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ON AN OPTIMAL SEARCHING SCHEDULE

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The problem how to divide a given amount of searching effort to various possible positions was initiated by B.O. Koopman [1], and several authors have been trying to generalize his theory.

In these theories, the search effort which corresponds to the number of units participating in the search multiplied by the search time is assumed divisible as one wishes, but in practice there would be occasions where the search effort connot be divided as the theories tell. If a considerable number of units is available, the searching will be scheduled fairly in accordance with the theoretical way of division. However, in the case that only one unit is asked for carrying out the search, for instance, the unit must scan here ane there so that appropriate time may be spent in respective places just as the theories require.

In the latter case, in particular if the places or regions where the object is supposedly located are more or less separated from each other, the time necessary for changing places is not a negligible factor, and there appears some difficulty in applying the existing theories to this situation. Therefore, it might be of practical use to set up a model where the transfer time between regions is explicitly taken into account. In the following, a theoretical model describing the situation outlined above and the resulting optimal time schedule of search are given.

ASSUMPTIONS

An object is known to be located somewhere in either of two regions with area A_1 and A_2 respectively, and the probability of its location being in a unit area of the region A_j is assumed constant and is given by ρ_j $(j=1, 2)$. The probability density ρ_j multiplied by the area A_j is the probability p_j that the object is located in the region, and it is assumed they are both positive and added to unity.

$$p_1+p_2=1, \quad p_j=\rho_j A_j>0 \quad (j=1, 2) \tag{1}$$

Concerning the detection probability, the following is postulated. An observer who is trying to detect the object is supposed to have chosen the correct region A_j in which the object is really located, and to have started searching, where the search effort is assumed to continue until the object is finally detected. The time required for the detection would be a random variable X, and its probability distribution function is assumed of negative exponential type with a mean $1/\lambda_j$. Namely,

$$P(X\leqq t)=1-e^{-\lambda_j t}, \quad t>0. \tag{2}$$

The postulate is equivalent to assume the well known random search, and in this case the parameter λ_j is given by

$$\lambda_j=\frac{VW}{A_j}, \quad (j=1, 2) \tag{3}$$

with the effective sweep width W and the velocity V of the observer. Since VWt is the area scanned by the observer in t, $\lambda_j t$ is the scanned area divided by the whole area of the region in question, and it is often called as searching effort density.

The searching schedule proposed in our model is now described: First, the observer chooses either one region, not knowing which is the correct region to be scanned. For the time being, this region will be named as A_1. Since the object is not necessarily in A_1, the observer should not persist in scanning this region too long. So, the observer is assumed to intermit his effort in the region when a prescribed time T_1^0 is elapsed before detection, and switch over to the other region for the next trial. In our model this prescribed time will be named as a shift time. As for the new region, A_2, a shift time T_2 is also prescribed, and if the added effort there does not yield success during this time, the observer is assumed to go back to the original region, A_1, to continue the similar procedure repeatedly until the detection is established. Here, the prescribed shift time T_j for region A_j is assumed constant for every cycle except the opening stage of scanning with $T_1{}^0$, because the situation of the observer just after a switch is supposed identical with that one cycle before.

The switch time between two regions would be considered as a random variable. We need not know its distribution function itself, but only its mean value; the one-way mean switch time is denoted as $\Theta/2$.

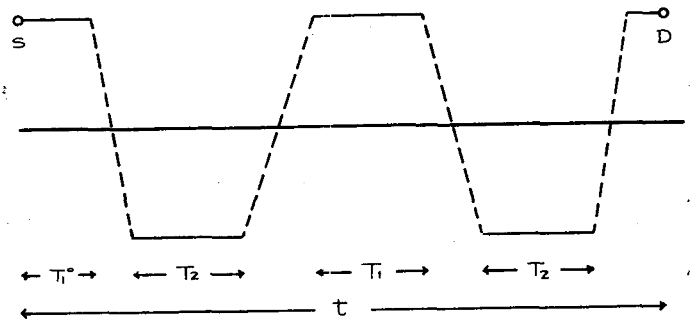

Fig. 1 Illustration of the searching schedule

The proposed search schedule is schematically illustrated in Fig. 1, where the horizontal solid lines represent the shift time in each region, and the broken lines correspond to the switch time. The letters S and D mean start and detection respectively, and in this case, the detection is established after four switches.

In general, the observer has some possibility of finding out the object on his way to the another region, but in this model, this chance is assumed not available for him. Namely, the switch time is really an idle time for the observer. The shift times, $T_1{}^0$, T_1 and T_2 must not be too long because the object may in fact be located somewhere in another region. On the other hand, they must not be too short since changing the region for scanning so frequently causes him to waste much switch time. These values should, therefore, be reasonably determined so that the searching schedule described above may be the most effective one.

In this study, the mean over-all time for detection (the expectation of the time from the beginning of the opening stage of scanning to the detection, the time necessary for the arrival to the chosen region being excluded) is adopted as the measure of effectiveness, and the three variables T_1^0, T_1 and T_2 which minimize the mean time will be sought in the following.

FORMULATION

For brevity's sake, such notations as

$$\lambda_j T_j = \phi_j \qquad (j=1,\ 2) \tag{4}$$

$$\lambda_j T_j^0 = \phi_j^0 \qquad (j=1,\ 2) \tag{5}$$

$$\lambda_j \Theta = \theta_j \qquad (j=1,\ 2) \tag{6}$$

are used from now on. ϕ_j is the shift time divided by the mean detection time $1/\lambda_j$ in A_j, and as mentioned above, this is also the fraction of the whole area A_j that is scanned by the observer during the shift time, T_j, *i.e.*, it stands for the effort density corresponding to T_j. ϕ_j^0 is similar quantity for the opening scanning, and the meaning of θ_j is the effort density which would be attained if the idle time Θ were applied to the search effort in A_j.

As the first step, let us suppose the observer has arrived at the correct region A_j and started searching, where the schedule of search is characterized by a set of shift times (T_1, T_2), the opening stage of scanning being excluded. Let the mean conditional detection time (from the arrival mentioned above till the detection) be denoted as $E(t_j)$. To calculate $E(t_j)$, two cases are considered separately; the first case where the object is detected within the prescribed time T_j, the probability of this case being given by $1-e^{-\phi_j}$, and the second case where the detection fails with probability $e^{-\phi_j}$. As to the second case, the observer has to waste time in this manner: The mean switch time $\Theta/2$ is required for the switch-over, and scanning time $T_k(k \neq j)$ in the other region (in fact the scanning is completely fruitless since the object is really located in the original region, A_j) is added, and finally $\Theta/2$ is again required for switching back to the original region. When the observer comes back, the initial situation is again established, and the mean detection time after this time point will be given by $E(t_j)$ itself.

On the other hand, the mean detection time for the first case is given by the formula,

$$\frac{\int_0^{T_j} \lambda_j t e^{-\lambda_j t} dt}{\int_0^{T_j} \lambda_j e^{-\lambda_j t} dt} = \frac{1}{\lambda_j} \frac{1 - e^{-\phi_j} - \phi_j e^{-\phi_j}}{1 - e^{-\phi_j}}.$$

Therefore, the mean conditional time $E(t_j)$, $(j=1, 2)$ are found to satisfy the relations,

$$\begin{cases} E(t_1) = \frac{1}{\lambda_1}(1 - e^{-\phi_1}) - \frac{\phi_1}{\lambda_1} e^{-\phi_1} + e^{-\phi_1}\left(\frac{\phi_1}{\lambda_1} + \frac{\phi_2}{\lambda_2} + \Theta + E(t_1)\right) \\ E(t_2) = \frac{1}{\lambda_2}(1 - e^{-\phi_2}) - \frac{\phi_2}{\lambda_2} e^{-\phi_2} + e^{-\phi_2}\left(\frac{\phi_2}{\lambda_2} + \frac{\phi_1}{\lambda_1} + \Theta + E(t_2)\right) \end{cases} \tag{7}$$

and $E(t_j)$ are derived as

$$\begin{cases} E(t_1) = \frac{1}{\lambda_1} + \frac{1}{\lambda_2} \frac{\phi_2 + \theta_2}{e^{\phi_1} - 1} \\ E(t_2) = \frac{1}{\lambda_2} + \frac{1}{\lambda_1} \frac{\phi_1 + \theta_1}{e^{\phi_2} - 1}. \end{cases} \tag{8}$$

We can now get the formulation concerning the whole schedule of search. At the opening stage, the observer begins with the region A_1, and only this stage of scanning is different from the following ones. For the opening scanning, it must be noted that the shift time, T_1^0, is not same as the later value T_1, and that the probability that A_1 is the correct region is given by p_1. The over-all mean detection time $E(t)$ can be easily derived by a similar consideration to what led the relation (7), and the required formula

$$E(t)=p_1\left\{\frac{1}{\lambda_1}(1-e^{-\phi_1^0})+e^{-\phi_1^0}\left(E(t_1)+\frac{\phi_2}{\lambda_2}+\frac{\theta_2}{\lambda_2}\right)\right\}+p_2\left(E(t_2)+\frac{\phi_1^0}{\lambda_1}+\frac{\theta_1}{2\lambda_1}\right) \tag{9}$$

will be readily obtained.

OPTIMALIZATION

The three variables, ϕ_1^0, ϕ_1 and ϕ_2 which characterize the proposed searching schedule are now determined so as to minimize the expected detection time (9). The necessary conditions are given by

$$\frac{\partial E(t)}{\partial \phi_1^0}=0, \tag{10 a}$$

$$\frac{\partial E(t)}{\partial \phi_1}=0, \tag{10 b}$$

$$\frac{\partial E(t)}{\partial \phi_2}=0, \tag{10 c}$$

and the fisrt condition is found to yield

$$\lambda_1 p_1\left(E(t_1)-\frac{1}{\lambda_1}+\frac{\phi_2}{\lambda_2}+\frac{\theta_2}{\lambda_2}\right)e^{-\phi_1^0}=p_2,$$

i.e.,

$$\rho_1 e^{-\phi_1^0}=\frac{1-e^{-\phi_1}}{\phi_2+\theta_2}\rho_2\,. \tag{11}$$

The second condition together with the relation (11) just established give $\exp(\phi_1)=\exp(\phi_2)$, and so

$$\phi_1=\phi_2$$

is obtained. Finally, eliminating ϕ_1^0 from the last condition (10 c) and (11) we can easily get

$$(\phi_1+\theta_1)(\phi_2+\theta_2)=(e^{\phi_2}-1)^2e^{-\phi_2},$$

and combining these three relations, we arrive at three equations in three variables ϕ_1^0, ϕ_1 and ϕ_2,

$$\phi_1=\phi_2\equiv\phi, \tag{12 a}$$

$$(\phi+\theta_1)(\phi+\theta_2)=4\sinh^2\left(\frac{\phi}{2}\right), \tag{12 b}$$

$$\rho_1 e^{-\phi_1{}^0}=\frac{1-e^{\phi}}{\phi+\theta_2}\rho_2\,. \tag{12 c}$$

It is worth while to note that from (12 b) $\phi_1(=\phi_2)$ is regarded as a function of θ_1 and θ_2 only, and this relation substituted to (12 c) gives $\phi_1{}^0$ its expression which is a function of ρ_1/ρ_2 as well as θ_1 and θ_2.

In order that the root $\phi_1{}^0$ is positive, the inequality,

$$\rho_1>\frac{1-e^{-\phi}}{\phi+\theta_2}\rho_2 \tag{13}$$

should be satisfied by ρ_j, where it is supposed the above mentioned $\phi=\phi(\theta_1,\ \theta_2)$ is substituted to the right hand member of (13). When this inequality is valid, (12)s determine a unique set of positive roots, $\phi_1{}^0$, ϕ_1 and ϕ_2.

Whether or not the solution just obtained really gives the mean detection time a minimum value is not yet shown. To do this, we must evaluate three determinants, the elements of which are composed of six partial derivatives of the second order, and see whether their signs are all positive. This, after somewhat laborious computation, is found to be the case ; a searching schedule corresponding to a minimum expected detection time is thus obtained. The minimum value has, however, no concise expression. Inserting (12 a) and (12 c) to the expression (9), we obtain

$$E_{\min}=\frac{1}{\lambda_1}+p_2\left(E(t_2)+\frac{\phi_1{}^0}{\lambda_1}+\frac{\theta_1}{2\lambda_1}\right) \tag{14}$$

for the minimum value, where $E(t_2)$ and $\phi_1{}^0$ is given by (8) and (12).

Here another possibility of searching schedule should be examined. Thus far, we have supposed the search effort is begun with the region A_1, giving the above result. If the searching shedule had the alternative form, *i.e.*, if the opening effort of scanning were applied to the region A_2 and similar procedure followed, the solution would be all the same as above but with the subscripts 1 and 2 exchanged : The solution equations

$$\phi_1=\phi_2\equiv\phi \tag{15 a}$$

$$(\phi+\theta_1)(\phi+\theta_2)=4\sinh^2\left(\frac{\phi}{2}\right), \tag{15 b}$$

$$\rho_2 e^{-\phi_2^0}=\frac{1-e^{-\phi}}{\phi+\theta_1}\rho_1, \tag{15 c}$$

give a set of positive roots (ϕ_2^0, ϕ_2, ϕ_1) so long as the inequality,

$$\rho_2>\frac{1-e^{-\phi}}{\phi+\theta_1}\rho_1 \tag{16}$$

holds, and the minimum value is given by

$$E'_{\min}=\frac{1}{\lambda_2}+p_1\left(E(t_1)+\frac{\phi_2^0}{\lambda_2}+\frac{\theta_2}{2\lambda_2}\right). \tag{17}$$

Comparing (13) with (16), we see there are two solutions available for some region of the parameters ρ_j, and there is a definite value of ρ_1 in this region where two minimum values (14) and (17) coincide with each other. Denoting it as ρ_1^0, we can verify the inequalities,

$$E_{\min}\gtrless E'_{\min} \qquad \text{as} \quad \rho_1\lesseqgtr\rho_1^0$$

and it is concluded that if

$$\rho_1>\rho_1^0 \tag{18}$$

the solution given by (12 a) through (12 c) is the optimal one we have been seeking. If the inequality fails to hold, the optimal schedule is characterized by the roots of (15)s.

Equating (14) with (17) we obtain the expression of ρ_1^0, which seems, however, to be given no closed form:

$$\rho_1^0=\frac{1}{A_1}\,\frac{\dfrac{\phi}{\lambda_1}+\dfrac{\theta_1}{2\lambda_1}(e^{\phi}+1)+\dfrac{1+\phi_1^0}{\lambda_1}(e^{\phi}-1)}{\dfrac{\phi}{\lambda_1}+\dfrac{\phi}{\lambda_2}+\left(\dfrac{\theta_1}{2\lambda_1}+\dfrac{\theta_2}{2\lambda_2}\right)(e^{\phi}+1)+\left(\dfrac{1+\phi_1^0}{\lambda_1}+\dfrac{1+\phi_2^0}{\lambda_2}\right)(e^{\phi}-1)} \tag{19}$$

with ϕ, ϕ_1^0 and ϕ_2^0 given by (12 b), (12 c) and (15 c) respectively.

APPROXIMATION AND EXAMPLE

The optimal solution obtained has interesting properties.

(1) The shift times T_1 and T_2 should be prescribed so that they are proportional to the area of each region ($T_1/T_2=A_1/A_2$), and T_j is given a value with only Θ, λ_1 and λ_2. This follows from the relations (12) and

it is independent upon the value of p_j and also the fact which region is chosen for the opening trial.

(2) As to the opening trial, the region to be begun with is determined by such a relation as (18), which corresponds to the case of A_1 being chosen. The shift time of the opening trial, T_j^0, is determined from (12 c) or (15 c) which depends on the ratio, ρ_1/ρ_2, as well as Θ, λ_1 and λ_2.

(3) The effort density for the opening scanning, ϕ_j^0, is smaller than the density corresponding to the shift time of the later stage if both (13) and (16) hold, since we have

$$\phi_1^0+\phi_2^0=\phi \tag{20}$$

which is easily derived by multiplying (12 c) by (15 c) and using the relation (12 b). On the other hand, if (16) fails to hold, ϕ_1^0 is larger than ϕ, telling the fact the object has rather high possibility in the region A_1 and considerable amount of effort should be applied first to A_1 before another region is tried to scan. Similar statement is also possible as for the case where the inequality (13) fails to hold.

It is natural to suppose the switch time Θ is fairly short in most cases where such a searching schedule as proposed above is adopted. Supposing θ_j/ϕ is considerably smaller than unity and ϕ is of order 1 and expanding (12 b) in series, we get an approximating formula for the effort density ϕ,

$$\phi \fallingdotseq 2\left(3\,\frac{\theta_1+\theta_2}{2}\right)^{\frac{1}{3}}. \tag{21}$$

For instance, (21) gives $\phi=3.36$ for $\theta_1=\theta_2=2$, whereas the correct value is 3.46, the difference being only about five percent. The degree of approximation will naturally be much improved as the value of θ is decreased. Fig. 2 shows ϕ-θ curve (assuming $\theta_1=\theta_2=\theta$) obtained from (12 b). It must be remembered ϕ increases rapidly for smaller value of θ, telling that even when only a short time is required for the transfer between the regions, frequent switchover is never advisable.

A numerical example will be given here. Let the two regions be equal in their area, and let the location of an object being in each region

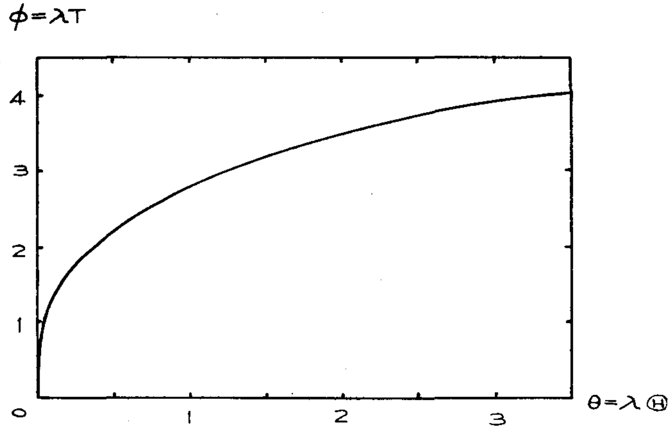

Fig. 2 ϕ versus θ curve (12 b) with $\theta_1=\theta_2\equiv\theta$

be equally probable. As to the detection probability, the mean time for detecting the object, $1/\lambda_1=1/\lambda_2=8$ hours is given, and the idle time necessary for the one-way switch is known to be 3/2 hours in the mean.

Since $\lambda_1=\lambda_2$, θ_1 is equal to θ_2 and is 3/8. From Fig. 2, ϕ is found to be approximately 2, and $\phi_1{}^0=\phi_2{}^0=\phi/2=1$ is easily obtained from the relations (12 c), (15 c) together with (19). The searching schedule is as follows:

Either of two regions may be chosen for the opening trial. The observer should first scan the region, say A_1, till 8 hours of time is elapsed. If the object is not detected by this time, the observer switches over to the another region A_2, and remains there for 16 hours. If the detection is not still established, he should come back again to the originol region, and further 16 hours' searching should be added there. If the detection is still unsucessful, trials should be repeated subsequently in the similar way until the object is finally detected. The expected time of detection from the beginning is about 18 hours.

DISCUSSIONS

B.O. Koopman [1] first set up the problem of the allocation of search effort, and several articles [2], [3], [4], with the aim of generalizing his theory have been published since then. The original form of the problem and its solution are as stated below.

An object is in a unknown position, but its probability p_1 and p_2 of being in each one of two regions A_1, A_2 is given. The law of detection is of exponential type in the search effort density ϕ as was assumed in our model, and a limited total amount of search effort Φ is available.

It is the aim of this problem to divide the given amount Φ between two regions so as to maximize the over-all chance of finding out the object. Namely, let ϕ_1 and ϕ_2 be the effort density applied to the two regions respectively, then under the side conditions,

$$A_1\phi_1+A_2\phi_2=\Phi, \qquad \phi_1, \phi_2 \geqq 0, \tag{22}$$

the function

$$P(\phi_1, \phi_2)=p_1(1-e^{-\phi_1})+p_2(1-e^{-\phi_2}) \tag{23}$$

is maximized with respect to ϕ_1 and ϕ_2.

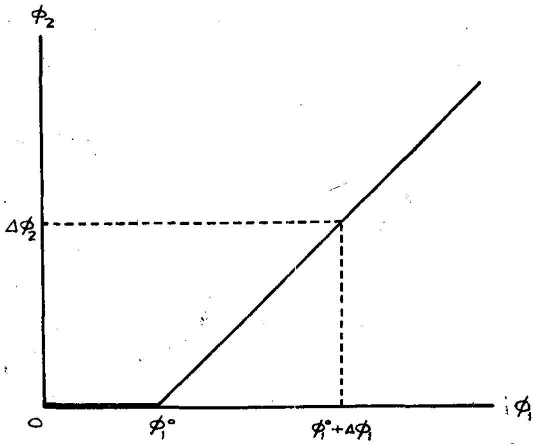

Fig. 3 Trajectory of the optimal point

The optimal set of values (ϕ_1, ϕ_2) obtained by him will be here given an intuitive interpretation using a graphical representation of the trajectory of the point (ϕ_1, ϕ_2). Let the total amount Φ be at first considered as a parameter. When Φ is varied from a small value to infinity, the corresponding optimal solution (ϕ_1, ϕ_2) moves in the $\phi_1\phi_2$ plane and is found to draw such a line as is shown in Fig. 3, where $\rho_1 > \rho_2$ is assumed. The solution to a fixed value of Φ is of course given by the intersection of this trajectory and the straight line (22).

It is noteworthy that at first the optimal point runs on the ϕ_1 axis until the point $\phi_1 = \phi_1^0$ is reached, where ϕ_1^0 is given by

$$\rho_1 e^{-\phi_1^0} = \rho_2 , \tag{24}$$

and then it changes its direction. The straight line on which the optimal point moves after the above-mentioned point is passed has a slope unity, and in this stage the optimal increment of the effort density in two regions are the same:

$$\Delta\phi_1 = \Delta\phi_2 . \tag{25}$$

The optimal search schedule in our model has some analogy in its properties with the above-mentioned solution. Of course, the setting up of the problem is different; one adopted the expected over-all detection time as the measure, and the other the over-all detection probability. Moreover, the idle time during the inter-regional movement plays an important role in our model, and the two solutions are naturally different from each other. Nevertheless there is some close correspondency between them.

Our stage of opening scan corresponds to the portion of the trajectory which is on the ϕ_1 axis, and the relation (12 c) is compared with (24), where it is easily seen the former goes over to the latter relation in the limit $\Theta \to 0$. Our sequence of switchovers between two regions corresponds to the stage represented by the oblique line in Fig. 3, where the effort density is divided between two regions according to the relation (25) which has quitely close parallelism with (12 a) in our model.

Supposedly, our model and its modifications are worth investigating in more detail. For example, the possibility that the object finds out the observer prior to being detected would be an interesting feature possible in practice and attractive from the theoretical point of view. Hide-and-seek type competition is expected here, and such methods as the game theory and the dynamic programming-like approach might be also powerful to deal with the model. These problems will, we wish, be investigated in the near future.

REFERENCES

[1] B.O. Koopman, "The Optimum Distribution of Effort," Opns. Res. **1**, 52—63 (1952).

[2] B.O. Koopman, "The Theory of Search, III. The Optimum Distribution of Searching Effort," Opns. Res. **5**, 613—626 (1957).

[3] Jacques de Guenin, "Optimum Distribution of Effort: An Extension of the Koopman Basic Theory," Opns. Res. **9**, 1—7 (1961).

[4] Kazuo Tada, "Tansaku Doryoku no Haibun ni Kansuru Ichi-Kōsatsu," Management Science, Japan **7**, 81—86 (1964). (in Japanese).

Added in proof: After submitting the M.S. to the Society, the author was noticed the presence of the article by E.N. Gilbert, "Optimal Search Strategies," appeared in the Journal of the Society of Industrial and Applied Mathematics, Vol. 7, No. 4, 1959. In this paper, similar searching schedule problems are investigated, and a special case of our model is also dealt with as an example. Only the word "switch time" in Gilbert's paper being adopted to avoid confusion, the author dared not revise his M.S. Comparison of two papers and their generalization will be appeared elsewhere.