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PRIMAL DUAL METHOD OF PARAMETRIC PROGRAMMING AND IRI's THEORY ON NETWORK FLOW PROBLEMS

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INTRODUCTION

Primal dual algorithm of linear programming problems was first applied to the network flow problem by Ford and Fulkerson [2] [3]. In 1959, Kelley [1] pointed out that this is nothing but a method for solving parametric programming problem. In § 1, we shall describe the primal dual method of parametric programming in a general fashion. The content is essentially the same as that of [1], except for that the simplex method and the concept of basis are avoided, as they are not neccessary for our discussions and we treat the "general form" of the linear programming. This method was applied by Kelley [4] and Fulkerson [6] independently of each other, to a problem in planning and scheduling, which is now called CPM (Critical Path Method).

On the other hand, in 1960, Iri studied the network flow problem from an entirely different viewpoint. He developed a general algebraic and topological theory of electric circuit and noticed the analogy of the transportation problem with the circuit.

A few important points should be noted about Iri's theory. The first of them is his methodology. In his theory, the input voltage and total input current are increased alternatively starting from 0 so that the solutions of problems are found out. A technique called "Θ-matrix method" used at the voltage increasing steps forms the most important part in [5]. Iri's alternative increasing steps are regarded as an illustration of a method which is applicable to the general problem of parametric programming.

In § 2 we introduce this method, under the name "double parametrization method", and show that it is as equally efficient as the method in § 1 in the sense that the number of iterations to reach at a parameter value λ is the same for both the methods. In general, in formulating a parametric programming problem, various ways are possible according as which variable is taken as a parameter. For instance, if the input voltage is taken as the parameter of the network flow problem we get Ford and Fulkerson's method. A different approach, of course, is obtained if the total flow is taken as a parameter. In the former, the maximal flow is found by a labeling method which is well-known as one for solving the restricted primal problem, while in the latter, the maximal input voltage is found by the Θ-matrix method. Just as the labeling method in Fulkerton's theory gives the optimal solution of not only the restricted primal problem but also its dual problem, Θ-matrix method gives the optimal solutions of both the restricted primal and the dual problems simultaneously (this fact is not remarked in [5]). These two methods are discussed in § 3.

In § 4, we apply Kelley-Fulkerson's and Iri's methods to the problem of CPM in parallel to § 3. Iri's method in CPM has not yet been published. Iri himself, however, was aware of the possibility of the application as early as in 1961 and wrote an ALGOL program at the RAND Institute of JUSE, Tokyo. It is shown in § 5 that if we apply the primal dual method directly to the problem with many parameters, a certain very strong condition on the solutions of restricted primal problems is required.

However, if we regard the network flow problem with many sources as a multi-parametric programming problem with many input flows as parameters, the condition above stated is fulfilled. Thus it is the third feature of Iri's method that his Θ-matrix method can be applied to the network flow problem as a multi-parameter programming, as discussed in § 6.

Transportation problem of Hitchcock-type turns quite naturally to be a multi-parameteric programming, for which a numerical example solved by the method in §6 is attached in §7.

§1. PRIMAL DUAL METHOD OF PARAMETRIC PROGRAMMING IN GENERAL FORM

Let P|λ and D|λ denote respectively the following parametric programming problem and its dual problem.

P|λ

$$x_j \geqq 0 \qquad \text{if } j \in S \qquad \text{(P1)}$$

$$\left.\begin{array}{ll} \sum_j a_{ij}x_j \geqq b_i, & \text{if } i \in T, \\ (\text{or } \sum_j a_{ij}x_j - u_i = b_i,\ u_i \geqq 0, & \text{if } i \in T,) \\ \sum_j a_{ij}x_j = b_i, & \text{if } i \notin T, \end{array}\right\} \qquad \text{(P2)}$$

$$\text{minimize } f(x) = \sum_j (c_j + \lambda d_j)x_j, \qquad \text{(P3)}$$

D|λ

$$\left.\begin{array}{ll} \sum_i a_{ij}y_i \leqq c_j + \lambda d_j, & \text{if } j \in S, \\ (\text{or } \sum_i a_{ij}y_i + w_j = c_j + \lambda d_j,\ \ w_j \geqq 0, & \text{if } j \in S,) \\ \sum_i a_{ij}y_i = c_j + \lambda d_j, & \text{if } j \notin S, \end{array}\right\} \qquad \text{(D1)}$$

$$y_i \geqq 0, \qquad \text{if } i \in T, \qquad \text{(D2)}$$

$$\text{maximize } g(y) = \sum_i y_i b_i, \qquad \text{(D3)}$$

where i ranges over the set of integers $\{1, 2, \cdots, m\}$ and j over $\{1, 2, \cdots, n\}$, and T (resp. S) is a given subset of $\{1, 2, \cdots, m\}$ (resp. $\{1, 2, \cdots, n\}$).

1.1. Our aim is to trace the optimal solution of P|λ or D|λ, when λ increases from λ_0, being given the optimal solution of P|λ_0 or D|λ_0.

Let (x_j, u_i) and (y_i, w_j) be the optimal solutions of P|λ and D|λ respectively and let us define the restricted primal RP|λ and its dual restricted problem RD|λ, based on (y_i, w_j), as follows.

RP|λ

$$x_j \geqq 0, \qquad \text{if } j \in S, \qquad \text{(RP1)}$$

$$\left.\begin{array}{lr} \sum_j a_{ij}x_j \geqq b_i\,, & \text{if } i\in T, \\ \text{(or } \sum_j a_{ij}x_j - u_i = b_i,\ u_i \geqq 0, & \text{if } i\in T,) \\ \sum_j a_{ij}x_j = b_i\,, & \text{if } i\notin T \end{array}\right\} \quad \text{(RP2)}$$

$$\left.\begin{array}{l} \sum_{j\in S} x_j w_j = 0\,, \\ \sum_{i\in T} u_i y_i = 0\,, \end{array}\right\} \quad \text{(RP3)}$$

$$\text{minimize} \quad f_1(x) = \sum_j d_j x_j\,. \quad \text{(RP4)}$$

RD$|\lambda$

$$\left.\begin{array}{l} \sum_i a_{ij}\sigma_i \leqq d_j\,, \quad \text{if} \quad w_j = 0\,, \quad \text{and} \quad j\in S\,, \\ \sum_i a_{ij}\sigma_i = d_j\,, \quad \text{if} \quad j\notin S \end{array}\right\} \quad \text{(RD1)}$$

$$\sigma_i \geqq 0\,, \quad \text{if} \quad y_i = 0 \quad \text{and} \quad i\in T\,, \quad \text{(RD2)}$$

$$\text{maximize} \quad h(\sigma) = \sum_i \sigma_i b_i\,. \quad \text{(RD3)}$$

Proposition 1.1.

A feasible solution (x_j, u_i) of P$|\lambda$ is optimal, if and only if it is a feasible solution of RP$|\lambda$.

Proof.

If (x_j, u_i) resp. (y_i, w_j) is a solution of P$|\lambda$ resp. D$|\lambda$, then we have easily $\sum_j (c_j+\lambda d_j)x_j = \sum_i b_i y_i + \sum_{j\in S} w_j x_j + \sum_{i\in T} u_i y_i$, and $\sum_{j\in S} w_j x_j \geqq 0$, $\sum_{i\in T} u_i y_i \geqq 0$. By the Duality Theorem, (x_j, u_i) is optimal, if and only if $\sum_{j\in S} w_j x_j = 0$ and $\sum_{i\in T} u_i y_i = 0$, that is, (x_j, u_i) is a feasible solution of RP$|\lambda$.

Proposition 1.2.

If (y_i') is a feasible solution of D$|\lambda+\theta$ for some $\theta>0$, and if (σ_i) satisfies $(y_i')=(y_i)+\theta(\sigma_i)$, then (σ_i) is a feasible solution of RD$|\lambda$.

Proof.

By our assumption, $(y_i+\theta\sigma_i)$, is a feasible solution of D$|\lambda+\theta$. So that,

$$\sum_i a_{ij}(y_i+\theta\sigma_i)\leqq c_j+(\lambda+\theta)d_j \qquad \text{for} \quad j\in S, \tag{1.1}$$

$$\sum_i a_{ij}(y_i+\theta\sigma_i)=c_j+(\lambda+\theta)d_j \qquad \text{for} \quad j\notin S, \tag{1.2}$$

$$x_i+\theta\sigma_i\geqq 0 \qquad \text{for} \quad i\in T \tag{1.3}$$

From (1.1) and (1.2)

$$(\sum_i a_{ij}\sigma_i-d_j)\theta\leqq w_j \qquad \text{for} \quad j\in S, \tag{1.4}$$

$$(\sum_i a_{ij}\sigma_i-d_j)\theta=0 \qquad \text{for} \quad j\notin S. \tag{1.5}$$

Hence σ_i is a feasible solution of RD$|\lambda$.

Proposition 1.3.

Let (σ_i) be a feasible solution of RD$|\lambda$ and put $(\beta_j)=(d_j-\sum_i a_{ij}\sigma_i)$, then $(y+\theta\sigma_i)$ is a feasible solution of D$|\lambda+\theta$ $(\theta>0)$, if and only if $0<\theta\leqq\theta_0$ where θ_0 is defined as follows.

$$\theta_1=\begin{cases}\min(-w_j/\beta_j;\ \beta_j<0,\ j\in S) & \text{if there exists } j \text{ such that } \beta_j<0,\\ \infty & \text{otherwise,}\end{cases}$$

$$\theta_2=\begin{cases}\min(-y_i/\sigma_i;\ \sigma_i<0,\ i\in T) & \text{if there exists } i \text{ such that } \sigma_i<0,\\ \infty & \text{otherwise,}\end{cases}$$

$$\theta_0=\min(\theta_1,\ \theta_2).$$

Proof.

Note that $\theta_1>0$ and $\theta_2>0$, because $\beta_j<0$ implies $w_j>0$ for $j\in S$, and $\sigma_i<0$ implies $y_i>0$ for $i\in T$, by RD1 and RD2. Now from the proof of Proposition 1.2, $(y_i+\theta\sigma_i)$ is a feasible solution of D$|\lambda+\theta$, if and only if (1.3) and (1.4) hold, that is if and only if $\theta\leqq\theta_1$ and $\theta\leqq\theta_2$.

Proposition 1.4.

Suppose that $0<\theta\leqq\theta_0$, where θ_0 is defined as in Proposition 1.3, and that (x_j) is a solution of RP$|\lambda$. Then (x_j) resp. $(y_i+\theta\sigma_i)$ is an optimal feasible solution of P$|\lambda+\theta$ resp. D$|\lambda+\theta$, if and only if (x_j) resp. (σ_i) is an optimal solution of RP$|\lambda$ resp. RD$|\lambda$.

Proof.

Proposition 1.1~1.3, together with the following relation and Duality

Theorem imply the proposition.

$$f(x)=\sum_j(c_j+(\lambda+\theta)d_j)x_j=\sum_i b_i(y_i+\theta\sigma_i)=g(y+\theta\sigma)$$

$$f_1(x)=\sum d_jx_j=\sum\sigma_ib_i=h(\sigma).$$

Proposition 1.5.

If the optimal solutions of $P|\lambda$ and $D|\lambda$ exist for some λ, the neccessary and sufficient condition for the existence of the optimal solutions of $P|\lambda'$ and $D|\lambda'$, $\lambda'>\lambda$, is the existence of the optimal solutions of $RP|\lambda$ and $RD|\lambda$.

Proposition 1.6.

An optimal solution of $RP|\lambda$ is a feassible solution of $RP|\lambda+\theta$ where $0<\theta\leqq\theta_0$.

Proof.

Let (x_j, u_i) resp. (σ_i) be the optimal solution of $RP|\lambda$ resp. $RD|\lambda$, then (y_i', w_j') defined by

$$y_i'=y_i+\theta\sigma_i,$$
$$w_j'=w_j+\theta\beta_j \qquad \text{for} \qquad j\in S, \tag{1.6}$$

is the optimal solution of $D|\lambda+\theta$. For, from D1 with $\lambda+\theta$

$$\sum_i a_{ij}(y+\theta\sigma_i)+w_j'=c_j+(\lambda+\theta)d_j, \qquad \text{for} \quad j\in S.$$

To prove that (x_j, u_i) is a feasible solution of $RP|\lambda+\theta$, it suffices to show that

$$\sum_{j\in S} x_jw_j'=0, \tag{1.7}$$

$$\sum_{i\in T} u_iy_i'=0. \tag{1.8}$$

Now

$$\sum_{j\in S} x_jw_j'=\sum_{j\in S} x_j(w_j+\theta\beta_j) \qquad \text{by (1.6)}$$

$$=\sum_{j\in S} x_jw_j+\theta\sum_{j\in S} x_j\beta_j$$

$$=\theta\sum_{j\in S} x_j\beta_j$$

and

$$\sum_{i\in T} u_i y_i' = \sum_{i\in T} u_i(y_i+\theta\sigma_i)$$
$$=\theta \sum_{i\in T} u_i \sigma_i .$$

While for an optimal solution (x_j, u_i) resp. (σ_i) of RP$|\lambda$ resp. RD$|\lambda$, we have $\sum_{j\in S} x_j\beta_j=0$ and $\sum_{i\in T} u_i\sigma_i=0$ because from RP1–3 and RD–2,

$$\sum_j d_j x_j = \sum_i \sigma_i b_i + \sum_{i\in T}\sigma_i u_i + \sum_{j\in S}\beta_j x_j$$

and

$$\sum_{i\in T}\sigma_i u_i = \sum_{i\in T \text{ and } y_i=0}\sigma_i u_i \geqq 0, \quad \sum_{j\in S}\beta_j x_j = \sum_{j\in S \text{ and } w_j=0}\beta_j x_j \geqq 0$$

therefore, $\sum_{i\in T}\sigma_i u_i=0$ and $\sum_{j\in S}\beta_j x_j=0$ by the Duality Theorem.

Thus we can formultate the following procedure to solve P$|\lambda$ or D$|\lambda$.

1. Start with an optimal feasible solution (x_j, u_i) resp. (y_i, w_j) of P$|\lambda$ resy. D$|\lambda$ for some λ.

2. Construct RP$|\lambda$ and RD$|\lambda$ making use of (y_i, w_j).

2a) If taere is no optimal solution of RP$|\lambda$ and RD$|\lambda$ (e.g. RP$|\lambda$ has no bounded solution) then, there exists no optimal solution of P$|\lambda'$ and $D|\lambda'$ for $\lambda'>\lambda$. In this case, give up the procedure.

2b) When we can get optimal solutions (x_j, u_i) resp. (σ_i) of RP$|\lambda$ resp. RD$|\lambda$, put $\beta_j=d_j-\sum_i a_{ij}\sigma_i$,

$$\theta_1=\begin{cases}\min(-w_j/\beta_j\,;\ \beta_j<0,\ j\in S), & \text{if there exists } j\in S \text{ such that } \beta_j<0,\\ \infty, & \text{otherwise,}\end{cases}$$

$$\theta_2=\begin{cases}\min(-y_i/\sigma_i\,;\ \sigma_i<0,\ i\in T), & \text{if there exists } i\in T \text{ such that } \sigma_i<0,\\ \infty, & \text{otherwise}\end{cases}$$

and

$$\theta_0=\min(\theta_1,\ \theta_2).$$

3.

3a) If $\theta_0=\infty$, then (x_j, u_i) resp. $(y_i+\theta\sigma_i,\ w_j+\theta\beta_j)$ is the optimal solution of $P|\lambda+\theta$ resp. $D|\lambda+\theta$ for any $\theta>0$. Thus, the procedure is terminated.

3b) If $\theta_0<\infty$ then (x_j, u_i) resp. $(y_i+\theta_0\sigma_i, w_j+\theta_0\beta_j)$ is an optimal solution of $P|\lambda+\theta_0$ resp. $D|\lambda+\theta$. In this case return to step 2 (and here, Proposition 1.6 is very useful), and continue the process.

1.2. Next we shall get the optimal tolution of $P|\lambda'$ or $D|\lambda'$, $\lambda'<\lambda$, starting with the optimal solution of $P|\lambda$ and $D|\lambda$.

This time we consider the following $RP'|\lambda$ and $RD|\lambda$.

$RP'|\lambda$

$$x_j\geqq 0, \quad \text{if } j\in S, \qquad \text{(RP1)}$$

$$\left.\begin{array}{ll} \sum_j a_{ij}x_j\geqq b_i, & \text{if } i\in T, \\ (\text{or } \sum_j a_{ij}x-u_i=b_i,\ u_i\geqq 0, & \text{if } i\in T,) \\ \sum_j a_{ij}x_j=b_i, & \text{if } i\notin T, \end{array}\right\} \qquad \text{(RP2)}$$

$$\left.\begin{array}{l} \sum_{j\in S} x_j w_j=0, \\ \sum_{i\in T} u_i y_i=0, \end{array}\right\} \qquad \text{(RP3)}$$

$$\text{maximize } f_1(x)=\sum_j d_j x_j, \qquad \text{(RP'4)}$$

$RD'|\lambda$

$$\left.\begin{array}{ll} \sum_i a_{ij}\sigma_i\geqq d_j, & \text{if } w_j=0 \text{ and } j\in S, \\ \sum_i a_{ij}\sigma=d_j, & \text{if } j\notin S, \end{array}\right\} \qquad \text{(RD'1)}$$

$$\sigma_i\leqq 0, \quad \text{if } y_i=0 \text{ and } i\in T, \qquad \text{(RD'2)}$$

$$\text{minimize } h(\sigma)=\sum_j \sigma_i b_i. \qquad \text{(RD'3)}$$

In this case θ_1, θ_2 and θ_0 are defined as follows.

$$\theta_1=\begin{cases} \min_{\beta_j>0} w_j/\beta_j, & \text{if there exists } j\in S \text{ such that } \beta_j>0, \\ \infty, & \text{otherwise,} \end{cases}$$

$$\theta_2 = \begin{cases} \min_{\sigma_i>0} y_i/\sigma_i, & \text{if there exists } i\in T \text{ such that } \sigma_i>0, \\ \infty, & \text{otherwise,} \end{cases}$$

$$\theta_0 = \min(\theta_1, \theta_2).$$

Moreover, for θ, $0<\theta\leqq\theta_0$, (x_j, u_i) resp. $(y_i-\theta\sigma_i, w_j-\theta\beta_j)$ is an optimal solution of $P|\lambda-\theta$ resp. $D|\lambda-\theta$.

§2. A METHOD OF DOUBLE PARAMETRIZATION

Again, we consider $P|\lambda$ and $D|\lambda$, and now we introduce a new variable μ in $P|\lambda$.

$P|\lambda$

$$x_j\geqq 0, \qquad \text{if } j\in S \qquad \text{(P1)}$$

$$\left.\begin{array}{ll} \sum_j a_{ij}x_j\geqq b_i, & \text{if } i\in T \\ (\text{or } \sum_j a_{ij}x_j-u_i=b_i, \quad u_i\geqq 0, & \text{if } i\in T), \\ \sum_j a_{ij}x_j=b_i, & \text{if } i\notin T, \end{array}\right\} \qquad \text{(P2)}$$

$$-\sum d_jx_j=\mu, \qquad \text{(P3)}$$

$$\text{minimize} \quad f(x, \mu)=\sum_j c_jx_j-\lambda\mu. \qquad \text{(P4)}$$

$D|\lambda$

$$\left.\begin{array}{ll} \sum_i a_{ij}y_i\leqq c_j+\lambda d_j, & \text{if } j\in S, \\ (\text{or } \sum_i a_{ij}y_i+w_j=c_j+\lambda d_j, \quad w_j\geqq 0, & \text{if } j\in S,) \\ \sum_i a_{ij}y_i=c_j+\lambda d_j, & \text{if } j\notin S, \end{array}\right\} \qquad \text{(D1)}$$

$$y_i\geq 0, \qquad \text{if } i\in T, \qquad \text{(DT)}$$

$$\text{maximize } g(y)=\sum_j b_iy_i.$$

$RP|\lambda$

$$x_j\geqq 0, \qquad \text{if } j\in S, \qquad \text{(RP1)}$$

$$\left.\begin{array}{ll} \sum_j a_{ij}x_j \geqq b_i, & \text{if } i\in T, \\ (\text{or } \sum_j a_{ij}x_j - u_i = b_i, \quad u_i \geqq 0, & \text{if } i\in T,) \\ \sum_j a_{ij}x_j = b_i, & \text{if } i\notin T, \end{array}\right\} \quad \text{(RP2)}$$

$$-\sum_j d_j x_j = \mu, \quad \text{(RP3)}$$

$$\left.\begin{array}{l} \sum_{j\in S} x_j w_j = 0, \\ \sum_{i\in T} u_i y_i = 0, \end{array}\right\} \quad \text{(RP4)}$$

$$\text{minimize} \quad -\mu. \quad \text{(RP5)}$$

RD$|\lambda$

$$\left.\begin{array}{ll} \sum_i a_{ij}\sigma_i \leqq d_j, & \text{if } w_j = 0 \text{ and } j\in S, \\ \sum_i a_{ij}\sigma_i = d_j & \text{if } j\notin S, \end{array}\right\} \quad \text{(RD)}$$

$$\sigma_i \geqq 0, \qquad \text{if } y_i = 0 \text{ and } i\in T, \quad \text{(RD2)}$$

$$\text{maximize} \quad h(\sigma) = \sum_i b_i\sigma_i. \quad \text{(RD3)}$$

Now in P$|\lambda$, we regard μ as a parameter, and consider the following problem

D*$|\mu$

$$x_j \geq 0, \qquad \text{if } j\in S, \quad \text{(D*1)}$$

$$\left.\begin{array}{ll} \sum_j a_{ij}x_j \geq b_i, & \text{if } i\in T, \\ (\text{or } \sum_j a_{ij}x_j - u_i = b_i, \quad u_i \geq 0, & \text{if } i\in T,) \\ \sum_j a_{ij}x_j = b_i, & \text{if } i\notin T, \end{array}\right\} \quad \text{(D*2)}$$

$$-\sum_j d_j x_j = \mu, \quad \text{(D*3)}$$

$$\text{minimize} \quad f^*(x) = \sum_j c_j x_j. \quad \text{(D*4)}$$

The primal problem $P^*|\mu$ which is the dual of $D^*|\mu$ is defined as follows. Here, λ is regarded as a variable corresponding to (D^*3).

$P^*|\mu$

$$\left.\begin{array}{lll} \sum_i a_{ij} y_i \leqq c_j + \lambda d_j, & & \text{if } j \in S, \\ (\text{or } \sum_i a_{ij} y_i + w_j = c_j + \lambda d_j, & w_j \geqq 0, & \text{if } j \in S,) \\ \sum_i a_{ij} y_i = c_j + \lambda d_j, & & \text{if } j \notin S, \end{array}\right\} \quad (P^*1)$$

$$y_i \geqq 0, \qquad \text{if } i \in T, \quad (P^*2)$$

$$\text{maximize} \quad g^*(y, \lambda) = \sum_i y_i b_i + \mu\lambda. \quad (P^*3)$$

$RP^*|\mu$

$$\left.\begin{array}{lll} \sum_i a_{ij} y_i + w_j = c_j + \lambda d_j, & w_j \geqq 0, & \text{if } i \in S, \\ \sum_i {}_i a_{ij} y_i = c_j + \lambda d_j, & & \text{is } j \in S, \end{array}\right\} \quad (RP^*1)$$

$$y_i \geqq 0, \qquad \text{if } i \in T, \quad (RP^*2)$$

$$\left.\begin{array}{l} \sum_{j \in S} w_j x_j = 0, \\ \sum_{i \in T} y_i u_i = 0, \end{array}\right\} \quad (RP^*3)$$

$$\text{maximize } \lambda. \quad (RP^*4)$$

$RD|^*\mu$

$$\xi_j \geqq 0, \qquad \text{if } x_j = 0 \text{ and } j \in S, \quad (RD^*1)$$

$$\left.\begin{array}{ll} \sum_j a_{ij} \xi_j \geqq 0, & \text{if } u_i = 0 \text{ and } i \in T, \\ \sum_j a_{ij} \xi_j = 0, & \text{if } \qquad i \notin T, \end{array}\right\} \quad (RD^*2)$$

$$-\sum d_j \xi_j = 1, \quad (RD^*3)$$

$$\text{minimize} \quad \sum_j c_j \xi_j. \quad (RD^*4)$$

Proposition 2. 1.

(x_j, μ) resp. (y_i) is the optimal solution of $P|\lambda$, resp. $D|\lambda$ if and only if (x_j) resp. (y_i, λ) is the optimal solution of $D^*|\mu$ resp. $P^*|\mu$.

resp. (y_i^*) is the optimal solutions of $P|\lambda^*$ resp. $D|\lambda^*$. The number of steps required by the double parametrszation method, that is, the number of the values of $\lambda_i(\lambda_0<\lambda_i<\lambda_n)$ for which the problem have to be solved, is the same to that of the method described in § 1.

§ 3. TRANSPORTATION NETWORK FLOW PROBLEM

Let N be a network with m branches having proper orientation and $n+1$ nodes $0, 1, 2, \cdots, n$. Let the source and the sink be denoted by 0 n respectively. Further, let B be the set of all orientated branches of N. Then the standard form of the transportation network flow problem which corresponds to $D^*|\mu$ in § 2 is the following.

$D^*|\mu$

$$\sum_{(i,\ j)\in B} x_{ij} = \sum_{(j,\ k)\in B} x_{jk} \quad \text{for every nodes } j(\neq 0,\ n) \qquad (D^*1)$$

$$0 \leqq x_{ij} \leqq c_{ij}, \qquad (D^*2)$$

$$\sum_{(0,\ j)\in B} x_{0j} = \sum_{(i,\ n)\in B} x_{in} = \mu, \qquad (D^*3)$$

$$\text{minimize} \sum_{(i,\ j)\in B} d_{ij}x_{ij} \qquad (D^*4)$$

where, $c_{ij} \geqq 0,\ \ d_{ij} \geqq 0 \ \text{ for } \ (i,\ j)\in B.$

And its dual is

$P^*|\mu$

$$w_{ij}' = d_{ij} + u_j - u_i + w_{ij} \geqq 0, \qquad \text{for } (i,\ j)\in B, \qquad (P^*1)$$

$$w_{ij} \geqq 0, \qquad \text{for } (i,\ j)\in B \qquad (P^*2)$$

$$u_0 - u_n = \lambda, \qquad (P^*3)$$

$$\text{maximize} \quad \mu\lambda - \sum_{(ij)\in B} c_{ij}w_{ij}. \qquad (P^*4)$$

we define further

Proof.

If (x_j, μ) resp. (y_i) is a feasible solution of $P|\lambda$, resp. $D|\lambda$, then (x_j) resp. (y_i, λ) is a fessible solution of $D^*|\mu$ resp. $P^*|\mu$. On account of the optimality of (x_j, μ) resp. (y_i) for $P|\lambda$ resp. $D|\lambda$, we have

$$\sum_j c_j x_j - \lambda\mu = \sum_i y_i b_i .$$

Hence

$$f^*(x) = \sum_j c_j x_j = \sum_i y_i b_i + \lambda\mu = g^*(y, \lambda),$$

by the duality theorem, our proposition follows immediately. Now suppose that (y_i) is the optimal solution of $D|\lambda$, and that (x_j, μ) resp. (σ_i) is an optimal solution of $RP|\lambda$ resp. $RD|\lambda$, and define θ_0 as in Proposition 1.3. By the method stated in § 1, (x_j, μ) resp. $(y_i' = (y_i + \sigma_i\theta_0)$ is an optimal solution of $P|\lambda'$ resp. $D|\lambda'$ where $\lambda' = \lambda + \theta_0$. Optimal solution of $RD|\lambda$ are not always unique, but we assume for a moment that θ_0 is uniquely determined by $RD|\lambda$, independently of various optimal solutions through which it is constructed. Then the following proposition holds.

Proposition 2.2.

If (y^*, λ^*) is the optimal solution of $RP^*|\mu$ corresponding to op. solution (x_j, μ) of $RP|\lambda$, then $\lambda^* = \lambda + \theta_0$.

Proof.

By the Proposition 1.6, the optimal solution (x_j, μ) of $RP|\lambda$ is a feasible solution of $RP|\lambda + \theta_0$, so we can easily see that (y_i', λ') is a feassible solution of $RP^*|\mu$ and we have $\lambda^* \geqq \lambda + \theta_0$. On the other hand, (y^*, λ^*), being an optimal solution of $RP^*|\mu$, is an optimal solution of $P^*|\mu$ by the Proposition 1.1. Therefore. (x_j, μ) resp. (y_i^*) is the optimal solution of $P|\lambda^*$ resp. $D|\lambda^*$ by Proposition 2.1. If we put $\lambda^* = \lambda + \theta$ and $y_i^* + \theta\sigma_i$, (σ_i) is an optimal solution of $RD|\lambda$ by Proposition 1.2. and 1.4. Hence we have $\theta \leqq \theta_0$ by Proposition 1.3 and our assumption about θ_0. Consequentely we have $\lambda^* = \lambda + \theta_0$. By double parametrization method, we mean a procedure which, starting with the optimal solution of (x_j, μ) resp. (y_i) of $P|\lambda$ resp. $D|\lambda$ for some λ. solves $RP|\lambda$ and $RP^*|\mu$, alternatively. At some stage of this procedure, if (y_i^*, λ^*) is an optimal solution of $RP^*|\mu$, (x_j, μ)

$$\text{minimize} \quad -\mu\lambda + \sum x_{ij} d_{ij}. \tag{P4}$$

And the corresponding restricted problems are

RP$|\lambda$

$$0 \leqq x_{ij} \leqq c_{ij}, \tag{RP1}$$

$$\sum_{(i,\, j)\in B} x_{ij} = \sum_{(j,\, k)\in B} x_{jk}, \qquad \text{for each nodes } j(\neq 0,\, n) \tag{RP2}$$

$$\sum_{(0,\, j)\in B} x_{0j} = \sum_{(i,\, n)\in B} x_{in} = \mu, \tag{RP3}$$

$$\left.\begin{array}{ll} x_{ij}=0, & \text{if} \quad w_{ij}'>0, \\ x_{ij}=c_{ij}, & \text{if} \quad w_{ij}>0, \end{array}\right\} \tag{RP4}$$

$$\text{maximize} \quad \mu, \tag{RP5}$$

and

RD$|\lambda$

$$\left.\begin{array}{ll} \sigma_i-\sigma_j-\rho_{ij}\leqq 0, & \text{if} \quad w_{ij}'=0, \\ \rho_{ij}\geqq 0, & \text{if} \quad w_{ij}=0, \end{array}\right\} \tag{RD1}$$

$$\sigma_0-\sigma_n=1, \tag{RD2}$$

$$\text{maximize} \quad -\sum c_{ij}\rho_{ij}. \tag{RD3}$$

we may assume that w_{ij} in $P^*|\mu$, $RP^*|\mu$ and D$|\lambda$ satisfy the following conditions

$$\text{if} \quad d_{ij}+u_j-u_i \geqq 0, \quad \text{then} \quad w_{ij}=0,$$

$$\text{and if} \quad d_{ij}+u_j-u_i<0, \quad \text{then} \quad w_{ij}=u_i-u_j-d_{ij}.$$

Therefore, we assume that $w_{ij}\cdot w_{ij}'=0$.

3.1. Ford and Fulkerson's method

Ford and Fulkerson's or Kelley's method for solving tronsportation network flow problem can be characterized as a method for solving P$|\lambda$ and D$|\lambda$ given in § 1. As an initial optimal solution of P$|\lambda$ resp. D$|\lambda$ for $\lambda=0$, we can take $x_{ij}=0$ and $\mu=0$, resp. $w_{ij}=0$ and $w_{ij}'=d_{ij}$ for all $(i,\, j)\in B$

RP*|μ

$$w_{ij}{}'=d_{ij}+u_j-u_i+w_{ij}\geqq 0, \qquad \text{for} \quad (i,\ j)\in B, \qquad \text{(RP*1)}$$

$$w_{ij}\geqq 0, \qquad \text{for} \quad (i,\ j)\in B, \qquad \text{(RP*2)}$$

$$u_0-u_n=\lambda, \qquad \text{(RP*3)}$$

$$\left.\begin{array}{ll} w_{ij}{}'=0, & \text{if} \quad x_{ij}>0, \\ w_{ij}=0, & \text{if} \quad x_{ij}<c_{ij}, \end{array}\right\} \qquad \text{(RP*4)}$$

$$\text{maximize} \quad \lambda. \qquad \text{(RP*5)}$$

RD*|μ

$$\left.\begin{array}{ll} \xi_{ij}\geqq 0, & \text{if} \quad x_{ij}=0 \\ \xi_{ij}\leqq 0, & \text{if} \quad x_{ij}=c_{ij}, \end{array}\right\} \qquad \text{(RD*1)}$$

$$\sum_{(i,\ j)\in B}\xi_{ij}=\sum_{(j,\ k)\in B}\xi_{jk}, \qquad \text{for each nodes} \quad j(\neq 0,\ n) \qquad \text{(RD*)}$$

$$\sum_{(0,\ j)\in B}\xi_{0j}=\sum_{(i\ ,n)\in B}\xi_{in}=1, \qquad \text{(RD*3)}$$

$$\text{minimize} \sum_{(i,\ j)\in B}\xi_{ij}d_{ij}. \qquad \text{(RD*4)}$$

Here, D|λ and P|λ described in § 1, § 2 are written as follows

D|λ

$$w_{ij}{}'=d_{ij}+u_j-u_i+w_{ij}\geqq 0, \qquad \text{for} \quad (i,\ j)\in B, \qquad \text{(D1)}$$

$$w_{ij}\geqq 0, \qquad \text{for} \quad (i,\ j)\in B, \qquad \text{(D2)}$$

$$u_0-u_n=\lambda, \qquad \text{(D3)}$$

$$\text{maximize} \quad -\sum_{(i,\ j)\in B}{}' c_{ij}w_{ij}. \qquad \text{(D4)}$$

P|λ

$$0\leqq x_{ij}\leqq c_{ij}, \qquad \text{for} \quad (i,\ j)\in B, \qquad \text{(P1)}$$

$$\sum_{(i,\ j)\in B}x_{ij}=\sum_{(j,\ k)\in B}x_{jk}, \qquad \text{for each nodes} \quad j(\neq 0,\ n) \qquad \text{(P2)}$$

$$\sum_{(0,\ j)\in B}x_{0j}=\sum_{(i,\ n)\in B}x_{in}=\mu, \qquad \text{(P3)}$$

and $u_i=0$ for all nodes i. An essential point of this method lies in the so called labeling process in solving RP$|\lambda$ and RD$|\lambda$.

3.1.1. Labeling method for RP$|\lambda$.

The labels of the form $(\pm i, h)$ are attached to nodes according to the following rules.

1. Label source 0 with the label $(* \infty)$.

2. Consider any labeled node i with the label $(\pm k, h)$ not yet scanned.

a. For any unlabeled nodes j such that $(i, j)\in B$, if $x_{ji}<c_{ij}$ and $w'_{ji}=0$ we attach the label $(+i \min[h, c_{ij}-x_{ij}])$ to j. Otherwise j is left unlabeled.

b. For any unlabeled node j such that $(j, i)\in B$, if $x_{ji}>0$ and $w_{ji}=0$ we attach the label $(-i \min[h, x_{ji}])$ to j. Otherwise j is left unlabeled. When then the process 2 is over for all j such that $(i, j)\in B$ or $(j, i)\in B$, i is scanned.

3. When the sink n has been labeled with (i, h), we have obtained the path $0=i_0, i_1,\cdots, i_l=n$ where i_k is labeled $(\pm i_{k-1}, h_k)$, then we change $x i_k i_{k+1}$ to $x i_k i_{k+1}+h$ if $(i_k\ i_{k+1})\in B$ and to $x i_{k+1} i_k-h$ if $(i_{k+1}, i_k)\in B$. Thus we have increased the total flow by h and return to process 2.

4. When the labeling process has terminated, if the sink n is not labeled, the maximal flow, i.e. an optimal solution of RP$|\lambda$, have been obtained. Next we will solve the restricted problem RD$|\lambda$. First, let I be the the set of all labeled nodes and J the set of all unlabeled nodes. They will be utilized in the course of solution.

3.1.2. The optimal solution for RD$|\lambda$.

σ_i and ρ_{ij} are defined as follows

$$\sigma_i=\begin{cases} 1, & \text{if } i\in I, \\ 0, & \text{if } i\in J, \end{cases} \tag{3.1.1}$$

$$\rho_{ij}=\begin{cases} 1, & \text{if } (i, j)\in IJ \text{ and } w_{ij}'=0, \\ -1, & \text{if } (i, j)\in JI \text{ and } w_{ij}>0, \\ 0, & \text{otherwise} \end{cases} \tag{3.1.2}$$

Proposition 3. 1.

(σ_i, ρ_{ij}) defined by (3.1.1) and (3.1.2) is an optimal solution of RD|λ.

Proof.

The feasibility of (σ_i, ρ_{ij}) for RD|λ is obvious. The optimality of (σ_i, ρ_{ij}) for RD|λ is proved as follows.

When $(i, j) \in IJ$, $w_{ij}'>0$ implies $x_{ij}=0$ and $w_{ij}'=0$ implies $x_{ij}=c_{ij}$, for otherwise j would be labeled from i. Altogether, we have $x_{ij}=c_{ij}\rho_{ij}$ for $(i, j)\in IJ$.

When $(i, j)\in J\cdot I$, $w_{ij}>0$ implies $x_{ij}=c_{ij}$ and $w_{ij}=0$ implies $x_{ij}=0$, for otherwise i would be labeled from j. Therefore, we have $x_{ij}=-c_{ij}\rho_{ij}$ for $(i, j)\in J\cdot I$.

On the other hand we have

$$\sum_{(0,j)\in B} x_{0j}=\sum_{IJ} x_{ij}-\sum_{JI} x_{ij}, \qquad \text{and} \qquad \sum_{(0,j)\in B} x_{0j}=\sum_{(i,j)\in B} c_{ij}\rho_{ij}.$$

Hence by the duality theorem (x_{ij}) resp. $(\sigma_i\, \rho_{ij})$ is an optimal solution of RP|λ resp. RD|λ.

3. 1. 3. Determination of θ_0

θ_0 described in § 1 is determined as follows.

$$\theta_1=\begin{cases} \min \dfrac{w_{ij}'}{\sigma_i-\sigma_j-\rho_{ij}} & \text{where } \sigma_i-\sigma_j-\rho_{ij}>0 \text{ if there is } (i, j)\in B \text{ such that } \sigma_i-\sigma_j-\rho_{ij}>0, \\ \infty & \text{if there is no } (i, j)\in B \text{ suth that } \sigma_i-\sigma_j-\rho_{ij}>0. \end{cases}$$

That is,

$$\theta_1=\begin{cases} \min w_{ij} & \text{where } (i, j)\in I\cdot J \text{ and } w_{ij}'>0, \\ \infty & \text{if there is no } (i, j)\in I\cdot J \text{ suth that } w_{ij}'>0, \end{cases}$$

$$\theta_2=\begin{cases} \min\limits_{\rho_{ij}} -\dfrac{w_{ij}}{\rho_{ij}}=\min w_{ij}, & (i, j)\in J\cdot I \text{ and } w_{ij}>0 \\ \infty & \text{if there is no } (i, j)\in J\cdot I \text{ such that } w_{ij}>0, \end{cases}$$

$\theta_0=\min(\theta_1, \theta_2)$.

3.1.4. In the course of the labeling process in 3.1.1, if it turns out that maximal flow become ∞, no optimal solution exists for $P|\lambda'$, $D|\lambda'$ with λ' larger than λ. If the maximal flow is finite, x_{ij} determined by labeling process, together with $u_i+\sigma_i\theta$ and $w_{ij}+\rho_{ij}\theta$ with $\theta\leqq\theta_0$ are optimal solutions of $P|\lambda+\theta$ and $D|\lambda+\theta$.

3.2. Iri's theory on network flow problem

Iri's original theory for solving network-flow problem is nothing but the method of double parametrization. The "voltage increasing step" in his theory exactly corresponds to the problem $RP^*|\mu$ and "the current increasing step" to $RP|\lambda$. In what follows we shall solve $P^*|\mu$ resp. $D^*|\mu$ by the method given in § 1, where Iri's "Θ-matrix method" will take an essential part in solving restricted problems. It is pointed out that it is utilized for the solution of $RD^*|\mu$ as well as $RP^*|\mu$.

3.2.1. Θ-matrix method for solving $RP^*|\mu$.

For any pair of two nodes (i, j) we define a matrix (θ^i_j), which will be called Θ-matrix,

$$\theta^i_j=\begin{cases} 0, & \text{if } i=j, \\ -d_{ij}, & \text{if } (i,\ j)\in B \text{ and } x_{ij}>0, \\ d_{ji}, & \text{if } (j\ i)\in B \text{ and } x_{ji}<c_{ji}, \\ \infty, & \text{otherwise.} \end{cases}$$

$\overset{k}{v}_i$ is defined for any nodes i recursively as follows.

$$\overset{0}{v}_i=\begin{cases} \infty, & \text{if } i\neq n, \\ 0, & \text{if } i=n, \end{cases}$$

$$\overset{k+1}{v}_i=\min_j\,(\theta^j_i+\overset{k}{v}_j).$$

Proposition 3.2 (Iri's theorem c.f. [5])

(a) $\overset{k}{v}_i(k=1, 2,\cdots)$ rapidly converges, i.e. we have for some $N(\leqq n-1)$

$$\overset{0}{v}_i>\overset{1}{v}_i>\cdots>\overset{N}{v}_i=\overset{N+1}{v}_i=\cdots=\overset{\infty}{v}_i \qquad \text{for any nodes } i$$

we put here $u_i = \overset{\infty}{v_i}$ (notice that $u_n = 0$).

(b) u_i and $w_{ij} = \max(u_i - u_j - d_{ij}, 0)$ is a feasible solution of RP*$|\mu$.

(c) For any feassible solution (u_i') of RP*$|\mu$ satisfying $u_n' = 0$, $u_j \geqq u_j'$ holds for any nodes i, and $u_j w_{ij} \lambda (= u_0)$ obtained by the above Θ-matrix method is the optimal solution of RP*$|\mu$.

3.2.2. The optimal solution of RD*$|\mu$ is obtained as soon as the solution of RP*$|\mu$ has been found by Θ-matrix method. By the definition of $u_i = \overset{\infty}{v_i}$, $u_0 = \lambda$ can be written in the following form, provided that $u_0 \neq \infty$

$$u_0 = \theta_0^{i_1} + \theta_{i_1}^{i_2} + \cdots + \theta_{i_{k-1}}^{i_k} + \cdots + \theta_{i_{m-2}}^{i_{m-1}} + \theta_{i_{m-1}}^{n}$$

where

$$\theta_{i_{k-1}}^{i_k} = \begin{cases} -d_{i_k i_{k-1}} & \text{if } (i_k, i_{k-1}) \in B, \\ d_{i_{k-1} i_k} & \text{if } (i_{k-1}, i_k) \in B, \end{cases}$$

Now, we define ξ_{ij}, for $(i, j) \in B$, by

$$\xi_{ij} = \begin{cases} -1, & \text{if } (i, j) = (i_k i_{k-1}) \text{ and } (i_k i_{k-1}) \in B \\ & \quad \text{in the above expression of } u_0, \\ 1, & \text{if } (i, j) = (i_{k-1} i_k) \text{ and } (i_{k-1} i_k) \in B \\ & \quad \text{in the above expression of } u_0, \\ 0, & \text{otherwise,} \end{cases}$$

the following proposition is straight forward from the definition of $\overset{k}{v_j}$ and from the fact that $\lambda = u_0 = \sum_{(i, j) \in B} d_{ij} \xi_{ij}$.

Proposition 3.3.

ξ_{ij} defined above is the optimal solution of RD*$|\mu$.

3.2.3. θ_0 is defined as follows

$$\theta_1 = \begin{cases} \min (c_{ij} - x_{ij}) & \text{where } \xi_{ij} = 1, \\ \infty & \text{if there is no (i, } j) \in B \text{ such that } \xi_{ij} = 1, \end{cases}$$

$$\theta_2 = \begin{cases} \min\limits_{\xi_{ij} = -1} x_{ij} & \text{if there is } (i, j) \in B \text{ such that } \xi_{ij} = -1, \\ \infty & \text{otherwise,} \end{cases}$$

$$\theta_0 = \min(\theta_1, \theta_2),$$

if $u_0 = \infty$ in the Θ-matrix method, then there is no optimal solution of $P^*|\mu'$, $D^*|\mu'$ for $\mu' > \mu$, if $u_0 < \infty$ then (u_i, w_{ij}), $x_{ij} + \theta\xi_{ij}$ is a optimal solution of $P^*|\mu+\theta$, $D^*|\mu+\theta$ where $0 < \theta \leqq \theta_0$.

§ 4. CPM

CPM (the critical path method) is the method for solving the following parametric linear programming

$D|\lambda$

$$y_{ij} + t_i - t_j \leqq 0 \quad \text{for} \quad (i, j) \in B, \tag{D1}$$

$$d_{ij} \leqq y_{ij} \leqq D_{ij}, \tag{D2}$$

$$t_n - t_0 = \lambda, \tag{D3}$$

$$\text{maximize} \quad U(\lambda) = \sum_{(i, j) \in B)} c_{ij} y_{ij}. \tag{D4}$$

Again B is the set of all branches of given network with $n+1$ nodes and m brances. Here branches are called activities or jobs, t_i are node-times, i.e. starting times of jobs (i, j) for $(i, j) \in B$, and y_{ij} are durations for jobs (i, j). D_{ij} resp. d_{ij} can be interpreted as normal resp. crash duration for job (i, j), λ means the total duration of this scheduling. $U(\lambda) = \sum_{(i, j) \in B} c_{ij} y_{ij}$ where $c_{ij} \geqq 0$ is called project utility function. The dual problem of $D|\lambda$, considered as the primal has the following form

$P|\lambda$

$$f_{ij}, g_{ij}, h_{ij} \geqq 0 \quad \text{for} \quad (i, j) \in B, \tag{P1}$$

$$\sum_{(i, j) \in B} f_{ij} = \sum_{(j, k) \in B} f_{jk} \quad \text{for every nodes } j (\neq 0, n), \tag{P2}$$

$$\sum_{(0, j) \in B} f_{0j} = \sum_{(i, n) \in B} f_{in} = \mu, \tag{P3}$$

$$f_{ij} + g_{ij} - h_{ij} = c_{ij}, \tag{P4}$$

$$\text{minimize} \quad \lambda\mu + \sum_{(i, j) \in B} D_{ij}\, g_{ij} - \sum_{(i, j) \in B} d_{ij} h_{ij}. \tag{P5}$$

4.1. Kelley and Fulkerson's method.

4.1.1. To find an optimal solution of D$|\lambda$, for a sufficiently large λ. We put $y_{ij}=D_{ij}$, $t_0=0$, $t_j=\max_{(i,j)\in B}(y_{ij}+t_i)$ for $j(\neq 0)$, and $M=\max_{(i,n)\in B}(D_{in}+t_i)$. Then y_{ij}, t_i and $t_n=\lambda$ give an optimal solution of $D|\lambda$ for $\lambda\geqq M$.

4.1.2. Solution of RP$|\lambda$

RP$^{(1)}|\lambda$

$$f_{ij},\ g_{ij},\ h_{ij}\geqq 0 \qquad (\mathrm{RP}^{(1)}1)$$

$$\sum_{(i,j)\in B} f_{ij}=\sum_{(j,k)\in B} f_{jk} \quad \text{for} \quad j(\neq 0,\ n), \qquad (\mathrm{RP}^{(1)}2)$$

$$f_{ij}+g_{ij}-h_{ij}=c_{ij}, \qquad (\mathrm{RP}^{(1)}3)$$

$$\left.\begin{array}{lll} f_{ij}=0, & \text{if} & y_{ij}+t_i-t_j<0, \\ g_{ij}=0, & \text{if} & y_{ij}<D_{ij}, \\ h_{ij}=0, & \text{if} & y_{ij}>d_{ij}, \end{array}\right\} \qquad (\mathrm{RP}^{(1)}4)$$

$$\text{maximize} \quad \sum_{(i,n)\in B} f_{in}=\sum_{(0,j)\in B} f_{0j}. \qquad (\mathrm{RP}^{(1)}5)$$

RD$|\lambda$

$$\sigma_{ij}+\delta_i-\delta_j\geqq 0, \qquad \text{if} \quad y_{ij}+t_i-t_j=0, \qquad (\mathrm{RD}1)$$

$$\sigma_{ij}\geqq 0, \qquad \text{if} \quad y_{ij}=D_{ij}, \qquad (\mathrm{RD}2)$$

$$\sigma_{ij}\leqq 0, \qquad \text{if} \quad y_{ij}=d_{ij}, \qquad (\mathrm{RD}3)$$

$$-\delta_0+\delta_n=1, \qquad (\mathrm{RD}4)$$

$$\text{minimize} \quad \sum_{(i,j)\in B} c_{ij}\sigma_{ij}. \qquad (\mathrm{RD}5)$$

RP$^{(1)}|\lambda$ has various equivalent forms, that is

RP$^{(2)}|\lambda$

$$f_{ij}\geqq 0 \qquad (\mathrm{RP}^{(2)}1)$$

$$\sum_{(i,j)\in B} f_{ij}=\sum_{(j,k)\in B} f_{jk} \qquad \text{for} \quad j(\neq 0,\ n), \qquad (\mathrm{RP}^{(2)}2)$$

$$\left.\begin{array}{lll} f_{ij}=0, & \text{if} & D_{ij}+t_i-t_j<0, \\ f_{ij}\geqq c_{ij}, & \text{if} & D_{ij}+t_i-t_j>0, \\ f_{ij}\leqq c_{ij}, & \text{if} & d_{ij}+t_i-t_j<0, \end{array}\right\} \qquad (\mathrm{RP}^{(2)}3)$$

$$\text{maximize} \quad \sum_{(i,\ n)\in B} f_{in}\left(=\sum_{(0,\ j)\in B} f_{0j}\right). \qquad (\mathrm{RP}^{(2)}4)$$

$\mathrm{RP}^{(3)}|\lambda$

$$f_{ij}\geqq 0, \qquad (\mathrm{RP}^{(3)}1)$$

$$\sum_{(ij)\in B} f_{ij}=\sum_{(ik)\in B} f_j \qquad \text{for} \quad j(\neq 0,\ n), \qquad (\mathrm{RP}^{(3)}2)$$

$$\left.\begin{array}{lll} f_{ij}\leqq c_{ij}, & \text{if} & (i,\ j)\in Q_1\cap Q_2, \\ f_{ij}=c_{ij}, & \text{if} & (i,\ j)\in Q_1-(Q_2\cup Q_3\cup Q_4), \\ f_{ij}\geqq c_{ij}, & \text{if} & (i,\ j)\in Q_1\cap Q_4, \\ f_{ij}=0 & \text{if} & (i,\ j)\in B-Q_1, \end{array}\right\} \qquad (\mathrm{RP}^{(3)}3)$$

$$\text{maximize} \quad \sum_{(i,\ n)\in B} f_{in}\left(=\sum_{(0,\ j)\in B} f_{0j}\right), \qquad (\mathrm{RP}^{(4)}4)$$

here,

$$Q_1=\{(i,\ j)|\ y_{ij}+t_i-t_j=0\},$$

$$Q_2=\{(i,\ j)|\ y_{ij}=D_{ij}>d_{ij}\},$$

$$Q_3=\{(i,\ j)|d_{ij}=y_{ij}=D_{ij}\},$$

$$Q_4=\{(i,\ j)|\ y_{ij}<D_{ij}\}.$$

$\mathrm{RP}^{(4)}|\lambda$

$$f(i,\ j,\ k)\geqq 0 \qquad \text{for} \quad (i,\ j)\in B, \quad k=1,\ 2, \qquad (\mathrm{RP}^{(4)}1)$$

$$\sum_{(i,\ j)\in B}(f(i,j,1)+f(i,j,2))=\sum_{(j,\ k)\in B}(f(j,k,1)+f(j,k,2))$$

$$\text{for} \quad j(\neq 0,\ n), \qquad (\mathrm{RP}^{(4)}2)$$

$$f(i,\ j,\ k)\leqq c(i,\ j,\ k), \qquad k=1,\ 2, \qquad (\mathrm{RP}^{(4)}3)$$

$$\left.\begin{array}{lll} f(i,j,k)=c(i,j,k), & \text{if} & a(i,j,k)+t_i-t_j>0,\ k=1,2, \\ f(i,j,k)=0, & \text{if} & a(i,j,k)+t_i-t_j<0,\ k=1,2, \end{array}\right\} \qquad (\mathrm{RP}^{(4)}4)$$

$$\text{maximize} \sum_{(in)\in B} f(i, n, 1)+f(i, n, 2)), \qquad ((RP^{(4)}5)$$

here,

$$c(i, j, 1)=c_{ij}, \qquad a(i, j, 1)=D_{ij},$$

$$c(i, j, 2)=\infty, \qquad a(i, j, 2)=d_{ij}.$$

Proposition 4.1.

$$RP^{(1)}|\lambda, \quad RP^{(2)}|\lambda, \quad RP^{(3)}|\lambda, \quad RP^{(4)}|\lambda$$

are mutually equivalent problems.

Lemma.

In D$|\lambda$, we assume without any loss of generality, that $t_0=0$, $t_n=\lambda$ and $y_{ij}=\min(D_{ij}, t_j-t_i)$.

Using the lemma and putting

$$f_{ij}+g_{ij}-h_{ij}=c_{ij}, \qquad g_{ij}=\max(0, c_{ij}-f_{ij}),$$

$$h_{ij}=\max(0, f_{ij}-c_{ij}), \qquad f_{ij}=f(i, j, 1)+f(i, j, 2)$$

and

$$f(i, j, 1)=\min(c_{ij}, f_{ij}), \qquad f(i, j, 2)=\max(0, f_{ij}-c_{ij}).$$

It is easily seen that we can transform any one of four equivalents of RP$|\lambda$ into another. By the first relation of (RP$^{(4)}$4), $a(i, j, 2)+t_i-t_j>0$ implies $f(i, j, 2)=\infty$. Actually, since $a(i, j, 2)+t_i-t_j=d_{ij}+t_i-t_j\leqq 0$, the statement, with an always false premise, trivially holds. Kelley took up the form of RP$^{(3)}|\lambda$, and Fulkerson studied the form of RP$^{(4)}|\lambda$. It is to be noted that RP$^{(4)}|\lambda$ has the same form of maximum flow problem of RP$|\lambda$ in 3.1. Therefore, we max solve any one of the four equivalents by the labeling method in 3.1.

4.1.3. An optimal solution (σ_{ij}, δ_i) is constructed by the labeling method in RP$|\lambda$ analogously to the way given in 3.1.2.

Put

$$\rho_{ij}=\sigma_{ij}+\delta_i-\delta_j$$

and

$$\theta_1 = \min_{\rho_{ij}<0} (y_{ij}+t_i-t_j)/\rho_{ij},$$

$$\theta_2 = \min_{\sigma_{ij}>0} (y_{ij}-d_{ij})/\sigma_{ij},$$

$$\theta_3 = \min_{\sigma_{ij}<0} (y_{ij}-D_{ij})/\sigma_{ij},$$

$$\theta_0 = \min(\theta_1, \theta_2, \theta_3).$$

*.1.4. The optimal solution (f_{ij}, g_{ij}, h_{ij}) of RP$|\lambda$ resp. $(y_{ij}-\theta_0\sigma_{ij}, t_i-\delta_i\theta_0)$ is an optimal solution of P$|\lambda-\theta_0$ resp. D$|\lambda-\theta_0$.

4.2. Iri's method and CPM

The problems D*$|\mu$ or P*$|\mu$ in CPM can be defined by

D*$|\mu$

$$f_{ij},\ g_{ij},\ h_{ij} \geqq 0, \tag{D*1}$$

$$f_{ij}+g_{ij}-h_{ij}=c_{ij}, \tag{D*2}$$

$$\sum_{(i,\ j)\in B} f_{ij} = \sum_{(j,\ k)\in B} f_{jk} \quad \text{for} \quad j\ (\neq 0,\ n), \tag{D*3}$$

$$\sum_{(0,\ j)\in B} f_{0j} = \sum_{(i,n)\in B} f_{in} = \mu, \tag{D*4}$$

$$\text{minimize} \quad \sum_{(i,\ j)\in B} (D_{ij}g_{ij}-d_{ij}h_{ij}). \tag{D*5}$$

By putting

$$f_{ij} = f(i,\ j,\ 1)+f(i,\ j,\ 2),$$

$$f(i,\ j,\ 1)=\min(c_{ij},\ f_{ij})$$

and

$$f(i,\ j,\ 2)=\max(0,\ f_{ij}-c_{ij})$$

according to Fulkerson we have the following problem which is equivalent to D*$|\mu$.

D*'$|\mu$

$$\left.\begin{aligned} &0\leqq f(i,\ j,\ 1)\leqq c_{ij},\\ &0\leqq f(i,\ j,\ 2), \end{aligned}\right\} \tag{D*'1}$$

$$\sum_{(i,\ j)\in B} (f(i,\ j,\ 1)+f(i,\ j,\ 2))$$

$$= \sum_{(j,\ k)\in B} (f(j,\ k,\ 1)+f(j,\ k,\ 2)) \qquad \text{for} \quad j\ (\neq 0,\ n), \qquad \text{(D*'2)}$$

$$\sum_{(0,\ j)\in B} (f(0,\ j,\ 1)+f(0,\ j,\ 2)$$
$$= \sum_{(i,\ n)\in B} (f(i,\ n,\ 1)+f(i,\ n,\ 2))=\mu, \qquad \text{(D*'3)}$$

$$\text{minimize} \quad \sum_{(i,\ j)\in B} (-D_{ij}f(\text{i},\ j,\ 1)-d_{ij}f(i,\ j,\ 2)) \qquad \text{(D*'4)}$$

This problem has the same form to D*$|\mu$ in §3 except that there exist two branches from i to j and d_{ij} of (D*4) in §3 are non-positive in this case. But the entire theory of Iri can be applied to this case.

4.2.1. Θ-matrix method for solving RP*$|\mu$

RP*$|\mu$

$$y_{ij}+t_i-t_j \leqq 0, \qquad \text{(RP*1)}$$

$$d_{ij} \leqq y_{ij} \leqq D_{ij}, \qquad \text{(RP*2)}$$

$$\left.\begin{array}{ll} y_{ij}+t_i-t_j=0, & \text{if } f_{ij}>0, \\ y_{ij}=D_{ij}, & \text{if } g_{ij}>0, \\ y_{ij}=d_{ij}, & \text{if } h_{ij}>0, \end{array}\right\} \qquad \text{(RP*3)}$$

$$\text{minimize} \quad \lambda=t_n-t_0. \qquad \text{(RP*4)}$$

RD*$|\mu$

$$\left.\begin{array}{ll} \xi_{ij} \geqq 0, & \text{if } f_{ij}=0 \\ \eta_{ij} \geqq 0, & \text{if } g_{ij}=0, \\ \varepsilon_{ij} \geqq 0, & \text{if } h_{ij}=0, \end{array}\right\} \qquad \text{(RD*1)}$$

$$\xi_{ij}+\eta_{ij}-\varepsilon_{ij}=0, \qquad \text{(RD*2)}$$

$$\left.\begin{array}{ll} \sum_{(i,\ j)\in B} \xi_{ij} = \sum_{(j,\ k)\in B} \xi_{jk}, & \text{for } j\ (\neq 0,\ n), \\ \sum_{(0,\ j)\in B} \xi_{0j} = \sum_{(i,\ n)\in B} \xi_{in}=1, & \end{array}\right\} \qquad \text{(RD*3)}$$

$$\text{minimize} \quad \sum_{(i,\ j)\in B} D_{ij}\eta_{ij} - \sum_{(i,\ j)\in B} d_{ij}\varepsilon_{ij} \qquad \text{(RD*4)}$$

$$\theta_j^i = \begin{cases} 0 & \text{if } i=j \\ -D_{ji} & \text{if } (j, i)\in B \text{ and } f(j, i, 1)<c_{ji} \text{ (i.e. } f_{ji}<c_{ji}) \\ -d_{ji} & \text{if } (j, i)\in B \text{ and } f(j, i, 1)=c_{ji} \text{ (i.e. } f_{ji}\geqq c_{ji}) \\ D_{ij} & \text{if } (i, j)\in B \text{ and } f(i, j, 1)>0,\ f(i, j, 2)=0 \text{ (i.e. } 0<f_{ij}\leqq c_{ij}) \\ d_{ij} & \text{if } (i, j)\in B \text{ and } f(i, j, 2)>0 \text{ (i.e. } f_{ij}>c_{ij}) \\ \infty & \text{otherwise.} \end{cases}$$

$\overset{k}{\tau}_i$ is defined recursively for any nodes i, as follows.

$$\overset{0}{\tau}_i = \begin{cases} \infty, & \text{if } i\neq n, \\ 0, & \text{if } i=n, \end{cases}$$

$$\overset{k+1}{\tau}_i = \min_j(\theta_i^j + \overset{k}{\tau}_j).$$

Since $\overset{k}{\tau}_i$ converges to $\overset{\infty}{\tau}_i$, we put $t_i'=\overset{\infty}{\tau}_i$ for every nodes i, and put $t_i=t_i'-t_0'$ and $y_{ij}=\min(D_{ij},\ t_j-t_i)$. Then $(t_i\ y_{ij})$ is an optimal solution of $\mathrm{RP}^*|\mu$, (i.e. minimizing $\lambda=t_n$) satisfying $t_0=0$.

4.2.2. $\overset{\infty}{\tau}_0=t_0'$ can be represented in the form $\sum_{i,j} \pm D_{ij}\pm d_{ij}$ provided that $t_0'\neq\infty$. If $+D_{ij}$ resp. $-D_{ij}$ appears under the summation we put $\eta_{ij}=1$ resp. $\eta_{ij}=-1$. On the other hand, if $+d_{ij}$ resp. $-d_{ij}$ appears, we put $\varepsilon_{ij}=-1$ resp. $\varepsilon_{ij}=1$ and $\xi_{ij}=\varepsilon_{ij}-\eta_{ij}$. Otherwise $\eta_{ij}=\varepsilon_{ij}=0$. Thus $(\xi_{ij},\ \eta_{ij},\ \varepsilon_{ij})$ is an optimal solution of $\mathrm{RD}^*|\mu$ and if we put

$$\theta_1 = \min_{\xi_{ij}<0}(-f_{ij}/\xi_{ij}),$$

$$\theta_2 = \min_{\eta_j=-1} g_{ij},$$

$$\theta_3 = \min_{\epsilon_{ij}=-1} h_{ij},$$

$$\theta_0 = \min(\theta_1,\ \theta_2,\ \theta_3),$$

$$(f_{ij}+\theta\xi_{ij},\ h_{ij}+\theta\eta_{ij},\ h_{ij}+\theta\varepsilon_{ij})$$

is an optimal solution of $\mathrm{D}^*|\mu+\theta$, where $0<\theta\leqq\theta_0$. Further, if t_0' obtained obtained by Θ-matrix method is infinite, then there is no optimal solution of $\mathrm{D}^*|\mu'$ for $\mu'>\mu$.

§ 5. MULTI-PARAMETRIC PROGRAMMING

Now, we consider the following problem with P parameters.

$P|\lambda_1, \cdots, \lambda_p$

$$x_j \geqq 0, \qquad \text{if } j \in S, \tag{P1}$$

$$\left.\begin{aligned} &\sum_j a_{ij}x_j \geqq b_i, && \text{if } i \in T, \\ (\text{or } &\sum_j a_{ij}x_j - u_i = b_i, \quad u_i \geqq 0, && \text{if } i \in T,) \\ &\sum_j a_{ij}x_j = b_i, && \text{if } i \notin T, \end{aligned}\right\} \tag{P2}$$

$$\text{minimize} \quad \sum_j (c_j + \lambda_1 d_j{}^1 + \cdots + \lambda_p d_j{}^p) x_j. \tag{P3}$$

$D|\lambda_1, \cdots, \lambda_p$

$$\left.\begin{aligned} &\sum_i a_{ij} y_i + w_j = c_j + \sum_{l=1}^{p} \lambda_l d_j^l, \quad w_j \geqq 0, && \text{if } j \in S. \\ &\sum_i a_{ij} y_i = c_j + \sum_{l=1}^{p} \lambda_l d_j^l && \text{if } j \notin S, \end{aligned}\right\} \tag{D1}$$

$$y_i \geqq 0, \qquad \text{if } i \in T, \tag{D2}$$

$$\text{maximize} \quad \sum_i y_i b_i . \tag{D3}$$

Given one optimal solution of (y_i, w_j) of $D|\lambda_1, \cdots, \lambda_p$ we shall give a sufficient condition which ensures a procedure to solve $D|\lambda_1+\theta_1, \cdots, \lambda_p+\theta_p$. For this purpose we introduce variables $(\sigma_i^1), \cdots, (\sigma_i^p)$ and a p restricted dual problem as follows,

$RD^l(l=1, 2, \cdots, p)$

$$\left.\begin{aligned} &\sum_i a_{ij}\sigma_i^l \leqq d_j^l, && \text{if } w_j = 0, \; j \in S, \\ &\sum_i a_{ij}\sigma_i^l = d_j^l, && \text{if } j \notin S, \end{aligned}\right\} \tag{RD^{l}1}$$

$$\sigma_i^l \geqq 0, \qquad \text{if } y_i = 0, \; i \in T, \tag{RD^{l}2}$$

$$\text{maximize} \quad \sum_i \sigma_i^l b_i.$$

The dual problem of RD^l far each $l=1,\cdots,p$ is given by

RP^l

$$x_j \geqq 0 \qquad j \in S, \qquad \text{(RP1)}$$

$$\left.\begin{array}{ll} \sum_j a_{ij}x_j - u_i = b_i, \quad u_i \geqq 0, & \text{if } i \in T, \\ \sum_j a_{ij}x_j = b_i, & \text{if } i \notin T, \end{array}\right\} \qquad \text{(RP2)}$$

$$\left.\begin{array}{l} \sum_{j\in S} x_j w_j = 0, \\ \sum_{i\in T} u_i y_i = 0, \end{array}\right\} \qquad \text{(RP3)}$$

$$\text{minimize} \quad \sum_j d_j^l x_j. \qquad \text{(RP}^l\text{4)}$$

Generally speaking, the optimal solutions of RP^l depend on l, but in some particular cases, single solution (x_j) happens to be the optimal for RP^l $l=1, 2, \cdots, p$ simultaneously. As a condition which plays an essential role here, and is somewhat stronger than the assumption made throughout this paper, we assume the following condition C.

[Condition C]; There exists a simultaneous optimal solution (x_j) of RP^l, $l=1,3,\cdots,p$. The following proposition clearly hold.

Proposition 5.1.

Suppose thas (y_i) is an optimal solution of $D|\lambda_1,\ldots,\lambda_p$, and σ_i^l are optimal solution of RD^l's. If the dondition C holds for RP^l and if (x_j) is the simultaneous optimal solution of all RP^l, then (x_j) resp. $(y_i+\sum_{l=1}^{p}\sigma_i^l\theta_l)$ is the optimal solution of $P|\lambda_1+\theta_1,\ldots,\lambda_p+\theta_p$ resp. $D|\lambda_1+\theta_1,\ldots,\lambda_p+\theta_p$. Where $\theta_1,\cdots,\theta_p$ satisfy the following inequalities

$$\sum_{l=1}^{p}\theta_l\,\beta_j^l \geqq -w_j, \qquad \text{if } w_j>0 \text{ and } j\in S, \qquad (5.1)$$

$$\sum_{l=1}^{p}\theta_l\,\sigma_i^l \geqq -y_i, \qquad \text{if } y_i>0 \text{ and } i\in T, \qquad (5.2)$$

where

$$\beta_j^l = d_j^l - \sum a_{ij}\,\sigma_i^l.$$

§ 6. TRANSPORTATION NETWORK FLOW PROBLEM WıTH MANY SOURCES

Let N be a network with m branches and $n+p$ nodes which contains p sources $0_1, \dots, 0_p$, and one sink n. Let B be the set of all branches of N. We consider the following transportation netword flow problem with p sources as a multi-parametric problem. As previously, we also formulate the other problems related to it.

$D^*|\mu_1, \dots, \mu_p$

$$\sum_{(i,j)\in B} x_{ij} = \sum_{(j,k)\in B} x_{jk} \quad \text{for every nodes } j\,(\neq 0_l, n), \tag{D*1}$$

$$0 \leqq x_{ij} \leqq c_{ij}, \tag{D*2}$$

$$\left.\begin{aligned} &\sum_{(0_l,j)\in B} x_{0_l j} = \mu_l, \qquad (l=1, 2, \dots, p) \\ &\sum_{(i,n)\in B} x_{in} = \mu_1 + \dots + \mu_p, \end{aligned}\right\} \tag{D*3}$$

$$\text{minimize} \quad \sum_{(i,j)\in B} d_{ij} x_{ij}. \tag{D*4}$$

$P^*|\mu_1, \dots, \mu_p$

$$w'_{ij} = d_{ij} + u_j - u_i + w_{ij} \geqq 0 \quad \text{for} \quad (i,j)\in B, \tag{P*1}$$

$$w_{ij} \geqq 0 \quad \text{for} \quad (i,j)\in B, \tag{P*2}$$

$$\text{maximize} \quad \sum_{l=1}^{p} \mu_l (u_{0_l} - u_n) - \sum_{(i,j)\in B} c_{ij} w_{ij}. \tag{P*3}$$

RP^{*l}

$$w'_{ij} = d_{ij} + u_j - u_i + w_{ij} \geqq 0 \quad \text{for} \quad (i,j)\in B, \tag{RP*1}$$

$$w_{ij} \geqq 0 \quad \text{for} \quad (i,j)\in B, \tag{RP*2}$$

$$\left.\begin{aligned} &w'_{ij} = 0, \quad \text{if} \quad x_{ij} > 0, \\ &w_{ij} = 0, \quad \text{if} \quad x_{ij} < c_{ij}, \end{aligned}\right\} \tag{RP*3}$$

$$\text{maximize} \quad u_{0_l} - u_n. \tag{RP*l4}$$

RD*^{l}

$$\left.\begin{array}{lll} \xi^l_{ij} \geqq 0 & \text{if} \ \ x_{ij}=0, & (i, j)\in B, \\ \xi^l_{ij} \leqq 0 & \text{if} \ \ x_{ij}=c_{ij}, & (i, j)\in B, \end{array}\right\} \qquad (\mathrm{RD}^{*l}1)$$

$$\sum_{(i, j)\in B} \xi^l_{ij} = \sum_{(j, k)\in B} \xi_{jk} \quad \text{for each nodes} \ \ j(\neq 0_l, \mathrm{n}), \qquad (\mathrm{RD}^{*l}2)$$

$$\left.\begin{array}{l} \sum_{(0_l, j)\in B} \xi^l_{0_l j}=1, \\ \sum_{(i, n)\in B)} \xi^l_{(i,\ n)}=1, \end{array}\right\} \qquad (\mathrm{RD}^{*l}3)$$

$$\text{minimize} \quad \sum_{(i, j)\in B} \xi^l_{ij} d_{ij} . \qquad (\mathrm{RD}^{*l}4)$$

Fortunately, the condition C is satisfied by RP*^{l}. Because, particular feasible solutions u_l obtained by Iri's Θ-matrix method happen to be the maximal one among all feasible solution of RP*^{l} for all l (cf. Proposition 3.2). Therefore, the optimal solution of RD*^{l} can be constructed similarly as in 3.2.2. We express u_{0_l} as $u_{0_l}=\sum \pm d_{ij}$, where (i, j) ranges over some subset of B. For the $(i, j)\in B$ for which $+d_{ij}$ appears under the summation we put $\xi^l_{ij}=1$. For those for which $-d_{ij}$ appears, we put $\xi^l_{ij}=-1$. Otherwise we put $\xi^l_{ij}=0$. Then, it is easy tosee that (ξ^l_{ij}) is an optimal feasible solution of RD*^{l}. $\theta_1, \cdots, \theta_p$ are determined by

$$\sum_{l=1}^{p} \xi^l_{ij}\theta_l \geqq -x_{ij} \qquad \text{for} \quad x_{ij}>0, \qquad (6.1)$$

$$\sum_{l=1}^{p} \xi^l_{ij}\theta_l \leqq c_{ij}-x_{ij} \qquad \text{for} \quad c_{ij}-x_{ij}>0 \qquad (6.2)$$

It seems to be natural to impose the following conditions on θ_l's adding to (6.1) and (6.2)

$$\theta_1 \geqq 0, \ \cdots, \theta_p \geqq 0 \qquad (6.3)$$

and

$$\text{maximize} \quad \theta_1+\theta_2+\cdots+\theta_p . \qquad (6.4)$$

Having got an optimal solution $(x_j+\sum_{l=1}^{p}\xi_{ij}^{l}\theta_l)$ of $D^*|\mu_1+\theta_1, \cdots, \mu_p+\theta_p$, we now take it as a starting point from which we carry on the procedure of solving RP^{*l} by the Θ-matrix method.

Remark 1. We may consider another multi-parametric problem $D|\lambda_1, \cdots, \lambda_p$ with $(\lambda_1, \cdots, \lambda_p)=(u_{0_1}-u_n, \cdots, u_{0p}-u_n)$ as parameters. In this case RP_l may be interpreted as that the maxima of all input flows $(\mu_1, \cdots, \mu_p)=(\sum_{(0_1 j)\in B} x_{0_1 j}, \cdots, \sum_{(0p, j)\in B} x_{0p j})$ are looked for. But here C is not satisfied by RP^l, that is, in general there doesn't exist the simultaneous maximal flows.

Remark 2. CPM problems with many starting nodes can also be solved by this method.

§7. A NUMERICAL EXAMPLE OF CAPACITATED HITCHCOCKPROBLEM TREATED AS A MULTI-PARAMETRIC PROGRAMMING

Hitchcook problem is

$$\sum_{j=1}^{n} x_{ij}=a_i, \qquad i=1, 2, \cdots, m,$$

$$\sum_{i=1}^{m} x_{ij}=b_j, \qquad j=1, 2, \cdots, n,$$

where

$$\sum_i a_i=\sum_j b_j,$$

$$0\leqq x_{ij}\leqq c_{ij},$$

$$\text{minimize} \quad \sum d_{ij}\, x_{ij}.$$

We regard above capacitated Hitchcock problem as the following multi-parametric programming with parameters $\mu_1, \mu_2, \cdots, \mu_m$

$D^*|\mu_1, \cdots, \mu_m$

$$\sum_j x_{ij}=\mu_i,$$

$$\sum_i x_{ij} \leqq b_j,$$

$$0 \leqq x_{ij} \leqq c_{ij},$$

$$\text{minimize} \quad \sum d_{ij} x_{ij}.$$

$P^*|\mu_1, \cdots, \mu_m$

$$v_j \geqq 0,$$

$$w_{ij} \geqq 0,$$

$$d_{ij} + v_j - u_i + w_{ij} \geqq 0,$$

$$\text{maximize} \quad \sum \mu_i u_i - \sum_j b_j v_j - \sum c_{ij} w_{ij},$$

if we add $\mu = \mu_1 + \cdots + \mu_m$, then we have one-parameter programming $D^*|\mu$.

7.1. Aprocedure for solving of $P^*|\mu_1, \cdots, \mu_m$ or $D^*|\mu_1, \cdots, \mu_m$ is as follows.

a. Fiast of all we put $x_{ij}=0$ for all i, j, $u_i = v_j = 0$ and $w_{ij}=0$, so we have the optimal solution of $D^*|o, \cdots, o$, $P^*|o, \cdots, o$.

b. To solve $RP^*|\mu_1, \cdots, \mu_m$ and RD^*

RP^{*l}

$$v_j \geqq 0,$$

$$w_{ij} \geqq 0,$$

$$d_{ij} + v_j - u_i + w_{ij} \geqq 0,$$

$$d_{ij} + v_j - u_i + w_{ij} = 0, \qquad \text{if} \quad x_{ij} > 0,$$

$$w_{ij} = 0, \qquad \text{if} \quad x_{ij} < c_{ij},$$

$$\text{maximize } u_l, \quad l = 1, 2, \cdots, m.$$

$RD^{*(l)}$

$$\sum_j \xi_{ij}^l = \begin{cases} 0 & \text{if} \quad i \neq l, \\ 1 & \text{if} \quad i = l, \end{cases}$$

$$\xi_{ij}^l \geqq 0, \qquad \text{if} \quad x_{ij} = 0,$$

$$\xi_{ij}^l \leqq 0, \qquad \text{if} \quad x_{ij} = c_{ij},$$

$$\sum_i \xi^l_{ij} \leqq 0, \qquad \text{if} \quad \sum_i x_{ij} = b_j,$$

$$\text{minimize} \quad \sum_{i,j} \xi^l_{ij} d_{ij}.$$

c. Simultaneous optimal solutions of RP*^{l} for $l=1, \cdots, m$, is determined as follows.

$$\overset{0}{\beta}_j = \begin{cases} 0, & \text{if} \quad \overline{b_j} > 0, \quad \text{where} \quad \overline{b_j} = b_j - \sum_i x_{ij}, \\ \infty, & \text{if} \quad \overline{b_j} = 0, \end{cases}$$

$$\overset{2k+1}{\alpha_i} = \min_{j(x_{ij}<c_{ij})} (d_{ij} + \overset{2k}{\beta_j}),$$

$$\overset{2k+2}{\beta_j} = \min \{\overset{2k}{\beta_j}, \min_{i(x_{ij}>0)} (\overset{2k+1}{\alpha_i} - d_{ij})\},$$

$$\begin{cases} u_i = \overset{\infty}{\alpha_i}, \\ v_j = \overset{\infty}{\beta_j}. \end{cases}$$

d. Optimal solution of RD*^{l} is determined as follows. When we represent u_l in the form $\sum \pm d_{ij}$, if $+d_{ij}$ resp. $-d_{ij}$ appears in $\sum \pm d_{ij}$, then we put $\xi^l_{ij}=1$ resp. $\xi^l_{ij}=-1$, otherwise $\xi^l_{ij}=0$.

e. Determination of θ_l's

We determine θ_l under the conditions

$$\sum_{l=1}^{m} \theta_l \sum_i \xi^l_{ij} \leqq \overline{b_j}, \qquad \text{if} \quad \overline{b_j} > 0,$$

$$\sum_l \theta_l \xi^l_{ij} \geqq -x_{ij}, \qquad \text{if} \quad x_{ij} > 0,$$

$$\sum_l \theta_l \xi^l_{ij} \leqq c_{ij} - x_{ij}, \quad \text{if} \quad x_{ij} < c_{ij}$$

and $\theta_l \geqq 0$, making $\sum_l \theta_l$ as large possible.

f. To change flows

μ_l change to $\mu_l + \theta_l$

x_{ij} change to $x_{ij} + \sum_l \xi^l_{ij} \theta_l$

$$\sum x_{ij}d_{ij} \text{ change to } \sum x_{ij}d_{ij} + \sum_l \theta_l \sum_{i,j} \xi^l_{ij} d_{ij}.$$

Remark 1. As easily seen, $\sum_j \xi^l_{ij}=0$ or 1 for j such that $\bar{b}_j>0$, $\sum_l \theta_l \sum_j \xi^l_{ij} \leqq \bar{b}_j$ are very simple form, but the author is not aware of any simple algorithm other than simplex method to determine θ_l's.

Remark 2. The Hdtchcock problem can be treated as a one-parameter problem P*$|\mu$ and D*$|\mu$ dealt with in 3.2, by adding another source node o and m branches $(o1), \cdots, (om)$ related to it. An optimal solution of RP*$|\mu$ is given by u_i and v_j found in C together with $u_0 = \min_{(a_i>0)} u_i$ where $a_i = a_i - \sum_j x_{ij}$. While that of RD*$|\mu$, $\xi^{(0)}_{ij}$ is equal to ξ^l_{ij} with l for which $u_0 = u_l$. Further, an optimal solution of D*$|\mu+\theta$ is given by $x_{ij} + \theta \xi^{(0)}_{ij}$. θ_0 is characterized as the maximal θ satisfying $\theta \sum_i \xi^{(0)}_{ij} \leqq \bar{b}_j$ for j such that $\bar{b}_j>0$, and $0 \leqq x_{ij} + \theta \xi^{(0)}_{ij} \leqq c_{ij}$.

Values of d_{ij}, c_{ij} in capacitated Hitchcock Problem are given as follows.

Table of d_{ij}, a_i,

a_i \ b_i	3	5	4	6	3
9	10	20	5	9	10
4	3	10	8	30	6
8	1	20	7	10	4

Table of c_{ij}

2	3	5	5	1
1	8	2	1	1
3	1	2	2	3

Step 0 Initial solution of x_{ij} and μ_l

μ_l	$\bar{a}_i$ \ $\bar{b}_j$	3	5	4	6	3
0	9	0	0	0	0	0
0	4	0	0	0	0	0
0	8	0	0	0	0	0

Step 1.

The case solved as a multi-parametric problem

(a) $u_i=\sum \pm d_{ij}$, $u_1=d_{13}$, $u_2=d_{21}$, $u_3=d_{31}$

(b) conditions for θ's

$$\sum_{l=1}^{m} \theta_l \sum_i \xi_{ij}^{(l)} \leqq \bar{b}_j \quad \text{for} \quad \bar{b}_i>0$$

$$\theta_2+\theta_3 \leqq 3, \qquad \theta_1 \leqq 4$$

$$\sum_l \theta_l \xi_{ij}^{(l)} \leqq -x_{ij} \quad \text{for} \quad x_{ij}>0$$

$$\sum_l \theta_l \xi_{ij}^{(l)} \leqq c_{ij}-x_{ij} \quad \text{for} \quad c_{ij}-x_{ij}>0$$

$$\theta_1 \leq 5 \qquad \theta_2 \leq 1 \qquad \theta_3 \leq 3$$

$$\theta_l \leqq \bar{a}_l$$

$$\theta_1 \leq 9 \qquad \theta_2 \leq 4 \qquad \theta_3 \leq 8$$

(c) determination of θ_l and next μ_l

$$\theta_1=4, \qquad \theta_2=1, \qquad \theta_3=2$$

$$\mu_1=4, \qquad \mu_2=1, \qquad \mu_3=2$$

(d) change of x_{ij}

$$x_{ij} \rightarrow x_{ij}+\sum_l \theta_l \xi_{ij}^{(l)}$$

$$x_{13}=4, \qquad x_{21}=1, \qquad x_{31}=2$$

μ_i	$\bar{a}_i$ \ $\bar{b}_j$	0	5	0	6	3
4	5	0	0	4	0	0
1	3	1	0	0	0	0
2	6	2	0	0	0	0

Sept 2

(a) $u_1=d_{14}$, $u_2=d_{25}$, $u_3=d_{35}$

(b) $\theta_1\leqq 6$, $\theta_2+\theta_3\leqq 3$ $\theta_1\leqq 5$, $\theta_2\leqq 1$, $\theta_3\leqq 3$,

$\theta_1\leqq 5$, $\theta_2\leqq 3$, $\theta_3\leqq 6$

(c) $\theta_1=5$, $\theta_2=0$, $\theta_3=3$

(d) $x_{14}=5$, $x_{25}=0$, $x_{35}=3$, $\mu_1=9$, $\mu_2=1$, $\mu_3=5$

μ_i	$\bar{a}_i$ \ $\bar{b}_j$	0	5	0	1	0
9	0	0	0	4	5	0
1	3	1	0	0	0	3
5	3	2	0	0	0	3

Step 3

(a) $u_1=d_{15}+d_{31}+d_{22}-d_{21}-d_{35}$,

$u_2=d_{22}$,

$u_3=d_{31}+d_{22}-d_{21}$

(1) ξ^1_{ij}

0	0	0	0	1
−1	1	0	0	0
1	0	0	0	−1

(2) $\xi^2{}_{ij}$

0	0	0	0	0
0	1	0	0	0
0	0	0	0	0

(3) $\xi^3{}_{ij}$

0	0	0	0	0
−1	1	0	0	0
1	0	0	0	0

(b) $\theta_1+\theta_2+\theta_3 \leqq 5,\quad -\theta_1-\theta_2 \geqq -1,\quad -\theta_1 \geqq -3$

$\theta_1 \leqq 1,\quad \theta_1+\theta_3 \leqq 1,\quad \theta_1+\theta_2+\theta_3 \leqq 8,\quad \theta_1 \leq 0,\quad \theta_2 \leq 3,\quad \theta_3 \leq 3$

(c) $\theta_1=0,\quad \theta_2=3,\quad \theta_3=1$

$\mu_1=9+0=9,\quad \mu_2=1+3=4,\quad \mu_3=5+1=6$

(d) $x_{15}=0+0=0,\quad x_{31}=2+1=3,\quad x_{22}=0+3+1=4$

$x_{21}=1-1=0,\quad x_{35}=3-0=3.$

μ_i	$\bar{b}_j$ \ $\bar{a}_j$	0	1	0	1	0
9	0	0	0	4	5	0
4	0	0	4	0	0	0
6	2	3	0	0	0	3

Sept 4

(a) $u_1=d_{14},\quad u_2=d_{22},\quad u_3=d_{34}$

(b) $\theta_2 \leqq 1,\quad \theta_3 \leqq 1,\quad \theta_1 \leqq 0,\quad \theta_2 \leqq 4,\quad \theta_3 \leqq 2,\quad \theta_1 \leqq 0,\quad \theta_2 \leqq 0,\quad \theta_3 \leqq 2$

(c) $\theta_1=0,\quad \theta_2=0,\quad \theta_3=1,$

$\mu_1=9,\quad \mu_2=4,\quad \mu_3=7$

(d) $x_{34}=0+1=1$

μ_l	$\bar{a}_i$ \ $\overline{b_j}$	0	1	0	0	0
9	0	0	0	4	5	0
4	0	0	4	0	0	0
7	1	3	0	0	1	3

Step 5

(a) $u_1=d_{12}\quad u_2=d_{22}\quad u_3=d_{32}$

(b) $\theta_1+\theta_2+\theta_3\leqq 1$

$\theta_1\leq 3,\quad \theta_2\leq 4,\quad \theta_3\leq 1,\qquad \theta_1\leqq 0,\quad \theta_2\leqq 0,\quad \theta_3\leqq 1$

(c) $\theta_1=0,\quad \theta_2=0,\quad \theta_3=1,\qquad \mu_1=9,\quad \mu_2=4,\quad \mu_3=8$

(d) $x_{32}=0+1=1$

μ_l	$\bar{a}_i$ \ $\overline{b_j}$	0	0	0	0	θ
9	0	0	0	4	5	0
4	0	0	4	0	0	0
8	0	3	1	0	1	3

Step 1

The case solved as a single parametric problem

(a) $u_0=\sum \pm d_{ij},\qquad u_0=u_3=d_{31}$

(b) θ_0=maximum θ such that

$$\theta\sum_i \xi_{ij}^{(0)}\leqq\overline{b_j}\quad\text{for}\quad\overline{b_j}>0\quad\text{and}\quad\theta\,\xi_{ij}^{(0)}\geqq -x_{ij}\quad\text{for}\quad\xi_{ij}^{(0)}<0$$

$$\text{and}\quad\theta\,\xi_{ij}^{(0)}\leqq c_{ij}-x_{ij}\quad\text{for}\quad\xi_{ij}^{(0)}>0\qquad\theta_0=3$$

(c) $x_{ij} \to x_{ij} + \theta_0 \varepsilon_{ij}^{(0)}$, $\mu \to \mu + \theta_0$, $x_{31} = 3$, $\mu = 3$,

$\bar{a}_i$ \ $\bar{b}_j$	0	5	4	6	3
9	0	0	0	0	0
4	0	0	0	0	0
5	3	0	0	0	0

Step 2

(a) $u_0 = u_3 = d_{35}$

(b) $\theta_0 = 3$

(c) $\mu = 3 + 3 = 6$, $x_{35} = 0 + 3 = 3$

$\bar{a}_i$ \ $\bar{b}_j$	0	5	4	6	0
9	0	0	0	0	0
4	0	0	0	0	0
2	3	0	0	0	3

Step 3

(a) $u_0 = u_1 = d_{13}$

(b) $\theta_0 = 4$

(c) $\mu = 6 + 4 = 10$, $x_{13} = 0 + 4 = 4$

$\bar{a}_i$ \ $\bar{b}_j$	0	5	0	6	0
5	0	0	4	0	0
4	0	0	0	0	0
2	3	0	0	0	3

Step 4

(a) $u_0=u_1=d_{14}$ (b) $\theta_0=5$

(c) $\mu=10+5=15$, $x_{14}=0+5=5$

$\bar{a}_i$ \ $\bar{b}_j$	0	5	0	1	0
0	0	0	4	5	0
4	0	0	0	0	0
2	3	0	0	0	3

Step 5

(a) $u_0=u_3=d_{34}$ (b) $\theta_0=1$

(c) $\mu=15+1=16$, $x_{34}=0+1=1$

$\bar{a}_i$ \ $\bar{b}_j$	0	5	0	0	0
0	0	0	4	5	0
4	0	0	0	0	0
1	3	0	0	1	3

Step 6

(a) $u_0=u_2=d_{22}$ (b) $\theta_0=4$

(c) $\mu=16+4=20$, $x_{22}=0+4=4$

$\bar{a}_i$ \ $\bar{b}_j$	0	1	0	0	0
0	0	0	4	5	0
0	0	4	0	0	0
1	3	0	0	1	3

Step 7

(a) $u_0=u_3=d_{32}$

(b) $\theta_0=1$

(c) $\mu=20+1=21, \quad x_{32}=0+1=1$

$\bar{b}_j$ \ $\bar{a}_i$	0	0	0	0	0
0	0	0	4	5	0
0	0	4	0	0	0
0	3	1	0	1	3

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