QUEUES WITH POISSON INPUT AND MIXED-ERLANGIAN SERVICE TIME DISTRIBUTION WITH FINITE WAITING SPACE

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ABSTRACT

The problem considered in this paper is the steady state solution of the single server queuing system characterised by (i) Poisson input; (ii) first come, first served, queue discipline; (iii) mixed-Erlangian service time distribution, the mean service times in the phases of the service channel not being same; and (iv) finite waiting space. The system size distribution is obtained from which the important queue characteristic, such as the mean number of units in the system may be obtained. In particular, the results for the queuing system $M/E_2/1$, the mean service times in the two phases of the service channel not being same, have been deduced and to show the workability of the procedure outlined, graphs showing the behaviour of (i) probability of no delay, (ii) probability of loss, and (iii) mean number of units in the system have been sketched which enable one to compare the results of the present study to those known earlier.

FORMULATION OF THE PROBLEM

Units arrive at a single service facility according to a stationary Poisson stream with parameter λ. The service facility consists of a number, j, of phases. Each arriving unit chooses (or is allotted) a number, r, of phases, $r=1, 2, \cdots, j$, with probability C_r, in each one of which it is served before leaving the service facility. The service time distribution in the rth phase of the service facility is assumed to be exponential with parameter μ_r. At any time there is and can be only one unit present in the whole service channel consisting of the j phases. The incoming unit joins the queue or service channel according as the service channel is busy or not. The queue discipline is first come, first served. The system (queue+service) can accomodate only N units, so that when a unit arrives and finds N units already in the system, it goes away and is thus lost to the system.

A similar type of problem when $\mu_r=\mu$ for $r=1, 2, \cdots, j$ has already been study by Jain [1]. But we have no obvious reason to assume that the mean service times in the various phases of the service channel are same. Actually the purpose of introducing the probabilities, C_r, which accounts for the name mixed-Erlangian instead of Erlangian for the service time distribution, is to accelerate or decelerate the service rate. The same purpose can also be achieved by the introduction of μ_r instead of μ and the introduction of both C_r and μ_r facilitates the simulation of some distributions and also enhances the field of applications to the cases where the mean service rate in the various phases of the service channel is not same. It is primarily the second aspect, *i.e.*, the field of applications, which determines the scope of this paper.

It may also be remarked that the Laplace transform of the system size distribution in the transient case can also be calculated without any additional complications. But we are purposely avoiding it here since the inversion, even in very particular cases, is quite tedious and forbidding.

CONTITUITY EQUATIONS AND THEIR SOLUTION

Let us introduce $p(n, r)$ as the steady state probability that there are n units in the system the unit in the service channel being in the rth phase. Also let $p(0)$ denote the probability of theye being no unit in the system. Thus our phase space consists of the point corresponding to $p(0)$ and the elements of the Cartesian product $J\times R$ where J and R are the finite sets $\{1, 2, \cdots, N\}$ and $\{1, 2, \cdots, j\}$ respectively. The continuity equations connecting the $p(n, r)$ in the steady state case are:

$$-(\lambda+\mu_r)p(n, r)+\mu_{r+1}(1-\delta_{rj})p(n, r+1)+\lambda p(n-1, r)$$
$$+\mu_1 C_r p(n+1, 1)=0\,,\ (n=2, 3, \cdots, N-1) \tag{1}$$

$$-(\lambda+\mu_r)p(1, r)+\mu_{r+1}(1-\delta_{rj})p(1, r+1)+\lambda C_r p(0)+\mu_1 C_r p(2, 1)=0 \tag{2}$$

$$-\mu_r p(N, r)+\mu_{r+1}(1-\delta_{rj})p(N, r+1)+\lambda p(N-1, r)=0 \tag{3}$$

$$-\lambda p(0)+\mu_1 p(1, 1)=0 \tag{4}$$

where δ_{rj} is the Kronecker delta and epuations $(1\sim4)$ are valid for $r=1, 2, \cdots, j$.

Also we have the following equation stating the condition of normality

$$\sum_{n=1}^{N}\sum_{r=1}^{j} p(n, r)+p(0)=1 \tag{5}$$

Our problem is to solve the equations $(1\sim5)$ for the $Nj+1$ probabilities involved. Equations $(1\sim4)$ are $Nj+1$ linear homogeneous equations in as many unknowns and thus a non-trivial solution will exist provided the determinant formed by the coefficients of these probabilities is zero. But equation (5) states that a non-trivial solution exists and hence the determinant formed by the coefficients is zero. Herebelow, we solve this system of equations by using the technique of generating functions.

Let us define the generating functions

$$F_r(x)=\sum_{n=1}^{N} p(n, r)x^n\,.$$

Multiplying equations $(1\sim3)$ by appropriate powers of x, adding and using (4), we get

$$[-(\lambda+\mu_r)+\lambda x]F_r(x)+\mu_{r+1}(1-\delta_{rj})F_{r+1}(x)+\frac{\mu_1 C_r}{x}F_1(x)$$
$$+\lambda x^N p(N,r)(1-x)+\lambda C_r p(0)(x-1)=0\ \ (r=1, 2, \cdots, j)\,. \tag{6}$$

For $r=1, 2, \cdots, j-1$, we have from (6)

$$F_{r+1}(x)=A_r F_r(x)+B_r \tag{7}$$

where

$$A_r=\frac{1}{\mu_{r+1}}[\lambda(1-x)+\mu_r],$$

$$B_r=\frac{1}{\mu_{r+1}}\left[(x-1)\{\lambda x^N p(N,\ r)-\lambda C_r p(0)\}-\frac{\mu_1 C_r}{x}F_1(x)\right].$$

Using (7) repeatedly on the right hand side of (7), we obtain for $r=1, 2, \cdots, j-1$

$$F_{r+1}(x)=F_1(x)\prod_{i=1}^{r} A_i+\sum_{i=1}^{r}\left[\prod_{s=i+1}^{r} A_s\left\{B_i'-\frac{\mu_1 C_i F_1(x)}{\mu_{i+1}x}\right\}\right] \tag{8}$$

where

$$B_i'=B_i+\frac{\mu_1 C_i F_1(x)}{x\mu_{i+1}}.$$

Putting $r=j-1$ in (8) and $r=j$ in (6) and equating the two values of $F_j(x)$ thus obtained, we get

$$F_1(x)=\frac{x\left[\lambda C_j p(0)-\lambda x^N p(N,j)+\{\lambda x-(\lambda+\mu_j)\}\sum_{i=1}^{j-1}\left\{B_i''\prod_{s=i+1}^{j-1}A_s\right\}\right]}{C\prod_{p=1}^{j}(x-x_p)} \tag{9}$$

where

$$B_i''=(x-1)B_i',$$

C is the coefficient of x^j in the denominator and $x_1, x_2, \cdots, x_j$ are the j roots of the denominator different from unit which has been cancell both from the numerator as also from the denominator.

Since now $F_1(x)$ is a polynomial in x, it is analytic. Therefore the numerator on the right hand side of (9) must cancel the j zeros of the denominator, which gives rise to j homogeneous equations involving $j+1$ unknown probabilities.

Also the normalizing equation (5) gives

$$F_1(1)\left[1+\mu_1\sum_{r=1}^{j-1}\frac{1-\sum_{r=1}^{j-1}C_i}{\mu_{r+1}}\right]+p(0)=1 \tag{10}$$

where $F_1(1)$ may be obtained from (9) by putting $x=1$.

Thus the j equations due to the analyticity of the right side of (9) and equation (10) are sufficient to determine the $j+1$ unknown probabilities involved in $F_1(x)$ given by (9). Once $F_1(x)$ is determined the other generating functions may be obtained from (8).

Infinite Waiting Space

In case an infinite waiting space is allowed, *i.e.* $N\to\infty$, then $p(N, r)\to 0$ for $r=1, 2, \cdots, j$ and $F_1(x)$ in (9) becomes an infinite series and should converge at least in the region $|x|<1$. But we can easily prove by using Rouche's theorem that the j roots of the denominator of the right hand side of (9) all lie outside the region $|x|<1$. Also the numerator contains only one unknown probability, *viz.*, $p(0)$, which may now be calculated from the normalizing condition (10). Thus, we have

$$p(0)=\left[1+\frac{\lambda C_j}{C\prod_{p=1}^{j}(1-x_p)}\left\{1+\mu_1\sum_{r=1}^{j-1}\frac{1-\sum_{i=1}^{r}C_i}{\mu_{r+1}}\right\}\right]^{-1}$$

and therefore $F_1(x)$ and hence $F_r(x)$ for $r=1, 2, \cdots, j$ are completely determined in this case.

In the next section we show how the procedure outlined can be worked out for the queuing system $M/E_2/1$.

The Queuing System $M/E_2/1$

Substitute

$$j=2, \quad C_r=\delta_{r2}$$

in the analysis of the previous section. Then

$$F_1(x) = \frac{x}{\lambda^2(x-x_1)(x-x_2)}[\lambda\mu_2 p(0)-\lambda\mu_2 x^N p(N, 2) + \lambda x^N p(N, 1)\{\lambda x-(\lambda+\mu_2)\}] \tag{11}$$

where

$$x_1, x_2=\frac{1}{2\lambda}[\lambda+\mu_1+\mu_2\pm\sqrt{(\lambda+\mu_1+\mu_2)^2-4\mu_1\mu_2}]:$$

The three equations for the determination of $p(N, 1)$, $p(N, 2)$ and $p(0)$ in this case are

$$[(\lambda+\mu_2)-\lambda x_1]\lambda x_1^N p(N, 1)-\lambda\mu_2 x_1^N p(N, 2)-\lambda\mu_2 p(0)=0 \qquad (12)$$

$$[(\lambda+\mu_2)-\lambda x_2]\lambda x_2^N p(N, 1)+\lambda\mu_2 x_2^N p(N, 2)-\lambda\mu_2 p(0)=0 \qquad (13)$$

$$F_1(1)=\frac{\mu_2}{\mu_1+\mu_2} \qquad (14)$$

Equations (12~14) on solution give

$$p(0)=\frac{\lambda(1-x_1)(1-x_2)x_1^N x_2^N(x_2-x_1)}{(\mu_1+\mu_2)[x_1^N x_2^N(x_2-x_1)-(x_1^N-x_2^N)-(x_2^{N+1}-x_1^{N+1})]} \qquad (15)$$

$$p(N, 1)=\frac{\mu_2(x_2^N-x_1^N)p(0)}{\lambda x_1^N x_2^N(x_2-x_1)} \qquad (16)$$

$$p(N, 2)=\frac{[(\lambda+\mu_2)(x_1^N-x_2^N)+\lambda(x_2^{N+1}-x_1^{N+1})]p(0)}{\lambda x_1^N x_2^N(x_2-x_1)} \qquad (17)$$

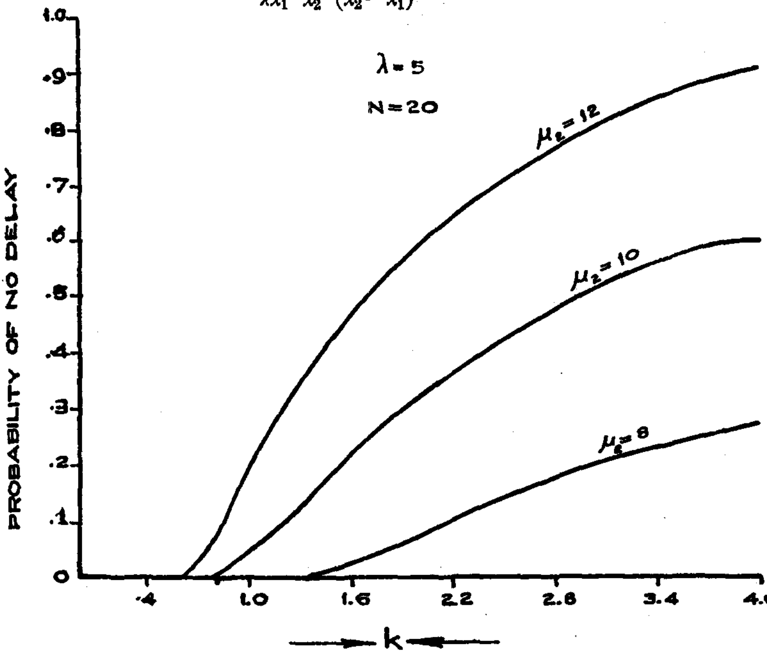

Fig. 1

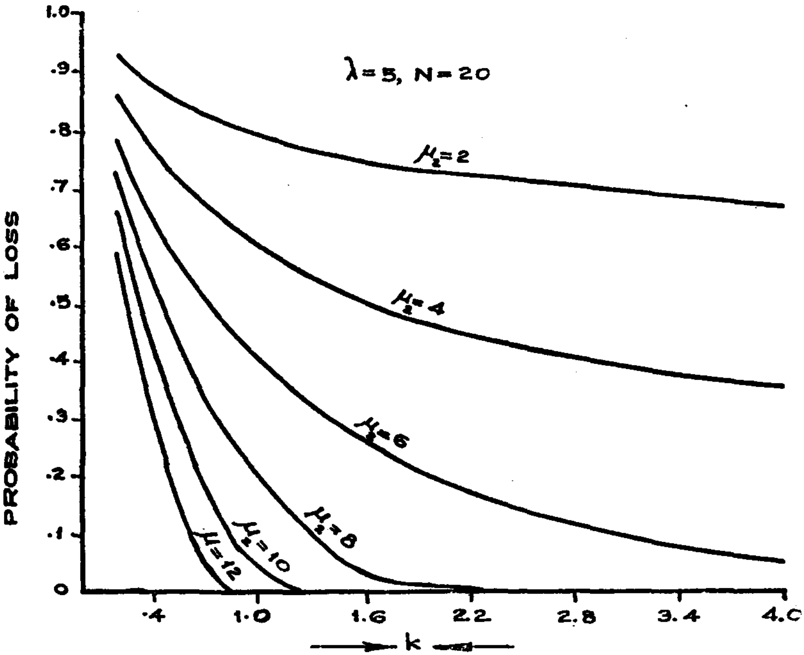

Fig. 2

Graphs are drawn for the following queue characteristics

(i) probability of no delay, *i. e.* $p(0)$, equation (15)

(ii) probability of loss, *i. e.* $p(N, 1)+p(N, 2)$, equations (16) and (17)

(iii) mean number, M, of units in the system by using the formula

$$M=\left\{\frac{d}{dx}[F_1(x)+F_2(x)]\right\}\bigg|_{x=1}$$

against the ratio $k=\mu_1/\mu_2$ of the two mean service rates. The results known earlier are just the points of intersections of these graphs with the line $k=1$. These graphs are drawn for $N=20$, $\lambda=5$, and various of μ_2 indicated in the graphs.

The numerical calculation also show that *the value of $p(0)$ does not change appreciably with N, whereas that of $p(N, 1)+p(N, 2)$ changes.*

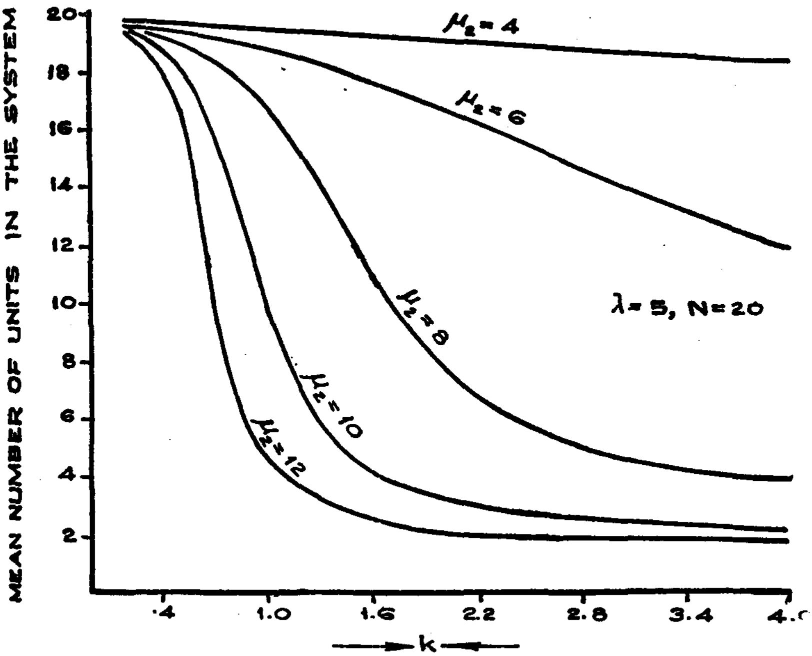

Fig. 3

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REFERENCES

1. Jain, H. C.: 'Queuing with Limited Waiting Space'. Nav. Res. Long. Quart. Vol. 9, Nos. 3 and 4, pp. 245—253, 1962.