} \tag{3.17}$$

each of which is associated with a constant L^i such that

$$\rho(T^i(\boldsymbol{\psi}), T^i(\boldsymbol{\psi}')) \leqq L^i \rho(\boldsymbol{\psi}, \boldsymbol{\psi}') \qquad i=1,2,\cdots,N \tag{3.18}$$

for arbitrary two elements

$$\begin{aligned} \boldsymbol{\psi} &= (\psi^1, \psi^2, \cdots, \psi^N) \epsilon \boldsymbol{\Psi}_N \\ \boldsymbol{\psi}' &= (\psi'^1, \psi'^2, \cdots, \psi'^N) \epsilon \boldsymbol{\Psi}_N \end{aligned} \tag{3.19}$$

then the operator

$$\begin{aligned} \boldsymbol{T}(\boldsymbol{\psi}) &= \boldsymbol{T}(\psi^1, \psi^2, \cdots, \psi^N) \\ &= (T^1(\psi^1, \psi^2, \cdots, \psi^N), \cdots, T^N(\psi^1, \psi^2, \cdots, \psi^N)) \epsilon \boldsymbol{\Psi}_N \end{aligned} \tag{3.20}$$

mapping $\boldsymbol{\Psi}_N$ into itself is associated with the Lipschitz constant evaluated as follows.

$$L = \max_{1 \leqslant i \leqslant N} \{L^i\} \tag{3.21}$$

Proof

Indeed, we see that

$$\begin{aligned} \rho(\boldsymbol{T}(\boldsymbol{\psi}), \boldsymbol{T}(\boldsymbol{\psi}')) &= \max_{1 \leqslant i \leqslant N} \rho(T^i(\boldsymbol{\psi}), T^i(\boldsymbol{\psi}')) \\ &\leqq \max_{1 \leqslant i \leqslant N} \{L^i \rho(\boldsymbol{\psi}, \boldsymbol{\psi}')\} = (\max_{1 \leqslant i \leqslant N} \{L^i\}) \rho(\boldsymbol{\psi}, \boldsymbol{\psi}'), \end{aligned}$$

from which (3.21) follows immediately.

Q. E. D.

Theorem 5

If we have an operator system of the form

$$\eta^1 = T^1(\phi^1, \phi^2, \cdots, \phi^N) \in \Psi^1$$

$$\eta^2 = T^2(\eta^1, \phi^2, \cdots, \phi^N) \in \Psi^2$$

(3.22)
$$\eta^i = T^i(\eta^1, \cdots, \eta^{i-1}, \phi^i, \cdots, \phi^N) \in \Psi^i$$

$$\eta^N = T^N({}^1\eta, \cdots, \eta^{N-1}, \phi^N) \in \Psi^N$$

instead of the operator system (3.17), with the Lipschitz constant given in (3.18), then the operator given by (3.22) written in the form.

(3.23)
$$\boldsymbol{\eta} = \boldsymbol{T}^*(\boldsymbol{\phi})$$
$$\boldsymbol{\phi} = (\phi^1, \cdots, \phi^N)$$
$$\boldsymbol{\eta} = (\eta^1, \cdots, \eta^N)$$

is associated with the Lipschitz constant

(3.24)
$$L^* = \max_{1 \leqslant i \leqslant N} \{C^i\}$$

where

$$C^i = L^i \max(1, \max_{1 \leqslant j \leqslant i-1} \{C^j\}), \qquad C^1 = L^1$$

so that, if every L^i $(i=1, 2, \cdots, N)$ is less than unity, we have

(3.25)
$$L^* < 1 \ .$$

Proof

This theorem may be proved inductively. Indeed, for $i=1$,

$$\rho(T^1(\phi), T^1(\phi')) \leqq L^1 \rho(\boldsymbol{\phi}, \boldsymbol{\phi}') = C^1 \rho(\boldsymbol{\phi}, \boldsymbol{\phi}')$$

If we assume that the relations

(3.26)
$$\rho(T^i(\boldsymbol{\eta}), T^i(\boldsymbol{\eta}')) \leqq C^i \rho(\boldsymbol{\phi}, \boldsymbol{\phi}')$$

hold for $i=1, \cdots, n-1$, then we have for $i=n$,

(3.27)
$$\begin{aligned}\rho(T^n(\boldsymbol{\eta}), T^n(\boldsymbol{\eta}')) &\leqq L^n \rho((\eta^1, \cdots, \eta^{n-1}, \phi^n, \cdots, \phi^N),\\ &\qquad (\eta'^1, \cdots, \eta'^{n-1}, \phi'^n, \cdots, \phi'^N))\\ &\leqq L^n \max((\max_{1 \leqslant i \leqslant n-1} \{C^i \rho(\boldsymbol{\phi}, \boldsymbol{\phi}')\}), \rho(\boldsymbol{\phi}, \boldsymbol{\phi}'))\\ &\leqq L^n \max((\max_{1 \leqslant i \leqslant n-1} \{C^i\}), 1) \rho(\boldsymbol{\phi}, \boldsymbol{\phi}') \ .\end{aligned}$$

Thus, the relations (3.26) hold for $i=1,\cdots,N$. And the remainder of the theorem follows from Theorem 4.

Q. E. D.

Let us now examine the exisitence of the solutions of the equations (2.10) and (2.18) evaluating the associated Lipschitz constants.

Consider first the equation (2.10). Denote by $T(f)$ the operator

$$T(f)=\max\begin{bmatrix} g^j(t), & j\in J: \text{the set of all superfices } j \text{ for which } \theta^j=0 \\ h(t;\theta^1,\cdots,\theta^N;f) & \end{bmatrix} \tag{3.28}$$

defined on Ψ. It will be shown that the operator $T(f)$ maps Ψ into itself and is associated with a Lipschitz constant less than unity. Thus, we have

Theorem 6

Equation (2.10) has a unique solution in Ψ, which is obtained as the limit of the sequence given by the recurrence relation

$$f_{n+1}=T(f_n)$$

with an arbitrary $f_0\in\Psi$.

Proof

Indeed, it may be readily seen that h is continuous as to t and $\boldsymbol{\theta}$, so that the operator (3.28) maps Ψ into itself. Moreover, it is readily verified that we have by (2.14)

$$\rho(h(t;\theta^1,\cdots,\theta^N;f),h(t;\theta^1,\cdots,\theta^N;f')) \tag{3.29}$$

$$\leqq\rho(f,f')\cdot\sup_{\theta\in\Theta}\int_{0+}^{\infty}(\exp(-\gamma\tau))\cdot\sum_{i=1}^{N}\frac{\varphi^i(\tau+\theta^i)\prod_{k=1}^{N}\int_{\tau+\theta^k}^{\infty}\varphi^k(r)\,dr}{\int_{\theta^i+\tau}^{\infty}\varphi^i(r)dr\prod_{k=1}^{N}\int_{\theta^k}^{\infty}\varphi^k(r)\,dr}\,d\tau\ .$$

But since by the assumption as to $\varphi^i(t)$ given in §2,

$$\int_{0+}^{\infty}e^{-\gamma\tau}\sum_{i=1}^{N}\frac{\varphi^i(\tau+\theta^i)\prod_{k=1}^{N}\int_{\tau+\theta^k}^{\infty}\varphi^k(r)\,dr}{\int_{\theta^i+\tau}^{\infty}\varphi^i(r)dr\prod_{k=1}^{N}\int_{\theta^k}^{\infty}\varphi^k(r)\,dr}\,d\tau \tag{3.30}$$

$$<\int_{0+}^{\infty}\sum_{i=1}^{N}\frac{\varphi^i(\tau+\theta^i)\prod_{k=1}^{N}\int_{\tau+\theta^k}^{\infty}\varphi^k(r)\,dr}{\int_{\theta^i+\tau}^{\infty}\varphi^i(r)dr\prod_{k=1}^{N}\int_{\theta^k}^{\infty}\varphi^k(r)\,dr}\,d\tau$$

and the integrand in the right hand side is the probability density function that the first proposal after time t would be made at time $t+\tau$ regardless of the kinds. Hence, the right hand side of the inequality in (3.30) is equal to unity. Reminding that

(3.31) $$\rho(T(f),\ T(f'))\leqq\rho(h(t\,;\ \theta^1,\cdots,\theta^N\,;\ f),\ h\ (t\,;\ \theta^1,\cdots,\theta^N\,;\ f')),$$

we see that the operator (3.28) is associated with a Lipschitz constant less than unity. The remainder of the theorem follows by Theorem 3.

Q. E. D.

Consider next the equation (2.18). Denote by $T^i(f^1,\cdots,f^N)$, $i=1, 2,\cdots,N$, the operators

(3.32) $$\max\begin{bmatrix}g^i(t)\\ h(t\,;\ \theta^1,\cdots,\theta^{i-1},\ 0,\ \theta^{i+1},\cdots,\theta^N\,;\ f^1,\cdots,f^2)\end{bmatrix}$$

defined on the product space $\boldsymbol{\Psi}_N$. It will be shown that each of the operators $T^i(f^1,\cdots,f^N)$ maps $\boldsymbol{\Psi}_N$ into Ψ^i and is associated with Lipschitz constant L^i less than unity. Thus we have

Theorem 7

Equation (2.18) has a unique solution in the product space $\boldsymbol{\Psi}_N$ as the limit of the sequence given by the system of the recurrence relations

(3.33)
$$\begin{aligned}f_{n+1}^1&=T^1(f_n^1,\cdots,f_n^N)\\ f_{n+1}^i&=T^i(f_n^1,\cdots,f_n^N)\\ f_{n+1}^N&=T^N(f_n^1,\cdots,f_n^N)\end{aligned}$$

or the system of the recurrence relations of the Seidel form

(3.34)
$$\begin{aligned}f_{n+1}^1&=T^1(f_n^1,f_n^2,\cdots,f_n^N)\\ f_{n+1}^2&=T^2(f_{n+1}^1,f_n^2,\cdots,f_n^N)\\ f_{n+1}^i&=T^i(f_{n+1}^1,\cdots,f_{n+1}^{i-1},f_n^i,\cdots,f_n^N)\end{aligned}$$

$$f_{n+1}^{N}=T^{N}(f_{n+1}^{1},\cdots,f_{n+1}^{N-1},f_{n}^{N}).$$

with an arbitrary $(f_0^1,\cdots,f_0^N)\in\boldsymbol{\Psi}_N$

Proof

Indeed, it may be readily seen that h is continuous as to t and $\boldsymbol{\theta}$, so that the operators in (3.32) map $\boldsymbol{\Psi}_N$ into Ψ^i, $i=1,\cdots,N$. Moreover, it is readily verified that we have by (2.14)

$$\rho(h(t;\theta^1,\cdots,\theta^N;f^1,\cdots,f^N),\ h(t;\theta^1,\cdots,\theta^N;f'^1,\cdots,f'^N)) \tag{3.35}$$

$$\leqq\rho(f,f')\ (\sup_{\theta\in\Theta}\int_{0+}^{\infty}e^{-r\tau}\sum_{i=1}^{N}\frac{\varphi^i(\tau+\theta^i)\prod_{k=1}^{N}\int_{\tau+\theta^k}^{\infty}\varphi^k(r)\,dr}{\int_{\theta^i+\tau}^{\infty}\varphi^i(r)dr\prod_{k=1}^{N}\int_{\theta^k}^{\infty}\varphi^k(r)\,dr}),$$

while the second factor in the right hand side is less than unity as it was shown in the proof of Theorem 6. Reminding that

$$\rho(T^i(\boldsymbol{f}),\ T^i(\boldsymbol{f}'))\leqq\rho(h(t;\boldsymbol{\theta};\boldsymbol{f}),\ h(t;\boldsymbol{\theta};\boldsymbol{f}')) \tag{3.36}$$

we see that the operator $T^i(\boldsymbol{f})$ is associated with the Lipschitz constant L^i less than unity. The remainder of the theorem follows by Theorems 4 and 5 and the fact that the fixed points of the operators in (3.17) and (3.22) coincide as it might be readily verified.

Q. E. D.

Example 1

Consider the case of *Poisson arrival*, *i.e.*, $\varphi^i(\tau)$ are of exponential form:

$$\varphi^i(\tau)=\lambda^i e^{-\lambda^i\tau}\qquad \lambda^i>0,\ \tau\geqq0,\quad i=1,2,\cdots,N\ . \tag{3.37}$$

Since

$$\frac{\varphi^i(\tau+\theta^i)}{\int_{\theta^i+\tau}^{\infty}\varphi^i(t)\,dt}\cdot\prod_{k=1}^{N}\frac{\int_{\tau+\theta^k}^{\infty}\varphi^k(t)dt}{\int_{\theta^k}^{\infty}\varphi^k(t)dt}=\lambda^i\ \exp(-\sum_{k=1}^{N}\lambda^k\tau)\ , \tag{3.38}$$

$h(t;\theta^1,\cdots,\theta^N;\boldsymbol{f})$ and thus $\boldsymbol{f}$ are independent of the parametres $\theta^1,\cdots,\theta^N$. Hence

$$h(t;\boldsymbol{f})=\int_{0+}^{\infty}e^{-r\tau}(\exp(-\sum_{k=1}^{N}\lambda^k\tau))\sum_{i=1}^{N}\lambda^i f^i(t+\tau)d\tau \tag{3.39}$$

from which we see that

$$L^i \leqq \frac{\sum_{i=1}^{N} \lambda^i}{\gamma + \sum_{i=1}^{N} \lambda^i} < 1 . \tag{3.40}$$

The remainder of this section will be devoted to the evaluation of the error associated with the approximated solution of the functional equation via approximated functional equation. In other words, we have to consider, in some cases, the approximated system of functional equations instead of the original system (2.18):

$$\begin{aligned} &f'^i(t\,;\,\theta^1,\cdots,\theta^{i-1},\,\theta^{i+1},\cdots,\theta^N) \\ &=\max\begin{bmatrix} g'^i(t) \\ h'(t\,;\,\theta^1,\cdots,\theta^i=0,\cdots,\theta^N\,;\,f'^1,\cdots,f'^N) \end{bmatrix} \\ &i=1,2,\cdots,N \\ &t_0 \leqq t, \quad \boldsymbol{\theta}=(\theta^1,\cdots,\theta^N)\in\Theta, \quad g'^i(t)\in\Gamma \end{aligned} \tag{3.41}$$

where

$$\begin{aligned} &h'(t\,;\,\theta^1,\cdots,\theta^i=0,\cdots,\theta^N\,;\,f'^1,\cdots,f'^N) \\ &=\sum_{i=1}^{N}\int_{0+}^{\infty} e^{-\gamma\tau} f'^i(t+\tau\,;\,\theta^1+\tau,\cdots,\theta^{i-1}+\tau,\,\theta^{i+1}+\tau,\cdots,\theta^N+\tau) \\ &\quad\times\left(\prod_{k=1}^{N}\frac{\int_{\tau+\theta^k}^{\infty}\varphi'^k(t)dt}{\int_{\theta^k}^{\infty}\varphi'^k(t)dt}\right)\frac{\varphi'^i(\tau+\theta^i)}{\int_{\theta^i+\tau}^{\infty}\varphi'^i(t)dt}\,d\tau \end{aligned} \tag{3.42}$$

with

$$\begin{aligned} &g^i(t)\approx g'^i(t)\in\Gamma \\ &\varphi^i(t)\approx\varphi'^i(t)\in\Omega. \end{aligned} \tag{3.43}$$

Theorem 8

If

$$\max_{1\leqslant i\leqslant N}\sup_{\boldsymbol{\theta}\in\Theta}\sup_{\tau>0}\left|\frac{\varphi^i(\tau+\theta^i)}{\int_{\theta^i+\tau}^{\infty}\varphi^i(t)dt}\prod_{k=1}^{N}\frac{\int_{\tau+\theta^k}^{\infty}\varphi^k(t)dt}{\int_{\tau+\theta^k}^{\infty}\varphi^k(t)dt} - \frac{\varphi'^i(\tau+\theta^i)}{\int_{\theta^i+\tau}^{\infty}\varphi'^i(t)dt}\prod_{k=1}^{N}\frac{\int_{\tau+\theta^k}^{\infty}\varphi'^k(t)dt}{\int_{\theta^k}^{\infty}\varphi'^k(t)dt}\right|\leqq\delta_1 \tag{3.44}$$

and

(3.45) $\rho(\boldsymbol{g},\ \boldsymbol{g}')=\delta_2$

where

$$\boldsymbol{g}=(g^1(t),\cdots,g^N(t))$$
$$\boldsymbol{g}'=(g'^1(t),\cdots,g'^N(t)) ,$$

then the distance between the solutions of the systems of functional equations (2.18) and (3.41) is evaluated as follows.

(3.46)
$$\rho(\boldsymbol{f},\ \boldsymbol{f}')$$
$$\leqq\frac{1}{1-L}$$
$$\times\max\left[\delta_1\left|\sum_{i=1}^{N}\int_{0+}^{\infty}e^{-r\tau}f'^i(t+\tau\ ;\ \theta^1+\tau,\cdots,\theta^i=0,\cdots,\theta^N+\tau)\,d\tau\right|,\ \delta_2\right]$$

Proof

In fact, by the corollary of Theorem 3,

$$\rho(\boldsymbol{f},\ \boldsymbol{f}_1)\leqq\frac{L}{1-L}\rho(\boldsymbol{f}_1,\ \boldsymbol{f}_0)$$

from which, it follows

$$\rho(\boldsymbol{f},\ \boldsymbol{f}_0)\leqq\rho(\boldsymbol{f},\ \boldsymbol{f}_1)+\rho(\boldsymbol{f}_1,\ \boldsymbol{f}_0)\leqq\frac{1}{1-L}\rho(\boldsymbol{f}_1,\ \boldsymbol{f}_0)$$

If we take $\boldsymbol{f}_0$ is equal to the solution $\boldsymbol{f}'$ of the equation (3.41)

$$\boldsymbol{f}_0=\boldsymbol{f}'$$

we obtain

$$\rho(h'(t\ ;\ \boldsymbol{\theta}\ ;\ \boldsymbol{f}'),\ h(t\ ;\ \boldsymbol{\theta}\ ;\ \boldsymbol{f}'))$$
$$\leqq\left|\sum_{i=1}^{N}\int_{0+}^{\infty}e^{-r\tau}f'^i(t+\tau\ ;\ \theta^1+\tau,\cdots,\theta^i=0,\cdots,\theta^N+\tau)\right.$$
$$\times\left(\frac{\varphi^i(\tau+\theta^i)}{\int_{\theta^k+\tau}^{\infty}\varphi^i(t)dt}\prod_{k=1}^{N}\frac{\int_{\tau+\theta^k}^{\infty}\varphi^k(t)dt}{\int_{\theta^k}^{\infty}\varphi^k(t)dt}\right.$$
$$\left.\left.-\frac{\varphi'^i(\tau+\theta^i)}{\int_{\theta^i+\tau}^{\infty}\varphi^i(t)\,dt'}\prod_{k=1}^{N}\frac{\int_{\tau+\theta^k}^{\infty}\varphi'^k(t)dt}{\int_{\theta^k}^{\infty}\varphi'^k(t)dt}\right)d\tau\right|$$
$$\leqq\delta_1\left|\sum_{i=1}^{N}\int_{0+}^{\infty}e^{-r\tau}f'^i(t+\tau\ ;\ \theta^1+\tau,\cdots,\theta^i=0,\cdots,\theta^N+\tau)d\tau\right| .$$

But since

$$\rho(\boldsymbol{f}_1, \boldsymbol{f}') \leqq \max[\rho(\boldsymbol{g}, \boldsymbol{g}'), \rho(h'(t; \boldsymbol{\theta}; \boldsymbol{f}'), h(t; \boldsymbol{\theta}; \boldsymbol{f}')),],$$

we have

$$\rho(\boldsymbol{f}, \boldsymbol{f}') = \rho(\boldsymbol{f}, \boldsymbol{f}_0) \leqq \frac{1}{1-L}\rho(\boldsymbol{f}_1, \boldsymbol{f}_0) = \frac{1}{1-L}\rho(\boldsymbol{f}_1, \boldsymbol{f}')$$

$$\leqq \frac{1}{1-L}\max[\delta_1|\sum_{i=1}^{N}$$

$$\int_{0+}^{\infty} e^{-\delta\tau} f'^i(t+\tau; \theta^1+\tau, \cdots, \theta^i=0, \cdots, \theta^N+\tau)d\tau|, \delta_2]$$

Q. E. D.

Remark that since $\boldsymbol{f}'$ is the solution of the approximated system of functional equations, we may evaluate the right hand side of the inequality (3.46).

§ 4. SOME PRORERTIES OF THE SOLUTION

Let us now examine some properties of the solution of the functional equation (2.18), in case some restrictions are laid on the functions $g^i(t)$. We consider two cases.

1° $g^i(t)$ are periodical.

Theorem 9

If there exists a positive number T, for which

(4.1) $\quad g^i(t+T) = g^i(t), \quad i=1, \cdots, N$

for all $t \geqq t_0$, then the solution of (2.18) is also periodical, *i.e.*,

(4.2) $\quad f^j(t+T) = f^j(t), \quad j=1, \cdots, N.$

Proof

In fact, if we choose an initial function

$$\boldsymbol{f}_0(t) = (f_0^1(t), f_0^2(t), \cdots, f_0^N(t))$$

for which

(4.3) $\quad f_0^j(t+T) = f_0^j(t), \quad j=1, \cdots, N$

for all $t \geqq t_0$ for the sequence given by (3.23), it may be readily seen from the operator (3.32) that

(4.4) $$f^j_n=(t+T)=f^j_n(t)\,, \quad j=1,\cdots,N$$

for all finite n. And, by Theorem 7 of §3, the sequence tends to a limit which is the unique solution of the system of the functional equations (2.18). Denoting this limit by $\boldsymbol{f}(t)=(f^1(t),\cdots,f^N(t))$ we have by (4.2) and (4.4),

(4.5) $$\begin{aligned}\rho(\boldsymbol{f}(t+T),\boldsymbol{f}(t))&\leqq\rho(\boldsymbol{f}(t+T),\boldsymbol{f}_n(t))+\rho(\boldsymbol{f}_n(t),\boldsymbol{f}(t))\\&=\rho(\boldsymbol{f}(t+T),\boldsymbol{f}_n(t+T))+\rho(\boldsymbol{f}_n(t),\boldsymbol{f}(t))\\&\leqq 2\rho(\boldsymbol{f}_n(t),\boldsymbol{f}(t))\,.\end{aligned}$$

But since the last term tends to zero as $n\to\infty$,

(4.6) $$\rho(\boldsymbol{f}(t+T),\boldsymbol{f}(t))=0\,.$$

i.e.,

(4.7) $$f^i(t+T)=f^i(T)\,, \quad j=1,\cdots,N$$

for all $t\geqq t_0$.

Q. E. D.

Example 2

If we assume that the '*arrival*' distributions are of exponential form,

(4.8), (3.37 bis) $$\varphi^i(\tau)=\lambda^i e^{-\lambda^i\tau}\,, \quad \lambda^i>0,\ \tau\geqq 0,\ i=1,\cdots,N$$

as in Example 1, §3 and moreover that $g^i(t)$ are periodical *i.e.*, there exists a positive T such that

(4.9) $$g^i(t+T)=g^i(t)\,, \quad i=1,\cdots,N$$

for all $t\geqq t_0$, we may simplify the relation (3.39) as follows.

(4.10) $$\begin{aligned}h(t;f)&=\int_{0+}^{\infty}e^{-\gamma\tau}(\exp(-\sum_{k=1}^{N}\lambda^k\tau))\sum_{i=1}^{N}\lambda^i f^i(t+\tau)d\tau\\&=\sum_{n=0}^{\infty}\int_0^T\exp(-(\gamma+\sum_{k=1}^{N}\lambda^k)(\tau+nT))\sum_{i=1}^{N}\lambda^i f^i(t+\tau)d\tau\\&=\frac{1}{1-\exp(-(\gamma+\sum_{k=1}^{N}\lambda^k)T)}\times\\&\quad\times\int_0^T\exp(-(\gamma+\sum_{k=1}^{N}\lambda^k)\tau)\sum_{i=1}^{N}\lambda^i f^i(t+\tau)d\tau\,.\end{aligned}$$

For $N=2$, the system of functional equations (2.18) takes the form,

$$f^1(t)=\max\Big[g^1(t) \\ \frac{1}{1-\exp(-(\gamma+\lambda^1+\lambda^2)T)} \times\int_0^T \exp(-(\gamma+\lambda^1+\lambda^2)\tau)(\lambda^1 f^1(t+\tau)+\lambda^2 f^2(t+\tau))d\tau \tag{4.11}$$

$$f^2(t)=\max\Big[g^2(t) \\ \frac{1}{1-\exp(-(\gamma+\lambda^1+\lambda^2)T)} \times\int_0^T \exp(-(\gamma+\lambda^1+\lambda^2)\tau)(\lambda^1 f^1(t+\tau)+\lambda^2 f^2(t+\tau))d\tau\,.$$

In order to have numerical results, let us now approximate the functions $g^1(t)$, $g^2(t)$, $f^1(t)$ and $f^2(t)$ by piecewise constant functions as follows

$$g^i(t)\sim g^{i,r} \qquad t\in[(r-1)\Delta\tau,\ r\Delta\tau)$$
$$f^i(t)\sim f^{i,r} \qquad t\in[(r-1)\Delta\tau,\ r\Delta\tau)$$
$$t_0=0 \qquad i=1,\ 2,\ m=T/\Delta\tau:\ \text{integer} \qquad r=1,\ \cdots,\ m\,.$$

Then (4.11) can be written as follows.

$$f^{1,i}=\max\Big[g^{1,i} \\ \frac{1}{1-\exp(-(\gamma+\lambda^1+\lambda^2)T)}\sum_{r=1}^{m}\left[\frac{\exp(-(\gamma+\lambda^1+\lambda^2)\tau)}{\gamma+\lambda^1+\lambda^2}\right]_{(r-1)\Delta\tau}^{r\Delta\tau} \times(\lambda^1 f^{1,s(r)}+\lambda^2 f^{2,s(r)})\,, \tag{4.13}$$

$$f^{2,i}=\max\Big[g^{2,i} \\ \frac{1}{1-\exp(-(\gamma+\lambda^1+\lambda^2)T)}\sum_{r=1}^{m}\left[\frac{\exp(-(\gamma+\lambda^1+\lambda^2)\tau)}{\gamma+\lambda^1+\lambda^2}\right]_{(r-1)\Delta\tau}^{r\Delta\tau} \times(\lambda^1 f^{1,s(r)}+\lambda^2 f^{2,s(r)})\,,$$

$$i=1,\cdots,m$$

where

$$s(r)=i+r \qquad i+r\leqq m$$
$$=i+r-m \qquad i+r>m\,.$$

We may propose three kinds of the recurrence relations to obtain the numerical solution as follows.

(i)

(4.14)

$$f^{1,i}_{n+1}=\max\Big[g^{1,i}\frac{1}{1-\exp(-(\gamma+\lambda^1+\lambda^2)T)}\times\sum_{r=1}^{m}C^r(\lambda^1 f^{1,s(r)}_n+\lambda^2 f^{2,s(r)}_n),$$

$$f^{2,i}_{n+1}=\max\Big[g^{2,i}\frac{1}{1-\exp(-(\gamma+\lambda^1+\lambda^2)T)}\times\sum_{r=1}^{m}C^r(\lambda^1 f^{1,s(r)}_n+\lambda^2 f^{2,s(r)}_n),$$

$i=1,\cdots,m$.

(ii)

(4.15)

$$f^{1,i}_{n+1}=\max\Big[g^{1,i}\frac{1}{1-\exp(-(\gamma+\lambda^1+\lambda^2)T)}\times\sum_{r=1}^{m}C^r(\lambda^1 f^{1,s(r)}_n+\lambda^2 f^{2,s(r)}_n),$$

$$f^{2,i}_{n+2}=\max\Big[g^{2,i}\frac{1}{1-\exp(-(\gamma+\lambda^1+\lambda^2)T)}\times\sum_{r=1}^{m}C^r(\lambda^1 f^{1,s(r)}_{n+1}+\lambda^2 f^{2,s(r)}_n),$$

$i=1,\cdots,m$.

(iii)

(4.16)

$$f^{1,i}_{n+1}=\max\Big[g^{1,i}\frac{1}{1-e^{-(\gamma+\lambda^1+\lambda^2)T}}\times(\lambda^1\sum_{r=1}^{m}C^r f^{1,s(r)}_{t(r)}+\lambda^2\sum_{r=1}^{m}C^r f^{2,s(r)}_n)$$

$$f^{2,i}_{n+2}=\max\Big[g^{2,i}\frac{1}{1-e^{-(\gamma+\lambda^1+\lambda^2)T}}\times(\lambda^1\sum_{r=1}^{m}C^r f^{1,s(r)}_{n+1}+\lambda^2\sum_{r=1}^{m}C^r f^{2,s(r)}_{t(r)})$$

where

(4.17) $$C^r = \left[\exp \frac{-(\gamma+\lambda^1+\lambda^2)}{\gamma+\lambda^1+\lambda^2}\right]_{(r-1)\Delta\tau}^{r\Delta\tau}$$

and

$$\begin{aligned} s(r) &= i+r & i+r &\leqq m \\ &= i+r-m & i+r &> m \\ t(r) &= n+1 & s(r) &< i \\ &= n & s(r) &\geqq i\,. \end{aligned} \tag{4.18}$$

Using different expressions, (i) is the simple iteration process, while (ii) corresponds to the Seidel process described in §3, (3.34). The method (iii) is also a Seidel process, from whose point of view, (ii) turn out to be '*blockwise*' Seidel process.

Numerical iterations corresponding to these methods are carried out with the values of parametres

$$\gamma = 0.5$$
$$\lambda^1 = 0.1$$
$$\lambda^2 = 0.3\ .$$

Thus the Lipschitz constants of these systems of operators are evaluated as 4/9. And

$$\Delta\tau = 0.1$$
$$T = 2.0$$

thus

$$m = 20\,.$$

Thus figures of merit are

(4.19) $$\begin{aligned} g^{1,r} &= -r+21 \\ g^{2,r} &= r \end{aligned} \qquad r=1,2,\cdots,20\ .$$

And the initial approximations are

(4.20) $$\begin{aligned} f_0^{1,r} &= 10 \\ f_0^{2,r} &= 10 \end{aligned} \qquad r=1,2,\cdots,20\,.$$

With these values, we have obtained the results as shown schematically in Fig. 2. The numbers n of iteration to reach the precision of 0.2×10^{-6} *i.e.*

(4.24) $\rho(f_n, f)<0.2\times 10^{-6}$

(*cf.* (3.16)) are 8, 7 and 6 for the methods (i), (ii) and (iii) respectively.

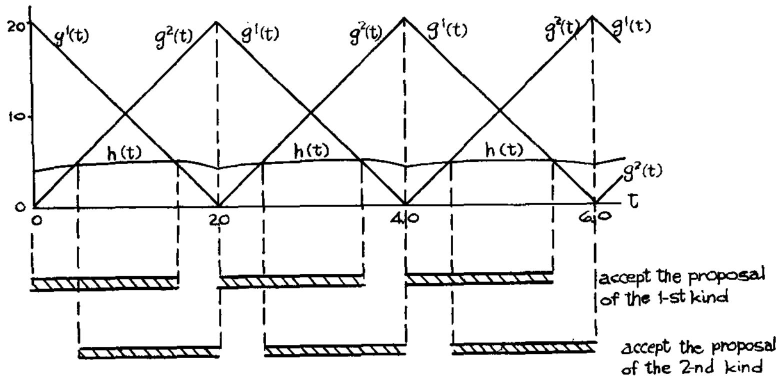

Fig. 2

2° $g^i(t)=0$ $\quad t\geqq T>t_0$

Theorem 10

If there exists a number $T>t_0$ such that

(4.25) $g^i(t)=0 \qquad t\geqq T, \qquad i=1,\cdots,N$,

then

(4.26) $f^j(t)=0 \qquad t\geqq T, \qquad j=1,\cdots,N$.

Proof

In fact, if we choose the initial functions $f_0^j(t)$, $j=1, \cdots, N$ which are

(4.27) $f_0^j(t)=0, \qquad t\geqq T, \qquad j=1,\cdots,N$

for the suquence given by the operator (3.28), it may be readily verified that

(4.28) $f_n^j(t)=0, \qquad t\geqq T, \qquad j=1,\cdots,N$.

And, by Theorem 7 of §3, the sequence tends to a limit which is the unique solution of the system of the functional equations (2.18). Denoting this limit by $\boldsymbol{f}(t)=(f^1(t),\cdots,f^N(t))$, we have by (3.13),

$$\text{(4.29)} \qquad \max_j \sup_{t \geqq T} |f^j(t)| = \max_j \sup_{t \geqq T} |f^j_n(t) - f^j(t)| \leqq \rho(\boldsymbol{f}_n, \boldsymbol{f}) \leqq \frac{L^n}{1-L} \rho(\boldsymbol{f}_1, \boldsymbol{f}_0)$$

where L is the Lipschitz constant associated with the operator system (3.28) evaluated to be less than unity in the proof of Theorem 6, § 3. Thus, the left hand side of (4.29) tends to zero as $n \to \infty$, from which the statement of the theorem follows immediately.

Q. E. D.

Numerical iteration was carried out by the methods which corresponds to the method (iii) in 1° of this section for the parametres

$N=2$

$\lambda^1=0.1$

$\lambda^2=0.3$

$\gamma=0.7$

$T=2.0$.

The figures of merit $g^1(t)$ and $g^2(t)$ are as shown schematically in Fig. 3. The numerical iteration was repeated 6 iterations to reach the results shown in Fig. 3.

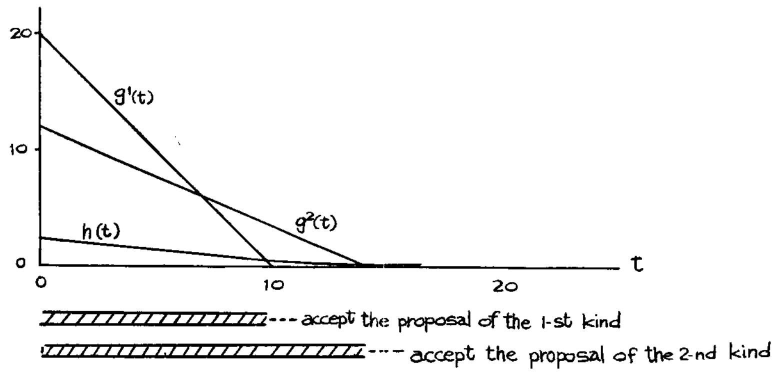

Fig. 3

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