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THE ASYMPTOTIC BEHAVIOR OF THE QUEUEING PROCESSES WITH A POISSON INPUT AND EXPONENTIALLY DISTRIBUTED SERVICE TIMES

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I. INTRODUCTION

In many applications of the queueing theory, we are concerned solely with the limiting or equilibrium probability distributions since the solution of non-equilibrium queue is far from convenient for its use. But we must evaluate the time required to approach very closely to the equilibrium state. If the time is not so long, some results concerning the steady state may practically give good approximations for non-equilibrium state.

Davis [1] tried to estimate the time using a notion of the build-up time T of waiting lines, and conjectured that it takes time $2T$ to $3T$ before the mean number in the system has reached a value within, say, 10 per cent of the steady state value.

Recently Morimura [2] proposed an analogous indicator representing the above time and discussed it in the case M/G/1.

Quite often, however, we want more explicit information about the behavior of the queue over fairly short time. This leads us to the asymptotic evaluation of time dependent solutions.

In this paper we shall study a simple queue in which the system is characterized by Poisson arrivals with constant mean rate, λ, exponential service times with common mean, $1/\mu$, and a finite number K of available channels.

2. THE ASYMPYOTIC FORMULAE FOR M/M/1

In 1952, A.B. Clark [3] studied the birth and death process to write down the solution for $P_{in}^{(t)}$, the probability that there are n items in the system at time t given that there were i waiting at time zero.

$$P_{in}^{(t)}=e^{-(\lambda+\mu)t}\Big[\Big(\sqrt{\frac{\mu}{\lambda}}\Big)^{i-n}I_{i-n}(2\sqrt{\lambda\mu}\,t)+\Big(\sqrt{\frac{\mu}{\lambda}}\Big)^{i-n+1}I_{i+n+1}(2\sqrt{\lambda\mu}\,t)+\Big(1-\frac{\lambda}{\mu}\Big)\Big(\frac{\lambda}{\mu}\Big)^{n}\sum_{k=i+n+2}^{\infty}\Big(\sqrt{\frac{\mu}{\lambda}}\Big)^{k}I_k(2\sqrt{\lambda\mu}\,t)\Big]. \tag{2.1}$$

where $I_n(x)\equiv i^{-n}J_n(ix)$ is the modified Bessel function of first kind. The series will now be examined. From standard textbooks

$$\exp\Big[\frac{1}{2}x\Big(y+\frac{1}{y}\Big)\Big]=\sum_{n=-\infty}^{\infty}y^nI_n(x).$$

We assume in what follows that the system is empty at time zero. Then

$$\frac{P_n(t)}{P_n(\infty)}=1+e^{-(1+\rho)\mu t}\Big[(1-\rho)\rho^{-1}\rho^{-\frac{n}{2}}I_n(2\sqrt{\rho}\,\mu t)+(1-\rho)^{-1}\rho^{-\frac{n+1}{2}}I_{n+1}(2\sqrt{\rho}\,\mu t)-\sum_{r=-1-n}^{\infty}\rho^{\frac{r}{2}}I_r(2\sqrt{\rho}\,\mu t). \tag{2.2}$$

where $\rho\equiv\lambda/k\mu$ is the traffic intensity. For notational convenience, we have just written $P_n(t)$ in place of $P_{0n}(t)$.

The rate of convergence of the above series depends on ρ and μt. Mathematical tables which has been published before, to our knowledge, are not adequate for the calculation of Eq. (2.2).

Using Meissel's formula we obtain for large order r

$$I_r(x)=\frac{(rx)^r\exp\{r\sqrt{1+x^2}\exp(-V_r)\}}{e^r\Gamma(r+1)(1+x^2)^{1/4}\{1+\sqrt{1+x^2}\}^r}, \tag{2.3}$$

where

$$V_r(x)=\frac{1}{24r}\left\{\frac{2-3x^2}{(1+x^2)^{3/2}}-2\right\}-\frac{-4x^2+x^4}{16r^2(1+x^2)^3}$$
$$-\frac{1}{5760r^3}\left\{\frac{16+1512x^2-3654x^4+375x^6}{(1+x^2)^{9/2}}-16\right\}$$
$$+\frac{32x^2-288x^4+232x^6-13x^8}{128r^4(1+x^2)^6}+\dots.$$

Furthermore the recurrence formula

(2.4) $$22r/xI_r(x)=I_{r-1}(x)-I_{r+1}(x)$$

provides a very useful method. For the calculation of infinite series in Eq. (2.2), the formula (2.3) has been used for large order n and we recurrently obtain the functions for other order from Eq. (2.4) (Fig. 1).

The invaluable works pertaining to the approximations of exponential like functions was done by C. Hastings [4]. He gave the useful approximations of the negative exponential functions e^{-x} by the forms $[1+a_1x+\dots..+a_px^p]^{-2\tau}$ over $(0, \infty)$—best in the sense of minimum absolute error. The parametric forms studied there are also very useful in approximating the functions given by Eq. (2.2) whose asymptotic behavior is like that of decaying exponential as $x\to\infty$.

To the moderate accuracy for the asymptotic behavior, the following approximation formulae is obtained as a result:

(2.5) $$\frac{P_n(t)}{P_n(\infty)}\sim 1-\frac{n-\sqrt{\rho}(n+1)}{a_3(1+\sqrt{\rho})\rho^{1/2(n+1/2)}(1+a_1y+a_2y^2)^8y^{3/2}},$$

in the case of $$\rho=\left(\frac{n}{n+1}\right)^2$$

(2.6) $$\frac{P_n(t)}{P_n(\infty)}\sim 1-\frac{\left(\frac{n+1}{n}\right)^{n+1/2}}{a^4y^{5/2}(1+a_1y+a_2y^2)^8},$$

where

$$y=(1-\sqrt{\rho})^2\mu t,$$

and $a_1=0.124,\quad a_2=0.00894,\quad a_3=7.09,\quad a_4=28.3$

Fig. 1. depicts these approximations in dashed lines.

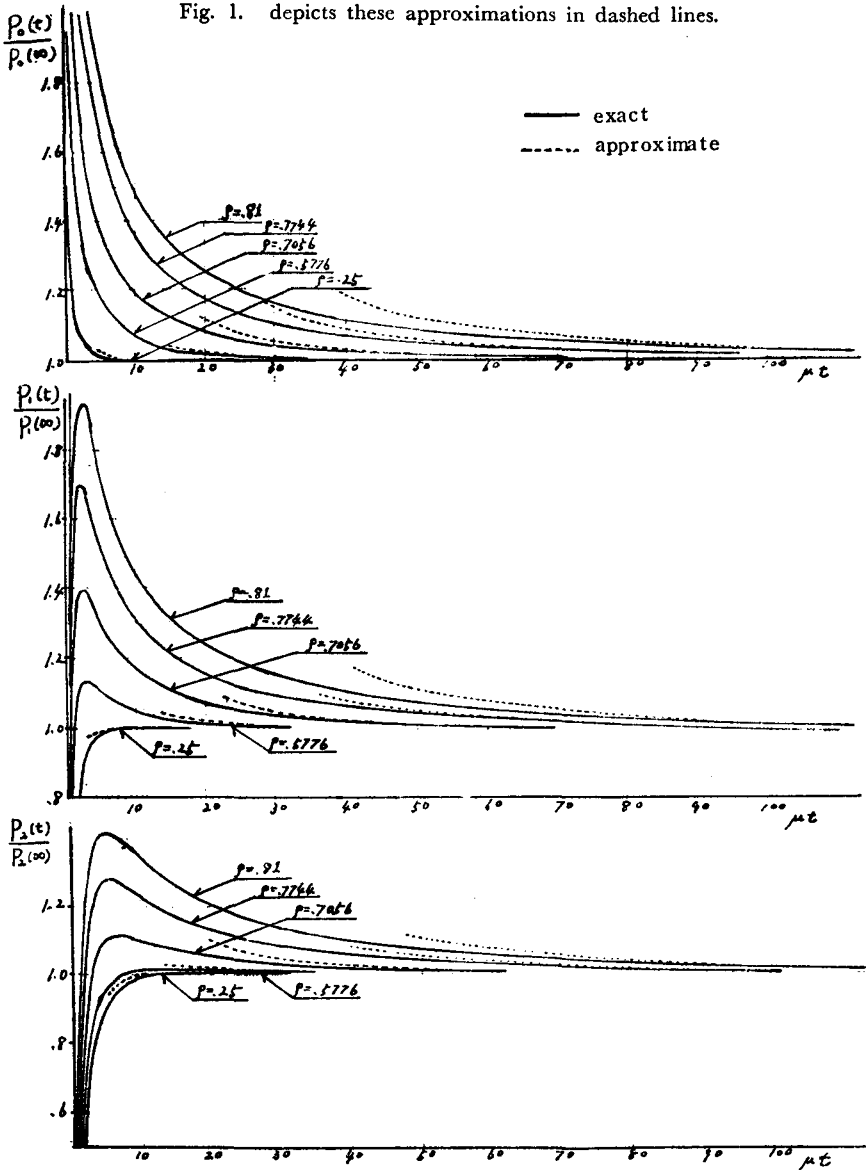

Fig. 1 Exact and approximate value of Pm(t) for n=0, 1, and 2

3. THE ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF M/M/K

3.1 A Probability Function Representing the Rate of Convergence

(a) THE CASE $K \geqq 2$

Let $Q_n(t)$ be the probability function defined by

$$Q(t)=P_n(t)/P_n(\infty)$$

The function $Q_n(t)$ can be interpreted as representing the rate of convergence.

The forward equations describing our problem are

$$(3.1.1) \qquad \frac{d}{dt}Q_0(t)=-\lambda Q_0(t)+\lambda Q_1(t)$$

$$(3.1.2) \qquad \frac{d}{dt}Q_n(t)=-(\lambda+n\mu)Q_n(t)+n\mu Q_{n-1}(t)+\lambda Q_{n+1}(t) \qquad (k>n\geq 1)$$

$$(3.1.3) \qquad \frac{d}{dt}Q_n(t)=-(\lambda+k\mu)Q_n(t)+k\mu Q_{n-1}(t)+\lambda\lambda Q_{n+1}(t) \qquad (k\leq n),$$

with initial conditions $Q_n(0)=\delta_{0n}/p_0$ (δ_{0n}: Kronecker symbol), where

$$p_0^{-1}=\sum_{n=0}^{k-1}\frac{(K\rho)^n}{n!}+\frac{(K\rho)^k}{K!(1-\rho)}$$

We introduce a function $Q(z,t)$ defined by

$$(3.1.4) \qquad Q(z,t)=\sum_{n=0}^{\infty}Q_n(t)z^n$$

and multiply Eqs. (3.1.2) and (3.1.3) by z^n and form the sum on the righthand sides over the appropriate ranges of n, including Eq. (3.1.1).

$$(3.1.5) \quad Z\frac{\partial}{\partial t}Q(z,t)=(\lambda-k\mu z)[(1-z)Q(z,t)-Q_0(t)]$$
$$+\sum_{n=1}^{k-1}(k-n)\mu z^{n+1}[Q_n(t)-Q_{n-1}(t)].$$

On taking Laplace-Stieltjes transform, in which typically $f^*(s)\equiv\int_0^\infty e^{-st}f(t)dt$, we now solve the equation with respect to $Q^*(z,s)$ and obtain

$$(3.1.6)\quad Q^*(z,s)=\frac{zp_0^{-1}-(\lambda-k\mu z)Q_0^*(s)+\sum_{n=1}^{k-1}(k-n)\mu z^{n+1}[Q_n^*(s)-Q_{n-1}^*(s)]}{sz-(\lambda-k\mu z)(1-z)}$$

Two zeros of denominator are

$$(3.1.7)\qquad \beta_i(s)=[\rho+1+s/k\mu\pm\sqrt{(\rho+1+s/k\mu)^2-4\rho}]/2 \qquad (i=1,2),$$

where

$$|\beta_1|>1 \quad \text{and } |\beta_2|<1,$$

so that

$$s=-k\mu(1-\beta_1)(1-\beta_2).$$

Now it follows from the definition of the Laplace-Stieltjes transform of that $Q^*(z,s)$ must converge everywhere within the unit circle $|z|=1$, provided $\mathrm{Re}(s)>0$. Thus in this region zeros of both numerator and denominator on the righthand side of Eq. (3.1.6) must coincide. Therefore,

$$(3.1.8)\qquad \beta_2 p_0^{-1}-(\lambda-k\mu\beta_2)Q_0^*(s)+\sum_{n=1}^{k-1}(k-n)\mu\beta_2^{n+1}[Q_n^*(s)-Q_{n-1}^*(s)]=0.$$

Laplace-Stieltjes transforms of Eqs. (3.1.1), (3.1.2), and (3.1.3) are

$$(3.1.9)\qquad \{(\lambda-k\mu\beta_2)(1-\beta_2)+\lambda\beta_2\}Q_0^*(s)-\lambda\beta_2 Q_1^*(s)={}_*\beta_2 p_0^{-1}$$

$$(3.1.10)\qquad \{(\lambda-k\mu\beta_2)(1-\beta_2)+(n\mu+\lambda)\beta_2\}Q_n^*(s)-n\mu\beta_2 Q_{n-1}(s)-\lambda\beta_2 Q_{n+1}^*(s)=0 \qquad (k>n\geq 1)$$

$$(3.1.11)\qquad \{\lambda+k\mu\beta_2^2\}Q_n^*(s)-k\mu\beta_2 Q_{n-1}^*(s)-\lambda\beta_2 Q_{n+1}^*(s)=0 \qquad (n\geq k),$$

From Eqs. (3.1.8), (3.1.9), and (3.1.10), $Q_n^*(s)$ $(n<K)$, meromorphic functions of β_2, are determined, and $Q_n^*(s)(n\geqq K)$ from Eq. (3.1.11).
Hence, $Q_n^*(s)$ has branch points at

$$s_0=-(\lambda+k\mu-2\sqrt{k\mu\lambda})$$

$$s_1=-(\lambda+k\mu+2\sqrt{k\mu\lambda})$$

The inversion of Q_n^* can now be carried out. The contour we choose is indicated in Fig. 2, where a branch cut has been made between branch points s_0, s_1, and $-\infty$.

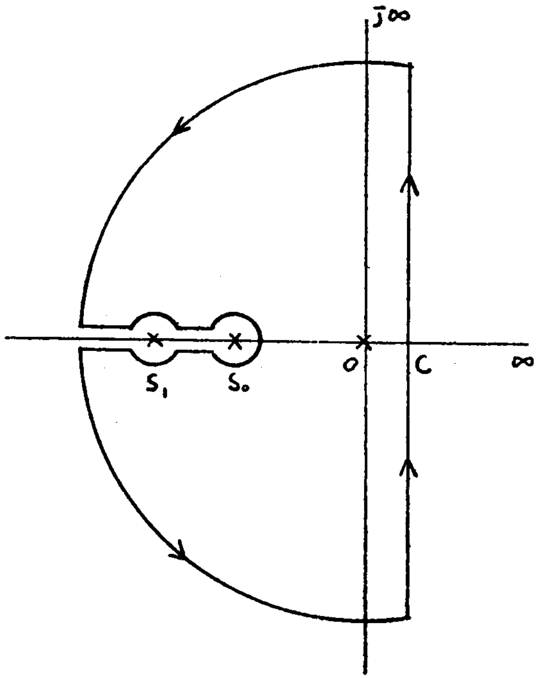

Fig. 2. Path of integration I

We assume that there is no pole, except $s=0$, in the sector $|\operatorname{arc}(s-s_0)| \leq \Psi$ ($\Psi=\frac{\pi}{2}+\varepsilon$; ε is arbitrarily small positive number (see Fig. 3)) Then we consider an integral along $\mathfrak{C}$, which consists of a small semicircle about s_0 and two lines $\operatorname{arc}(s-s_0)=\pm\Psi$. Taking account of the simple pole at $s=0$, we obtain

(3.1.12) $$Q_n(t)=1+\frac{1}{2\pi i}\int_{\mathfrak{C}} e^{ts}Q_n^*(s)ds.$$

(Lemma) [5]

Let $f(s)$ be an analytic function of s in the neighbourfood of a point $s=s_0$ in the sector $|\operatorname{arc}(s-s_0)| \leq \Psi$.

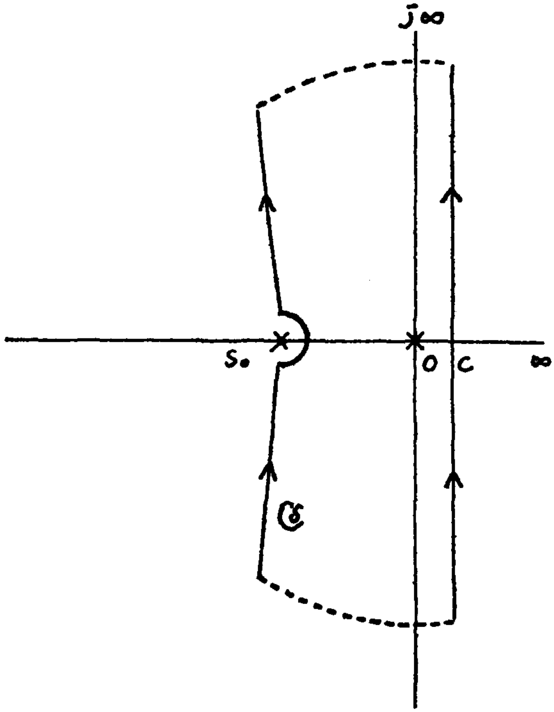

Fig. 3. Path of integration II

For real λ and arbitrary complex A, uniformly

$$f(s)\sim A(s-s_0)^{\lambda} \qquad (s\rightarrow s_0) \qquad (\lambda;\ \text{arbitrary real number}),$$

and, furthermore, along arc $(s-s_0)=\pm\Psi$

$$f(s)=O(e^{k|s|}), \qquad \text{for} \quad |s|\rightarrow\infty \qquad (k>0).$$

Then

$$F(t)=\frac{1}{2\pi i}\int_{\mathfrak{C}} e^{ts}f(s)\,ds\sim Ae^{s_0t}\frac{t^{-\lambda-1}}{\Gamma(-\lambda)} \qquad (t\rightarrow\infty)$$

From Eq. (3. 1. 12), we obtain

$$t[1-Q_n(t)]=\frac{1}{2\pi i}\int_{\mathfrak{C}} e^{ts}\frac{d\beta_2}{ds}\frac{dQ_n^*(s)}{d\beta_2}ds$$

Since $Q_n^*(s)$ is an analytic function of β_2 in the sector $|\operatorname{arc}(s-s_0)|\leq\Psi$ we can deduce

(3.1.13) $$\lim_{s\to s_0}\frac{dQ_n^*(s)}{d\beta_2}\lim_{\beta_2\to\sqrt{\rho}}\frac{dQ_n^*(s)}{d\beta_2}=A_n,$$

where A_n is real constant. From Eq. (3.1.7), we easily find that uniformly

(3.1.14) $$\frac{dQ_n^*(s)}{ds}\sim\frac{(k\mu\lambda)^{1/4}A_n}{-2k\mu(s-s_0)}\qquad(s\to s_0),$$

and along the large arcs of Fig. 3

$$\left|\frac{dQ_n^*(s)}{ds}\right|\to 0,\qquad(|s|\to\infty).$$

Then our function $\frac{dQ_n^*(s)}{ds}$ satisfies all conditions of the above lemma, which gives

(3.1.15) $$Q_n(t)\sim 1+\frac{A_n(k\mu\lambda)^{1/4}}{2k\mu\Gamma(1/2)}\cdot\frac{e^{s_0t}}{t^{3/2}}.$$

Observing that

$$s\beta_2=(\lambda-k\mu\beta_2)(1-\beta_2),$$

we obtain from Eq. (3.1.8)

(3.1.16) $$Q_0^*(s)=\frac{1-\beta_2}{s(1-\rho)}+\frac{1-\beta_2}{s}\left[p_0^{-1}-\frac{1}{1-\rho}+\sum_{n=1}^{k-1}(k-n)\mu\beta_2^n a_n^*(s)\right],$$

where

$$a_n^*(s)=Q_n^*(s)-Q_{n-1}^*(s)$$

$$\lim_{s\to 0}a_n^*(s)=a_n.$$

We multiply the righthand side of Eq. (3.1.16) by s, and find that

(3.1.17) $$\lim_{s\to 0}s\cdot\frac{1-\beta_2}{s(1-\rho)}=1.$$

(3. 1. 18)
$$\lim_{s\to 0} s\cdot\frac{1-\beta_2}{s}\left[p_0^{-1}-\frac{1}{1-\rho}+\sum_{n=1}^{k-1}(k-n)\mu\beta_2^n a_n^*(s)\right]$$
$$=(1-\rho)\left[p_0^{-1}-\frac{1}{1-\rho}+\sum_{n=1}^{k-1}(k-n)\mu\rho^n a_n\right].$$

On taking Laplace-Stieltjes transform in Eqs. (3. 1. 1) and (3. 1. 2),

(3. 1. 19) $sQ_0^*-p_0^{-1}=\lambda a_1^*(s)$

(3. 1. 20) $sQ_0^*==\lambda a_{n+1}^*(s)-n\mu a_n^*(s) \qquad (k>n\geq 1)$,

which give as $s\to 0$

(3. 1. 21) $1-p_0^{-1}=\lambda a_1$.

(3. 1. 22) $1=\lambda a_{n+1}-n\mu a_n \qquad (k>n\geq 1)$.

From these equations, we can obtain

(3. 1. 23)
$$a_n=\frac{(n-1)!}{\lambda(k\rho)^{n-1}}\cdot\left\{\sum_{m=1}^{n-1}\frac{1}{m!}(k\rho)^m+1-p_0^{-1}\right\},$$

and incidentally

$$\sum_{n=1}^{k-1}(k-n)\mu\rho^n a_n=\sum_{n=1}^{k-1}\left[\frac{(n-1)!}{k^{n-1}}-\frac{n!}{k^n}\right]\cdot\left[\sum_{m=1}^{n-1}\frac{(k\rho)^m}{m!}+1-p_0^{-1}\right]$$
$$=\sum_{m=1}^{k-2}\frac{(k\rho)^m}{m!}\sum_{n=m+1}^{k-1}\left[\frac{(n-1)!}{k^{n-1}}-\frac{n!}{k^n}\right]+(1-p_0^{-1})\sum_{n=1}^{k-1}\left[\frac{(n-1)!}{k^{n-1}}-\frac{n!}{k^n}\right]$$
$$=\frac{1}{1-\rho}-p_0^{-1}.$$

Then it follows that the righthand side of Eq. (3. 1. 8) vanishes and

(3. 1. 24)
$$Q_0^*(s)=\frac{\beta_2}{(\lambda-k\mu\beta_2)(1-\rho)}+o\left(\frac{1}{s}\right) \quad (s\to 0+),$$

where the formula

$$f(s)=o\left(\frac{1}{s}\right) \qquad (s\to 0+)$$

means that

$$sf(s)\to 0 \qquad (s\to 0+).$$

Substituting the above result into Eq. (3. 1. 6), we get

$$Q^*(z, s)=\frac{\lambda}{-k\mu(\lambda-k\mu\beta_2)(z-\beta_1)(1-\rho)}$$
$$+\frac{(\lambda-k\mu z)o\left(\frac{1}{s}\right)-\sum_{n=1}^{k-1}(k-n)\mu z^{n+1}[Q_n^*(s)-Q_{n-1}^*(s)]-z\left(p_0^{-1}-\frac{1}{1-\rho}\right)}{k\mu(z-\beta_1)(z-\beta_2)}.$$

Observing that

$$\lim_{s\to 0} s\sum_{n=1}^{k-1}(k-n)\mu z^{n+1}[Q_n^*(s)-Q_{n-1}^*(s)]$$
$$=\lim_{t\to\infty}\sum_{n=1}^{k-1}(k-n)\mu z^{n+1}[Q_n(t)-Q_{n-1}(t)]$$
$$=0,$$

we find

$$(3.1.25)\qquad Q^*(z, s)=\frac{-\lambda}{k\mu(\lambda-k\mu\beta_2)(z-\beta_1)(1-\rho)}+o\left(\frac{1}{s}\right)\qquad (s\to 0+)$$

so that

$$(3.1.26)\qquad Q_n^*(s)=\frac{\beta_2^{n+1}}{k\mu(1-\rho)\rho^n(\rho-\beta_2)}+o\left(\frac{1}{s}\right)\qquad (s\to 0+).$$

We represent by $\overline{Q}_n^*(s)$ the first term of the righthand side of Eq. (3.1.26) and observe that $\overline{Q}_n^*(s)$ has no singularities except the pole at $s=0$ and the branch points at s_0, s_1, Then we can shift the contour from the line $R_e(s)=C$ to $\mathfrak{C}$.
Therefore,

$$\overline{Q}_n(t)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\overline{Q}_n^*(s)e^{st}ds=1+\frac{1}{2\pi i}\int_{\mathfrak{C}}\overline{Q}_n^*(s)e^{st}ds.$$

It is easily seen that uniformly

$$\frac{d\overline{Q}_n^*(s)}{ds}\sim\frac{(k\mu\lambda)^{1/4}\{(n+1)\sqrt{\rho}-n\}}{-2(k\mu)^2(1-\rho)\rho^{\frac{n+1}{2}}(1-\sqrt{\rho})^2(s-s_0)^{1/2}}\quad (s\to s_0),$$

and so $\frac{d\overline{Q}_n^*(s)}{ds}$ satisfies the condition of the lemma. Thus,

$$(3.1.27)\qquad \overline{Q}_n(t)\sim 1+\frac{(k\mu\lambda)^{1/4}\{(n+1)\sqrt{\rho}-n\}}{2(k\mu)^2(1-\rho)\rho^{\frac{n+1}{2}}(1-\sqrt{\rho})^2\Gamma(1/2)}\cdot\frac{e^{s_0t}}{t^{3/2}}\quad (t\to\infty).$$

(b) THE CASE $K=1$

For $K=1$, a similar process leads us to the following analogous sets of equations

$$(3.1.28)\qquad \frac{d}{dt}Q_0(t)=-\lambda Q_0(t)+\lambda Q_1(t)$$

$$(3.1.29)\qquad \frac{d}{dt}Q_n(t)=-(\lambda+\mu)Q_n(t)+\mu Q_{n-1}(t)+\lambda Q_{n+1}(t)\quad (n\geq 1).$$

Furthermore, there is the same initial conditions

$$Q_n(0)=\delta_{0n}/p_0,$$

where

$$p_0=1-\rho.$$

The generating function $Q(z,t)=\sum_{n=0}^{\infty} Q_n(t)z^n$ must satisfy the following partial differential equation

$$(3.1.30)\qquad z\frac{\partial}{\partial t}Q(z,t)=(\lambda-z)[(1-z)Q(z,t)-Q_0(t)].$$

Therefore, $Q^*(z,s)$, the Laplace-Stieltjes transform of $Q(z,t)$, is given by

$$(3.1.31)\qquad Q^*(z,s)=\frac{zp_0^{-1}-(\lambda-\mu z)Q_0^*(z)}{sz-(1-z)(\lambda-\mu z)}$$

Let, the zeros of the denominator be

$$\beta_i(s)=[\rho+1+s/\mu\pm\sqrt{(\rho+1+s/\mu)^2-4\rho}]/2\qquad (i=1,2\ |\beta_1|>|\beta_2|),$$

so that

$$|\beta_1|>1,\qquad |\beta_2|<1.$$

Thus we have

$$\beta_2p_0^{-1}-(\lambda-\mu\beta_2)Q_0^*(s)=0,$$

which gives

$$(3.1.32)\qquad Q_0^*(s)=\frac{\beta_2p_0^{-1}}{\lambda-\mu\beta_2}.$$

If we substitute this into the $Q^*(z,s)$ formula, we find

(3. 1. 33) $$Q^*(z,\ s)=\frac{\lambda}{-\mu(\lambda-\mu\beta_2)(z-\beta_1)(1-\rho)},$$

so that

(3. 1. 34) $$Q_n^*(s)=\frac{\beta_2^{n+1}}{\mu(1-\rho)\rho^n(\rho-\beta_2)}.$$

From this after, in order to make distinction between the cases $K\geqq 2$ and $K=1$, we use superscript K to indicate the quantities relevant to K servers.

3. 2. THE BEST APPROXIMATIONS IN THE SENSE OF MINIMUM ABSOLUTE ERROR

When we change K, with ρ held constant, we have from Eqs. (3. 1. 26) and (3. 1. 34)

(8. 2. 1) $$Q_n^{k*}(s)=\frac{1}{k}Q_n^{1*}\left(\frac{s}{k}\right)+o\left(\frac{1}{s}\right).$$

The inverse transform of Eq. (3. 2. 1) is

(3. 2. 2) $$Q_n^k(t)=Q_n^1(kt)+\frac{1}{2\pi i}\int e^{ts}o\left(\frac{1}{s}\right)ds,$$

which, observing Eq. (3. 1. 15) and Eq. (3. 1. 27), gives the following approximation of moderate accuracy in the sense of minimum absolute error:

(3. 2. 3) $$Q_n^k(t)\approx Q_n^1(kt) \qquad (t\to\infty),$$

where the formula

$$f(t)\approx g(t)$$

means that the first term of the asymptotic expansion of the function $f(t)$ coincides with that of the function $g(t)$, up to a factor not containing t. These approximations are depicted and compared with exact solutions in

It is a great pleasure to acknowledge our thanks to colleagues of the Udagawa laboratory for the discussion on this subject.

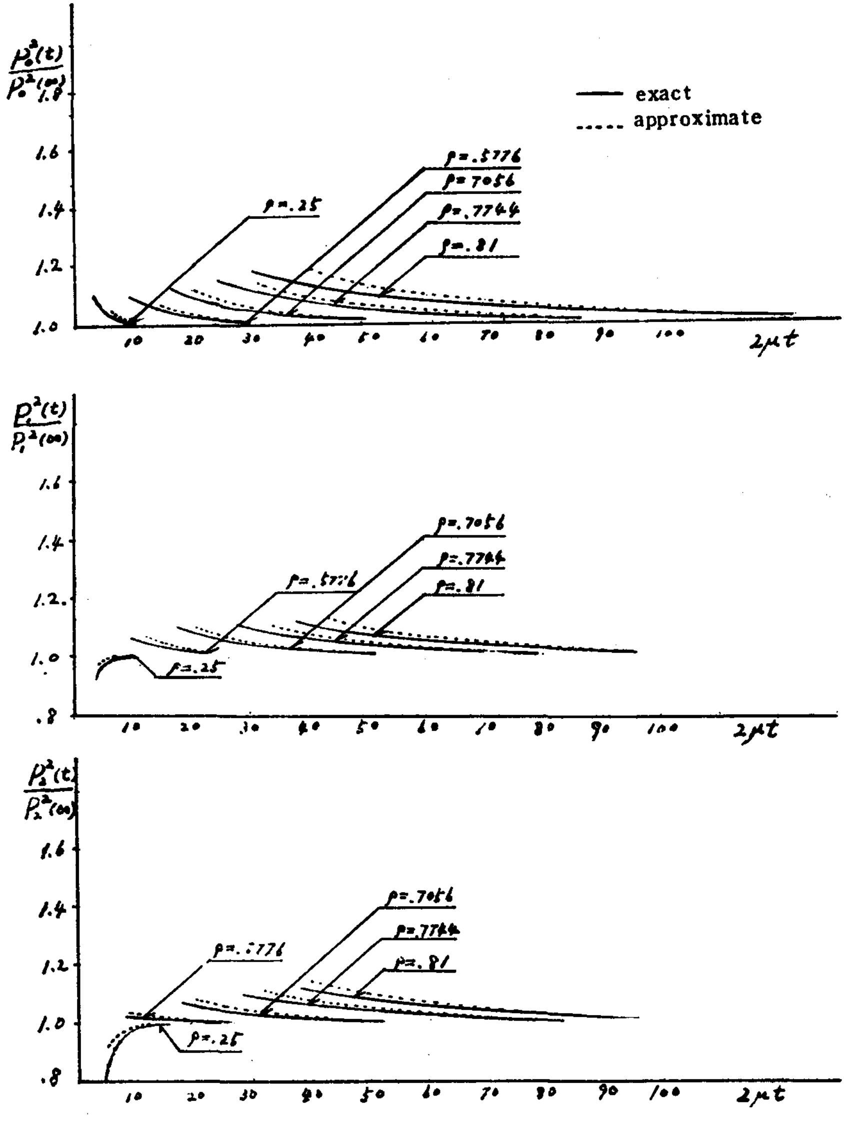

Fig. 4 Exact and approximate values of Pn(t) for K=2 and n=0, 1, and 2

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