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ON A QUEUING SYSTEM WITH EXTRA SERVICE

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I. INTRODUCTION

To describe the queuing system considering in this paper, we shall state a practical model comming about in an iron-manufacturing company.

We consider the case where ingots of a high temparature wait in a line to be produced by a process. Each ingot is produced in the order in which it arrives—first come, first served.

We assume that the temparature of a waiting ingot dropps in proportion to its waiting time. If the productional machine requires the high temparature of ingot before its production, its temparature falling in the waiting duration must be risen.

Thus ingots in line are produced after having spent time in proportion to the waiting time.

In this paper we shall consider the queuing system modified the above model. Recently, such a queuing system called the system with extra service was treated by F. D. Finch and F. Pollaczek. Finch [1] considered the system a customer who arrives to find the server idle does not commence service immediately on arrival but must wait a time before he commences service. Pollaczek [2] considered the system a customer who arrives to find the server busy does not commence service immediately on departure of the preceding customer but must wait a time before he commences service. Also in a tandem queue with blocking, a blocking time is considered as an extra time for the next service.

We discuss the following system. Customers arrive at the single counter at the instants $\tau_1, \tau_2, \cdots, \tau_n, \cdots$ where $\{\tau_n\}$ is a renewal process, that is the $g_n=\tau_{n+1}-\tau_n$ ($n\geqq 0$, $\tau_0=0$) are independently and identically distributed nonnegative random variables with common distribution function $A(x)$ andfinite expectation $\lambda^{-1}=\int_0^\infty x\,dA(x)$. Let the service time of the n-th customer (that is the customer arriving at τ_n) be R_n where $\{R_n\}$ $(n=1, 2, \cdots)$ is a sequence of identically and independently distributed non-negative random variables with common distribution $B(x)$ and finite expectation $\mu^{-1}=\int_0^\infty xdB(x)$. We suppose also that the sequence $\{R_n\}$ is independent of the input process $\{\tau_n\}$. Let customers be served in the order of their arrivals and let customers queue for service no customer departing until he has received service. If the n-th customer arrives to find the server busy, then he commences service after having spent a time in proportion to his waiting time, if he arrives to find the server idle, he commences service at once.

Let W_n be the time of completion of service for the $(n-1)$-th customer minus the time of arrival of the n-th customer, then

$$(1)\qquad W_{n+1}=\begin{cases} c\,W_n+R_n-g_n & (W_n\geqq 0)\\ R_n-g_n & (W_n<0)\end{cases}$$

woere c is a constant larger than 1.

The random variables $T_n=R_n-g_n$ $(n=1, 2, \cdots)$ are independently and identically distributed with common distribution function given by

$$(2)\qquad T(x)=\int_{-\infty}^\infty B(y+x)dA(y)$$

and finite expectation

$$\int_{-\infty}^\infty xdT(x)=\frac{1}{\mu}-\frac{1}{\lambda}.$$

We use the notation $a\vee b$ instead of max (a, b), then $W_n\vee 0$ is the waiting time for the n-th customer.

As a mathematical interest, c in (1) is taken as non-negative constant.

Our queuing system is a generalization of the system GI/G/1 studied by Lindley [3], which is the particular case $c=1$ in our case.

In the next section we provide criteria for the ergodicity of the process $\{W_n\}$.

Finally the author wishes to express his sincers thanke to Dr. H. Morimura for his valuable advice.

II. THE ERGODICITY OF THE SYSTEM

First, we assume $W_1 \leqq 0$, then (1) give

$$
\begin{aligned}
(3) \qquad W_{n+1} &= 0 \vee cW_n + T_n \\
&= T_n \vee [T_n + cW_n] \\
&= T_n \vee [T_n + c\{T_{n-1} \vee (T_{n-1} + cW_{n-1})\}] \\
&= T_n \vee [T_n + cT_{n-1}] \vee [T_n + cT_{n-1} + c^2 W_{n-1}] \\
&= \cdots\cdots \\
&= T_n \vee [T_n + cT_{n-1}] \vee [T_n + cT_{n-1} + c^2 T_{n-2}] \vee \cdots\cdots \\
&\quad \vee [T_n + cT_{n-1} + \cdots + c^{n-1} T_1] .
\end{aligned}
$$

$\{W_n, n \geqq 1\}$ is a Markov process with stationary transition probabilities. This fact will enable us to prove Lemma 2 below.

Lemma 1. $P(W_n \leqq t | W_1 = x) \leqq P(W_n \leqq t | W_1 = y)$
$= P(W_n \leqq t | W_1 = 0)$ for all t, $x \geqq 0$, $y < 0$ and $n \geqq 2$.

Proof: Fix a point ω in the sample space of $R_1, R_2, \cdots, R_n, g_1, \cdots, g_n$ and let

$$W_1(\omega, x) = x ;$$
$$W_j(\omega, x) = T_{j-1}(\omega) \vee [T_{j-1}(\omega) + cW_{j-1}(\omega, x)]$$

for $2 \leqq j \leqq n$. It is clear that if $x \geqq 0$ and $y < 0$, then

$$W_j(\omega, 0) = W_j(\omega, y) \leqq W_j(\omega, x) \quad \text{for each } j \geqq 2.$$

Hence the lemma is now seen to be true.

Lemma 2. $P(W_n \leqq t | W_1 = 0) \to F_0(t) \qquad (n \to \infty)$

where F_0 is a monotone non-decreasing function.

Proof: Let $H(x) = P(W_2 \leqq x | W_1 = 0)$. Then

$$(4) \qquad P(W_{n+1} \leqq t \mid W_1 = 0) = \int P(W_{n+1} \leqq t \mid W_1 = 0,\ W_2 = x) dH(x).$$

Since $\{W_n\}$ is a stationary Markov process and because of Lemma 1,

$$(5) \qquad P(W_{n+1} \leqq t \mid W_1 = 0,\ W_2 = x) = P(W_n \leqq t \mid W_1 = x)$$
$$\leqq P(W_n \leqq t \mid W_1 = 0) \qquad \text{for all } t.$$

Then, inserting (5) into (4) we have

$$P(W_{n+1} \leqq t \mid W_1 = 0) \leqq \int P(W_n \leqq t \mid W_1 = 0) dH(x)$$
$$= P(W_n \leqq t \mid W_1 = 0)$$

Thus $P(W_n \leqq t \mid W_1 = 0)$ is a monotone sequence and therefore converge to a limit which is called $F_0(t)$. The above-mentioned properties of F_0 are easily deduced.

Note. It is evident that for $y < 0$

$$P(W_n \leqq t \mid W_1 = y) \to F_0(t) \quad (n \to \infty).$$

Theorem 1. The function $F_0(t)$ defined in Lemma 2 is a probability distribution if any one of the following conditions hold:

(i) $T_1 = R_1 - g_1 \leqq 0$ with probability one,

(ii) $c = 0$,

(iii) $0 < c \leqq 1$ and $E(T_1) < 0$,

(iv) $0 < c < 1$, $E(T_1) \geqq 0$ and $\sigma^2(T_1) < \infty$.

Proof: The cases (i) and (ii) are trivial and then $F_0(t) = T(t)$. We consider the ca e (iii). The case where $c = 1$ and $E(T_1) < 0$ is well known as the system GI/G/1, which considered by Lindley [3].

If $0 < c < 1$, then from (3)

$$W_{n+1} = T_n \vee [cW_n + T_n]$$
$$\leqq T_n \vee [W_n + T_n].$$

Now we define the sequence $\{W_n^*\}$ by

(6) $$W_1^*=0,$$

and

(7) $$W_{n+1}^*=T_n\vee[W_n^*+T_n] \qquad (n=1,\ 2,\ \cdots),$$

then we have

$$W_n\leqq W_n^* \qquad (n=1,\ 2,\ \cdots).$$

According to the above inequality and Lindley's result [3] on the process $\{W_n^*\}$ with $E(T_1)<0$, we see our argument to be true.

Next, we consider the case (iv). We put in (3)

$$X_i=c^{i-1}T_{n-i+1} \qquad (i=1,\ 2,\ \cdots,\ n),$$

and

$$U_k=\sum_{i=1}^{k}X_i \qquad (k=1,\ 2,\ \cdots,\ n).$$

If $E(T_1)=0$, Kolmogorov's inequality holds;

(8) $$P(\max_{1\leqq k\leqq n}|U_k|\geqq\lambda)\leqq\frac{1}{\lambda^2}\sum_{k=1}^{n}\sigma^2(X_k) \qquad \text{for every} \quad \lambda>0,$$

since $\{X_i\}$ is a sequence of mutually independent random variables with $E(X_i)=0$ and $\sigma^2(X_i)<\infty$.

Hence (8) gives

$$P(\max_{1\leqq k\leqq n}|U_k|\geqq\lambda)\leqq\frac{\sigma^2(T_1)}{\lambda^2(1-c^2)}\equiv\varepsilon(\lambda) \qquad \text{(say)},$$

from which we have

(9) $$P(W_n\geqq t\,|\,W_1=0)\leqq P(\max_{1\leqq k\leqq n}|U_k|\geqq t)\leqq\varepsilon(t)$$

where $\varepsilon(t)\to 0$ as $t\to\infty$. That is, $\{W_n\}$ is bounded in probability. If $E(T_1)>0$, we put $X_i=c^{i-1}[T_{n-i+1}-E(T_{n-i+1})]$ and use the same way as above. Thus

$$P(W_n\geqq t+\frac{E(T_1)}{1-c}\,|\,W_1=0)$$

$$\leqq P(\max_{1\leqq k\leqq n}|U_k-E(U_k)|\geqq t)\leqq\frac{\sigma^2(T_1)}{t^2(1-c^2)}=\varepsilon(t),$$

that is,

$$P(W_n\geqq t|W_1=0)\leqq\varepsilon'(t)$$

where

$$\varepsilon'(t)\to 0 \quad \text{as} \quad t\to\infty.$$

Therefore the theorem is seen to be true.

Therem 2.

(i) If $c\geqq 1$ and $E(T_1)\geqq 0$, then $F_0(t)\equiv 0$ where the case $T_1=0$ with probability 1 is excluded.

(ii) If $c>1$ and $E(T_1)<0$, then $F_0(t)$ is not a probability distribution where the case $T_1\leqq 0$ with probability 1 is excluded.

Proof: First, we consider the case (i). From (3)

$$W_{n+1}=T_n\vee[cW_n+T_n]\geqq T_n\vee[W_n+T_n].$$

We define the sequence $\{W_n^*\}$ by (6) and (7), then

$$W_n\geqq W_n^* \qquad (n=1, 2, \cdots).$$

Applying the Lindley's result [3] to the process $\{W_n^*\}$ with $E(T_1)\geqq 0$, we see that $W_n\to+\infty$ in probability which proves that $F_0(t)\equiv 0$.

Also we can prove the fact by another way as follow:

We define W_n^{**} by

$$W_n^{**}=0\vee cW_n, \qquad (n=1, 2, \cdots).$$

Then

$$\begin{aligned}W_{n+1}^{**}&=0\vee cW_{n+1}=0\vee c[0\vee cW_n+T_n]\\&=0\vee c[W_n^{**}+T_n]\\&\geqq 0\vee[cT_n+W_n^{**}]\\&\geqq 0\vee[cT_n+0\vee(cT_{n-1}+W_{n-1}^{**})]\end{aligned}$$

$$=0\vee[cT_n]\vee[cT_n+cT_{n-1}+W^{**}_{n-1}]$$
$$=\cdots\cdots$$
$$\geqq c\{0\vee T_n\vee[T_n+T_{n-1}]\vee\cdots\vee[T_n+T_{n-1}+\cdots T_1]\}$$
$$=cW^{(L)}_{n+1},$$

where $W^{(L)}_{n+1}=0\vee W^{*}_{n+1}$ is the waiting time defined in Lindley's paper [3]. Since $W^{(L)}_n\to+\infty$ is probability under the conditions of the case (i) we see that W^{**}_n and W_n are also.

Next, we consider the case (ii). We put

(10) $$S_n=T_1+c^{-1}T_2+\cdots+c^{-n+1}T_n \quad (n=1, 2, \cdots),$$

then the process $\{-S_n\}$ forms an upper semi-martingale. Since

$$E|S_n|\leqq\frac{c}{c-1}E|T_1| \quad \text{for all } n,$$

according to the theorem 4. 1 *s*, p. 324 in [4], $\lim_{n\to\infty}S_n=S_\infty$ exists with probability one. We assume $P(T_1>0)>0$. Then there is a sufficient small positive number t_0 such that $P(T_1>t_0)>0$. Therefore for all $N\geqq1$,

(11) $$P(S_\infty>0)\geqq P(T_1>t_0,\ T_2>t_0,\ \cdots,\ T_N>t_0,$$
$$c^{-N}T_{N+1}+c^{-N-1}T_{N+2}+\cdots>-t_0(1+c^{-1}+\cdots+c^{-N+1}))$$
$$=P(T_1>t_0)^N\cdot P(c^{-N}T_{N+1}+c^{-N-1}T_{N+2}+\cdots>-t_0K+\varepsilon_N)$$

where $K=\frac{c}{c-1}$ and $\varepsilon_N=t_0Kc^{-N}\to0$ as $N\to\infty$.

We find an integer N such that

(12) $$\sum_{n=0}^{\infty}c^{-N-n}(N+n+1)^2\leqq D\,c^{-N}(N+1)^2<t_0K-\varepsilon_N$$

where D is a constant.

On the other hand, according to the Markov's inequality (see for example Loève [5], p. 158) we have

$$P(T_{N+n} \leqq -(N+n)^2) \leqq \frac{E|T_1|}{(N+n)^2} \quad (n=1, 2, \cdots).$$

Then if we take N statisfying the inequality (12),

$$\begin{aligned}(13) \qquad & P(c^{-N}T_{N+1}+c^{-N-1}T_{N+2}+\cdots>-t_0K+\varepsilon_N) \\ & \geqq P(T_{N+1}>-(N+1)^2,\ T_{N+2}>-(N+2)^2,\ \cdots) \\ & \geqq \prod_{n=0}^{\infty}\left(1-\frac{E|T_1|}{(N+n)^2}\right) \\ & >0,\end{aligned}$$

since $\sum_{n=1}^{\infty}(N+n)^{-2}<\infty$. By (11) and (13) we see that if $P(T_1>0)>0$, then $P(S_\infty>0)=\alpha>0$.

For any t and $\varepsilon>0$,

$$\begin{aligned} F_0(t) &= \lim_{n\to\infty} P(W_n \leqq t | W_1=0) \\ &\leqq \limsup_{n\to\infty} P(W_n \leqq c^{n-2}\varepsilon | W_1=0) \\ &\leqq \limsup_{n\to\infty} P(S_n \leqq \varepsilon | W_1=0) \\ &= P(S_\infty \leqq \varepsilon).\end{aligned}$$

Therefore

$$F_0(t) \leqq P(S_\infty \leqq 0) < 1-\alpha$$

that is, $F_0(t)$ is not a probability distribution.

Now we shall give two examples for which $P(S_\infty>0)=\alpha$ is evaluated.

Example 1. Let T_1 be a random variable defined by

$$T_1=\begin{cases} 1 & \text{with probability } p \\ -1 & \text{with probability } 1-p, \end{cases}$$

where $p<\frac{1}{2}$. Then $E(T_1)<0$. If $c>1$,

$$\begin{aligned} P(S_\infty>0) &\geqq P(T_1=1, \cdots, T_N=1) \\ &= P(T_1=1)^N = p^N \quad \text{for} \quad N>\log 2/\log c.\end{aligned}$$

Example 2. Let T_1 be a random variable obeying a normal probability law with parameters m and σ. Let $m<0$ and $c>1$. Then the characteristic function of T_1 is

$$\varphi(z)=\exp\left[izm-\frac{1}{2}(\sigma z)^2\right].$$

Therefore the characteristic function of S_∞ is

$$\psi(z)=\exp\left[izmK-\frac{1}{2}(\sigma Hz)^2\right],$$

where $K=\dfrac{c}{c-1}$ and $H^2=\dfrac{c^2}{c^2-1}$.

That is,

$$P(S_\infty>0)=\frac{1}{\sqrt{2\pi}\,\sigma H}\int_0^\infty e^{-\frac{1}{2}\left(\frac{y-mK}{\sigma H}\right)^2}dy$$
$$=1-\Phi\left(-\frac{mK}{\sigma H}\right),$$

where
$$\Phi(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}e^{-\frac{1}{2}y^2}\,dy.$$

Finally, we shall show that the limiting distribution, if it exists, is independent of the starting point W_1.

The cases where $T_1=0$ with probability one or $c=0$ are trivial and then

$$F_w(t)=F_0(t)=\begin{cases}1 & (t\geqq 0)\\ 0 & (t<0)\end{cases}$$

for any value w of W_1, where $F_w(t)=\lim_{n\to\infty}P(W_n\leqq t|W_1=w)$.

In what follows we exclude the case where $T_1=0$ with probability one.

If $0<c<1$, $E(T_1)<0$ and $W_1=w\leqq 0$, then $F_w(t)=F_0(t)$ because Lemma 1.

If $0<c<1$, $E(T_1)<0$ and $W_1=w>0$, $F_w(t)=F_0(t)$ because

$$\begin{aligned}P(W_{n+1}\leqq t|W_1=w)&=P(U_k\leqq t\ \text{ for }\ 1\leqq k<n,\ U_n\leqq t-c^n w)\\&\geqq P(U_k\leqq t\ \text{ for }\ 1\leqq k<n,\ U_n\leqq t-\varepsilon)\\&\geqq P(U_k\leqq t-\varepsilon\ \text{ for }\ 1\leqq k\leqq n)\\&\geqq F_0(t-\varepsilon)\end{aligned}$$

for any small number $\varepsilon>0$ and all $n\geqq\log(\varepsilon/w)/\log c$ where $U_k=\sum_{i=1}^{k} c^{i-1}$. T_{n-i+1}, that is,

$$\liminf_{n\to\infty} P(W_{n+1}\leqq t|W_1=w)\geqq F_0(t-\varepsilon).$$

Also $$P(W_{n+1}\leqq t|W_1=w)\leqq P(U_k\leqq t \text{ for } 1\leqq k\leqq n),$$

that is,

$$\limsup_{n\to\infty} P(W_{n+1}\leqq t|W_1=w)\leqq F_0(t).$$

Therefore

$$F_w(t)=F_0(t).$$

If $0<c<1$, $E(T_1)\geqq 0$ and $\sigma^2(T_1)<\infty$, then for any value w, $F_w(t)=F_0(t)$ by the same argument as the above.

If $c=1$ and $E(T_1)<0$, then $F_w(t)=F_0(t)$. This is well-known result by Lindley [3]. In above cases, if W_1 has any distribution $W_1(w)$, then

$$\lim_{n\to\infty}\int P(W_{n+1}\leqq t|W_1=w)dW_1(w)=F_0(t)$$

by Lebesgue's theorem on integrals. Uniqueness of the limiting distribution is obvious.

REFERENCES

1. P. D. Finch: A probability limit theorem with application to a generalizaticn of queuing theory, Acta. Math. Acad. Hung. 10, 317–325 (1959)
2. F. Pollaczek: Problèmes stochastiques posés par le phénomène de formation d'une queue d'attente à un quichet et par des phénomènes apparentes, Memorial des Sciences Mathématiques, Gauthier-Villars, Paris. (1957)
3. D. V. Lindley: The theory of queues with a single server, Proc. Cambridge Phil. Soc., 48, 277–289 (1952)
4. J. L. Doob: Stochastic Processes, John Wiley & Sons, Inc., New York. (1953)
5. M. Loève: Probability Theory, Van Nostrand, Princeton, (1955)