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AN EXPERIMENTS OF THE TIME SCHEDULE OF TRAINS

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I. INTRODUCTION

To simulate the situation of the activity of a station yard, the time schedule of the arrival or the departure of trains must be determined. We can use the time schedule of today for the simulation of the present situation, but to simulate the future situation, we must suppose the future time schedule. It is, however, very difficult. But if we can estimate the demand of transportation, we'll be able to determine the number of the arrival or the departure in each time interval, from the capacity of the rail road. So deviding a day into some proper time intervals, we suppose that the time of the arrival or the departure on each time interval is stochastically determined. But, it is to be desired that the distribution is alike form with the one of the present time schedule.

In this paper, by a simulation method, we shall show that the arrival or the departure of trains is realized under the following assumption.

Assumption: The number of the arrival of trains in a given time interval is fixed to be constant and any two consecutive trains have a constant time lag.

2. THE PROBABILITY DISTRIBUTION OF THE BETWEEN TWO CONSECUTIVE TRAINS.

We shall assume that n trains arrive at a station in a hour and the time lag between any two consecutive trains is τ minutes at least. It may be considered that the arrival time is random. As it is necessary τn minutes for n time lags in a hour, our problem is reduced to put n random points on a given time interval of the length $60-\tau n\equiv T$ minutes.

Now let the length of the sub-intervals devided by n independent random points,

$$X_1, X_2, \cdots, X_{n+1},$$

respectively. The apriori distributions of these random variables may be considered to have the same distribution, i.e. the uniform distribution over the interval $(0, T)$. Denote by $f_{n+1}(x)$ the apriori probability density of the sum

$$X_1+X_2+\cdots+X_{n+1},$$

and denote by

$$f(x_1|x)$$

the conditional probability density of X_1 under the condition

$$X_1+X_2+\cdots X_{n+1}=x$$

Then we have

$$f(x_1|x)=\frac{f_n(x-x_1)}{f_{n+1}(x)}$$

where f_n is the probability density of the sum

$$X_2+X_{3+}\cdots+X_{n+1}.$$

In the same manner, the conditional probability density of X_2 under the conditions

$$X_1+X_2+\cdots+X_{n+1}=x$$

and

$$X_1=x_1$$

is given by

$$f(x_2|x_1,\ x)=\frac{f_{n-1}(x-x_1-x_2)}{f_n(x-x_1)}$$

where f_{n-1} is the probability density function of

$$X_3+\cdots+X_{n+1}\,.$$

Also, the conditional probability density of X_i $(i=1, 2, \cdots, n)$ under the conditions

$$\begin{aligned} X_1+X_2+\cdots+X_{n+1}&=x\\ X_1&=x_1\\ X_2&=x_2\\ &\ \vdots\\ X_{i-1}&=x_{i-1}\end{aligned}$$

and

is given by

$$f(x_i|x_1,\ x_2,\ \cdots,\ x_{i-1},\ x)$$
$$=\frac{f_{n+1-i}(x-x_1-\cdots-x_i)}{f_{n+2-i}(x-x_1-\cdots-x_{i-1})} \tag{1}$$

where f_{n+1-i} is the probability density of the sum

$$X_{i+1}+X_{i+2}+\cdots+X_{n-1}, \qquad (i=1,\ 2,\ \cdots,\ n)$$

Next we must search the probability density functions

$$f_1,\ f_2,\ \cdots,\ f_{n+1}\,.$$

Here, we list the following known result:*)

Let $Y_1, \cdots, Y_n$ be independent random variables, having the same uniform distribution function in (0, 1). Then the probability density $g_n(x)$

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*) T. Uno, Mathematical Statistics, (In Japanese), p. 89, Kyoritsu Pub. Co.

of the sum $Y_1+\cdots+Y_n$ is given by

$$g_n(x)=\frac{1}{(n-1)!}\left[x^{n-1}-\binom{n}{1}(x-1)^{n-1}+\binom{n}{2}(x-2)^{n-1}-\cdots\right] \tag{2}$$

where $$x-1>0,\quad x-2>0,\ \cdots \qquad (0<x<n).$$

To applying this known result, put

$$Y_1=\frac{X_1}{T},\quad Y_2=\frac{X_2}{T},\quad \cdots,\quad Y_{n+1}=\frac{X_{n+1}}{T},$$

then the probability density of the sum

$$Y_{i+1}+\cdots+Y_{n+1}$$

is given by

$$f_{n+1-i}\left(\frac{x}{T}\right) \qquad (i=1,\ 2,\cdots,\ n).$$

Hence, by the above formula (2), we have

$$f_{n+1-i}\left(\frac{T-x_1-x_2-\cdots-x_i}{T}\right)=\frac{1}{(n-i)!}\left[\left(\frac{T-x_1-\cdots-x_i}{T}\right)^{n-i}\right]. \tag{3}$$

Therefore by (1) we see

$$f(x_i|x_1,\ x_2,\ \cdots,\ x_{i-1},\ T)$$

$$=\frac{\dfrac{1}{(n-i)!}\left(\dfrac{T-x_1-\cdots-x_i}{T}\right)^{n-i}}{\dfrac{1}{(n+1-i)!}\left(\dfrac{T-x_1-\cdots-x_{i-1}}{T}\right)^{n+1}}$$

$$=\frac{(n+1-i)}{\left(\dfrac{T-x_1-\cdots-x_{i-1}}{T}\right)^{n+1-i}}\left(\frac{T-x_1-\cdots-x_i}{T}\right)^{n-i}$$

Integrating the two side of the above formula,

$$\int_0^{\xi} f(x_i|x_1,\ x_2,\ \cdots,\ x_{i-1},\ T)\,dx_i$$

$$=\frac{n+1-i}{\left(\frac{T-x_1-\cdots x_{i-1}}{T}\right)^{n+1-i}}\int_0^{\xi}\left(\frac{T-x_1-\cdots x_i}{T}\right)^{n-i}dx_i$$

$$=1-\frac{[T-x_1-\cdots x_{i-1}-\xi]^{n+1-i}}{[T-x_1-\cdots-x_{i-1}]^{n+1-i}} \qquad (i=1,\ 2,\ \cdots,\ n). \qquad (4)$$

This formula is the conditional cumulative distribution of X_i under the condition

$$X_1=x_1$$
$$X_2=x_2$$
$$\vdots$$
$$X_{i-1}=x_{i-1}$$

and

$$X_1+X_2+\cdots+X_{n+1}=T \qquad (i=1,\ 2,\ \cdots,\ n).$$

3. AN EXPERIMENT BY SIMULATION METHOD

We had experimented 50 trials in the case of $n=5$, $\tau=5^{\text{min}}$ and $T=35^{\text{min}}$. The following table is the result.

Trial	x_1	x_2	x_1+x_2	x_3	$x_1+x_2+x_3$	x_4	$x_1+x_2+x_3+x_4$	x_5
1	2	17	19	1	20	0	20	13
2	12	7	19	9	28	2	30	5
3	7	6	13	4	17	1	18	15
4	12	5	17	12	29	1	30	3
5	1	6	7	13	20	0	20	12
6	1	6	7	2	9	14	23	8
7	5	6	11	11	22	6	28	7
8	6	3	9	6	15	16	31	1
9	0	2	2	6	8	4	12	14
10	2	3	5	7	12	8	20	0
11	2	1	3	9	12	10	22	1
12	4	8	12	0	12	5	17	1
13	5	8	13	5	18	10	28	0
14	13	2	15	12	27	3	30	2
15	1	10	11	5	16	6	22	10

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* I owe this experiment to M. Kanamatsu.

Trial	x_1	x_2	x_1+x_2	x_3	$x_1+x_2+x_3$	x_4	$x_1+x_2+x_3+x_4$	x_5
16	5	4	9	3	12	9	21	3
17	6	7	13	9	22	0	22	10
18	2	3	5	1	6	10	16	18
19	1	2	3	2	5	19	24	1
20	15	1	16	2	18	6	24	2
21	13	5	18	4	22	7	29	0
22	7	15	22	1	23	4	27	7
23	4	1	5	3	8	5	13	11
24	5	9	14	1	15	8	23	6
25	1	3	4	0	4	0	4	30
26	2	0	2	1	3	1	4	9
27	2	12	14	3	17	11	28	4
28	4	5	9	2	11	5	16	10
29	7	15	22	0	22	2	24	1
30	4	8	12	5	17	2	19	8
31	9	6	15	4	19	8	27	7
32	7	11	18	7	25	3	28	2
33	2	8	10	6	16	14	30	2
34	5	7	12	2	14	10	24	0
35	10	5	15	1	16	6	22	5
36	4	6	10	3	13	6	19	11
37	4	6	10	12	22	1	23	4
38	3	9	12	15	27	2	29	2
39	13	2	15	2	17	9	26	1
40	1	10	11	15	26	4	30	2
41	5	4	9	14	23	1	24	10
42	19	2	21	11	32	0	32	2
43	9	2	11	14	25	1	26	2
44	1	7	8	4	12	7	19	13
45	7	7	14	9	23	2	25	3
46	4	5	9	1	10	5	15	10
47	14	3	17	2	19	13	32	3
48	0	1	1	21	22	2	24	2
49	1	5	6	2	8	0	8	19
50	1	10	11	5	16	5	21	10

The result of these trials is given in Fig. 1. And we can see that the distribution of the time lags of any two consecutive trains may be approximated by a exponential distribution.

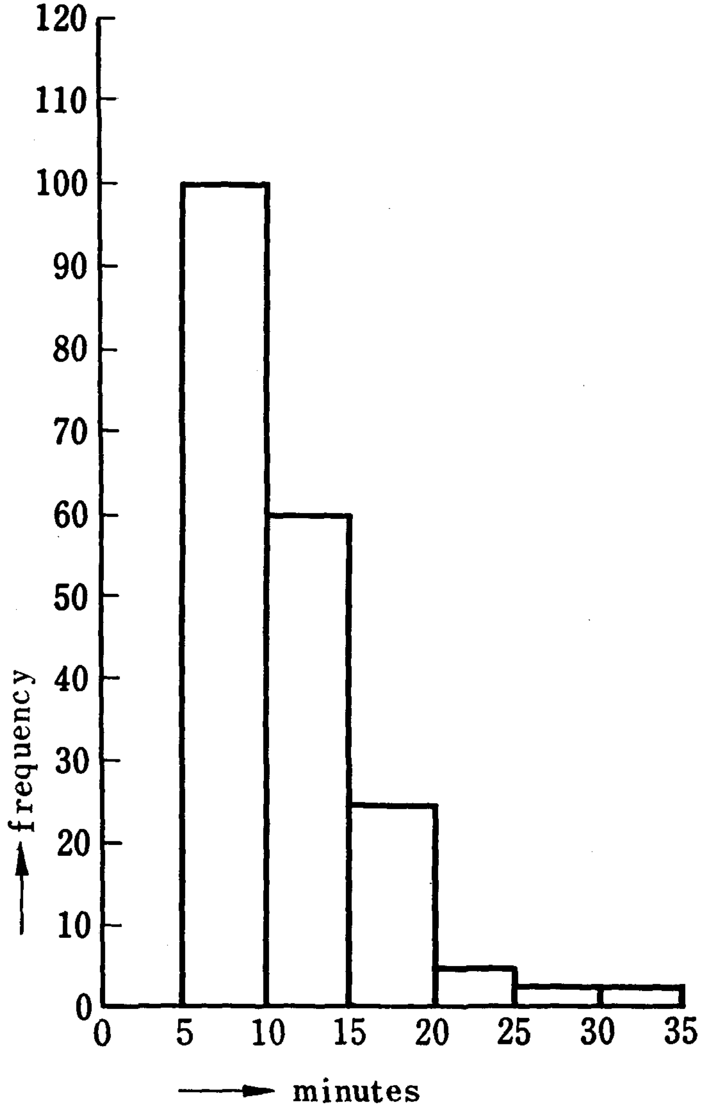

Fig. 1. The distribution of the time lags between two consecutive trains.