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COMPUTATIONAL SOLUTION TO THE m-MACHINE SCHEDULING PROBLEM

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§1. INTRODUCTION

In this paper we shall present the two formulations that lead to the optimal solution to the m-machine scheduling problem, that is the problem to find the optimal sequence of n-items that are processed on m-machines in minimal total elapsed time, where the processing requies that the machines be used by the same numerical order for any item and the items be sequenced identically on each machine, to obtain a practical rule in the present situations for this problem.

The finding of the truly optimal solution to this problem is essential for the estimation of the approximate solution to this problems as indicated by Giglio and Wagner. [1]

In the following we shall give two different algorithms and show the number of the fundamental operations such as additions, differences and comparisons for each of them.

§2. THE PROBLEM

Let m-machines be named by $M_1, M_2, \cdots, M_m$ and let m_{k,i_n} be the time required to process the items J_{i_n} on the machine M_k. Then, for a sequence $(J_{i_1}, J_{i_2}, \cdots, J_{i_n})$, let $T(i_1, i_2, \cdots, i_p)$, $(p=1, 2, \cdots, n)$ be the total elapesd time to process the items $(J_{i_1}, J_{i_2} \cdots, J_{i_p})$ in this order on the m-machines following the way above mentioned.

The main problem is to obtain the recurrence relations that give $T(i_1, i_2, \cdots, i_n)$ analytically.

§ 3. THE VALUE OF $T(i_1, i_2, \cdots, i_n)$

In the following we shall follow the somewhat similar formulations in my former paper. [2]

For the first item, clearly we obtain

$$T(i_1)=\sum_{k=1}^{m} m_{k, i_1}$$

Next we obtain [see Fig. 1]

$$T(i_1, i_2)=\sum_{p=1}^{2} m_{1, i_p}+g(i_2, t_{i_1})$$

where

$$g(i_2, t_{i_1})=\sum_{k=2}^{m} \{m_{k, i_2}+\max (\underline{t}_{k-1}^{(i_1)}-m_{k-1, i_2}, 0)\} \tag{1}$$

and

$$\begin{cases} \underline{t}_1^{(i_1)}=t_1^{(i_1)} \\ \underline{t}_k^{(i_1)}=t_{k-}^{(i_1)}\max (m_{k-1, i_2}-\underline{t}_{k-1}^{(i_1)}, 0) \end{cases}$$

$$(k=2, 3, \cdots, m-1)$$

$$t_k^{(i_1)}=m_{k+1, i_1} \ (k=1, 2, \cdots, m-1).$$

Similarly we obtain

$$T(i_1, i_2, i_3)=\sum_{p=1}^{3} m_{1, i_p}+g(i_3, t_{i_2})$$

where

$$g(i_3, t_{i_2})=\sum_{k=2}^{m} \{m_{k, i_3}+\max (\underline{t}_{k-1}^{(i_2)}-m_{k-1, i_3}, 0)\}$$

and

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$$\begin{cases} \underline{t}_1^{(i_2)} = t_1^{(i_2)} \\ \underline{t}_k^{(i_2)} = t_k^{(i_2)} - \max\left(m_{k-1,\, i_3} - \underline{t}_{k-1}^{(i_2)},\ 0\right) \end{cases}$$

$$(k=2,\, 3,\, \cdots,\, m-1)$$

$$t_k^{(i_2)} = (\text{the } k\text{-th term of (1)})$$

$$= m_{k+1,\, i_2} + \max\left(\underline{t}_k^{(i_1)} - m_{k,\, i_2},\ 0\right).$$

$$(k=1,\, 2,\, \cdots,\, m-1)$$

Gantt Chart

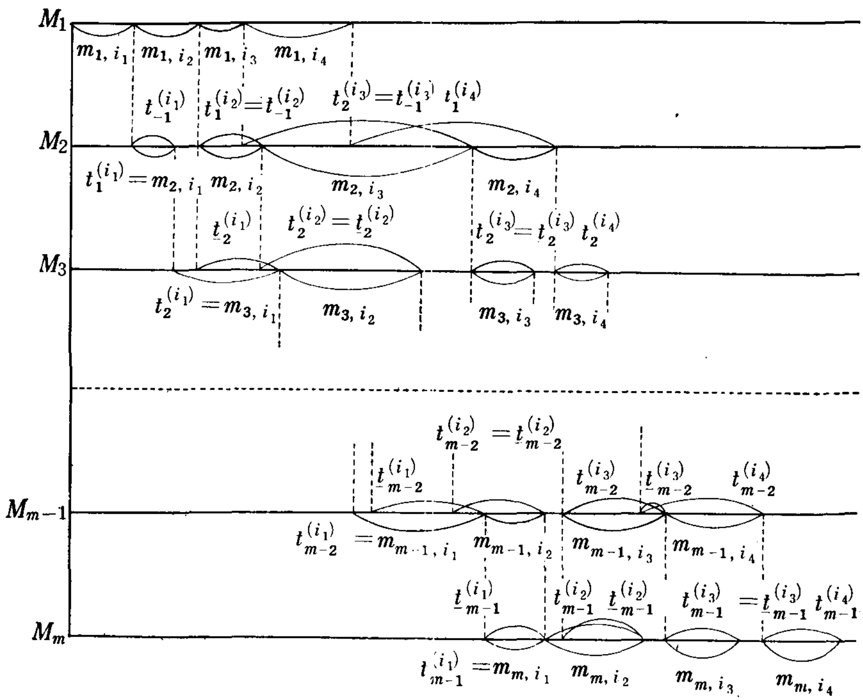

Fig. 1.

Generally we obtain

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$$T(i_1, i_2, \cdots, i_n)=\sum_{p=1}^{n} m_{1, i_p}+g(i_n, t_{i_{n-1}}) \tag{2}$$

where $$g(i_n, t_{i_{n-1}})=\sum_{k=2}^{m}\{m_{k, i_n}+\max(\underline{t}_{k-1}^{(i_{n-1})}-m_{k-1, i_n}, 0)\}$$

$$=\sum_{k=1}^{m-1} t_k^{(i_n)} \tag{3}$$

and $$\begin{cases} \underline{t}_1^{(i_{n-1})}=t_1^{(i_{n-1})} \\ \underline{t}_k^{(i_{n-1})}=t_k^{(i_{n-1})}-\max(m_{k-1, i_n}-\underline{t}_{k-1}^{(i_{n-1})}, 0) \end{cases} \tag{4}$$

$$(k=2, 3, \cdots, m-1)$$

$$t_k^{(i_{n-1})}=m_{k+1, i_{n-1}}+\max(\underline{t}_k^{(i_{n-2})}-m_{k, i_{n-1}}, 0) \tag{5}$$

$$(k=1, 2, \cdots, m-1).$$

Next, by substitute (3) for (2), we obtain

$$T(i_1, i_2, \cdots, i_n)=\sum_{p=1}^{n} m_{1, i_p}+\sum_{k=1}^{m-1} t_k^{(i_n)} \tag{6}$$

and as the term $\sum_{p=1}^{n} m_{1, i_p}$ is constant for any sequence of n items, we may look for the sequence $(J_{i_1}, J_{i_2}, \cdots, J_{i_n})$ that minimizes the term $\sum_{k=1}^{m-1} t_k^{(i_n)}$.

§4. ALGORITHM FOR CALCULATING $\sum_{k=1}^{m-1} t_k^{(i_n)}$ SUCCESSIVELY

Fom (4) and (5) for J_{i_p} $(p=1, 2, \cdots, n)$, we obtain the next recurrence relations:

for $p=1, 2, \cdots, n$;

$$t_k^{(i_p)}=m_{k+1, i_p}+\max(\underline{t}_k^{(i_{p-1})}-m_{k, i_p}, 0) \tag{7}$$

$$(t_k^{(i_1)}=m_{k+1, i_1})\ (k=1, 2, \cdots, m-1)$$

where $\underline{t}_k^{(i_0)}=0$; and

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$$\begin{cases} \underline{t}_1^{(i_{p-1})} = t_1^{(i_{p-1})} \\ \underline{t}_{k-1}^{(i_{p-1})} = t_{k-1}^{(i_{p-1})} - \max\,(m_{k-2,\, i_p} - \underline{t}_{k-2}^{(i_{p-1})},\ 0) \end{cases} \qquad (8)$$

$$(k=3,\ 4,\ \cdots,\ m)$$

Hence, to calculate the sum $\sum_{k=1}^{m-1} t_k^{(i_n)}$ for a sequence $(J_{i_1}, J_{i_2}, \cdots, J_{i_n})$ we may calculate successively in the following way. [Fig. 2]

(i) Put $t_k^{(i_1)}$ as equal to $m_{k+1,\, i_1}$ for J_{i_1},

(ii) then calculate $\underline{t}_k^{(i_1)}$ from (8) by putting $p=2$,

(iii) next, calculate $\underline{t}_k^{(i_2)}$ for J_{i_2} from (7) by putting $p=2$,

(iv) then determine $\underline{t}_k^{(i_2)}$ from (8) by putting $p=3$,

(v) continue (iii) and (iv) succeesively for each item J_{i_3}, J_{i_4}, $\cdots$, J_{i_n} in this orter,

(vi) finally, calculate the sum $\sum_{k=1}^{m-0} t_k^{(i_n)}$.

Fig. 2 shows the above successive calculations briefly.

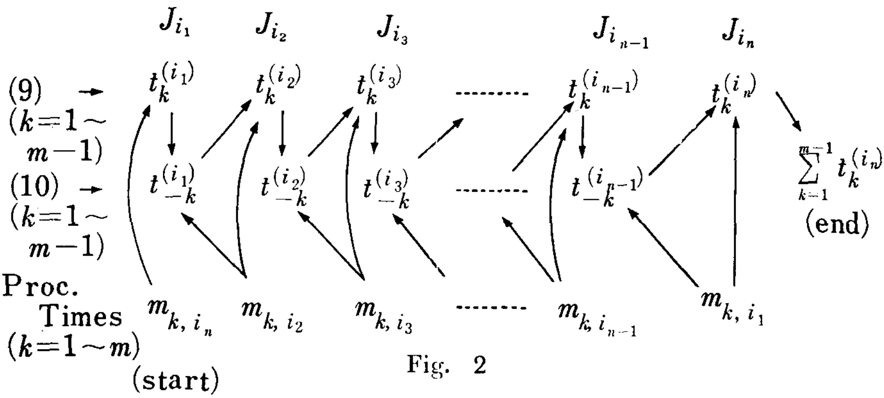

Fig. 2

This successive calculations for obtaining the sum $\sum_{k=1}^{m-1} t_k^{(i_n)}$ by recurrence relations (7) and (8) may be effective for modern computing equipment.

Next we shall summarize the number of fundamental operations such as additions, differences and comparison for the above successive

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calculations to obtain the sum $\sum_{k=1}^{m-1} t_k^{(i_n)}$ to each sequence $(J_{i_1}, J_{i_2}, \cdots, J_{i_n})$; that is,

a) the number of fundamental operations that is needed for obtaining the values $t_k^{(i_p)}(p=2, 3, \cdots, n)$ by (7) are $(3m-3)(n-1)$,

b) and the number corresponding to obtain the values $\underline{t}_k^{(i_p)}(p=1, 2, \cdots, n-1)$ by (8) are $(3m-6)(n-1)$,

c) and the number corresponding to obtain the final sum $\sum_{k=1}^{m-1} t_k^{(i_n)}$ are $(m-2)$,

hence, the total numbers of the fundamental operations are

$$3(2m-3)n-5m+7. \tag{9}$$

Then, to obtain the minimum of $\sum_{k=1}^{m-1} t_k^{(i_n)}$ among all sequences of n items, it needs the final numbers,

$$\left[\{3(2m-3)n-5m+7\}\cdot n!\right]+(n!-1) \tag{10}$$

and the optimal solution is a sequence of n items which gives the minimum of $\sum_{k=1}^{m-1} t_k^{(i_n)}$.

§ 5. COMPARISON WITH THE JOHNSON'S FORMULA FOR THREE-MACHINES CASE

Johnson has presented the next formula for three-machine scheduling problem. [3]

$$T(i_1, i_2, \cdots, i_n)=\sum_{p=1}^{n} m_{3, i_p}+\sum_{p=1}^{n} Y_{i_p}$$

where
$$\sum_{p=1}^{n} Y_{i_p}=\max_{1\leqslant u\leqslant v\leqslant n}(H_v+K_u) \tag{11}$$

and
$$\begin{cases} H_v=\sum_{p=1}^{v} m_{2, i_p}-\sum_{p=1}^{v-1} m_{3, i_p} & (v=1, 2, \cdots, n) \\ K_u=\sum_{p=1}^{u} m_{1, i_p}-\sum_{p=1}^{u-1} m_{2, i_p} & (u=1, 2, \cdots, n) \end{cases} \tag{12}$$

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then, we may calculate $\sum_{p=1}^{n} Y_{i_p}$ for each sequence, and select a sequence which gives the minimum $\sum_{p=1}^{n} Y_{i_p}$.

From (11) and (12), the numbers of the fundamental operations for calculating one $\sum_{p=1}^{n} Y_{i_p}$ are (n^2+6n-7).

On the other hand the numbers corresponding to (9) for $m=3$ are $(9n-8)$ and for $n \geqq 3$,

$$9n-8<n^2+6n-7.$$

Hence, for the n ($n \geqq 3$) items, the algorithm in Sec. 4 may be efficient for three-machine problem in order to obtain the optimal solution.

§6. ANOTHER ALGORITHM FOR OBTAINING THE OPTIMAL SOLUTION

In this section we shall give another recurrence relation that give the fewer number of operations than the relations given above.

6.1. Expression of $\sum_{k=1}^{m-1} t_k^{(i_n)}$

By substituting (4) for (5) in the section 3, we obtain

$$t_k^{(i_{n-1})}=m_{k+1,\, i_{n-1}}+\max\{t_k^{(i_{n-2})}-\max(m_{k-1,\, i_{n-1}}-\underline{t}_{k-1}^{(i_{n-2})},\ 0)-m_{k,\, i_{n-1}},\ 0\} \tag{13}$$

$$(k=1,\ 2,\ \cdots,\ m-1\ ;\ n\geq 2)$$

where, for $k=1$ and for $n=2$, we put

$$m_{0,\, i_{n-1}}=0,\quad \underline{t}_0^{(i_{n-2})}=0\ ;\quad t_k^{(i_0)}=0,\quad \underline{t}_{k-1}^{(i_0)}=0.$$

So that we obtain, by adding for $k=1,\ 2,\ \cdots,\ m-1$;

$$\sum_{k=1}^{m-1} t_k^{(i_{n-1})}=\sum_{k=2}^{m} m_{k,\, i_{n-1}}+\sum_{k=1}^{m-1}\max\{t_k^{(i_{n-2})}-\max(m_{k-1,\, i_{n-1}}-\underline{t}_{k-1}^{(i_{n-2})},\ 0)-m_{k,\, i_{n-1}},\ 0\},$$

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and this leads to the next formula,

$$\sum_{k=1}^{m-1} t^{(i_p)} = \sum_{k=2}^{m} m_{k,\, i_p} + \sum_{k=1}^{m-1} \max\{ t_k^{(i_{p-1})}$$

$$- \max(m_{k-1,\, i_p} - \underline{t}_{k-1}^{(i_{p-1})},\ 0) - m_{k,\, i_p},\ 0\} \tag{14}$$

$$(p=1,\ 2,\ \cdots,\ n)$$

where $m_{0,\, i_p}=0,\ \ \underline{t}_0^{(i_{p-1})}=0\,;\ \ t_k^{(i_0)}=0,\ \ \underline{t}_{k-1}^{(i_0)}=0.$

At the right hand side, by putting $t_k^{(i_{p-1})}$ outside of the maximum sign, we have

$$\sum_{k=1}^{m-1} t_k^{(i_p)} = \sum_{k=1}^{m-1} t_k^{(i_{p-1})} + \sum_{k=2}^{m} m_{k,\, i_p}$$

$$+ \sum_{k=1}^{m-1} \max\{ - \max(m_{k-1,\, i_p} - \underline{t}_{k-1}^{(i_{p-1})},\ 0) - m_{k,\, i_p},$$

$$- t_k^{(i_{p-1})}\}$$

hence we obtain

$$\sum_{k=1}^{m-1} t_k^{(i_p)} - \sum_{k=1}^{m-1} t_k^{(i_{p-1})} = \sum_{k=2}^{m} m_{k,\, i_p} + \sum_{k=1}^{m=1} \max\{ - \max(m_{k-1,\, i_p}$$

$$- \underline{t}_{k-1}^{(i_{p-1})},\ 0) - m_{k,\, i_p},\ - t_k^{(i_{p-1})}\}$$

$$= \sum_{k=2}^{m} m_{k,\, i_p} - \sum_{k=1}^{m-1} m_{k,\, i_p} + \sum_{k=1}^{m-1} \max\{\min(\underline{t}_{k-1}^{(i_{p-1})} - m_{k-1,\, i_p},\ 0),$$

$$m_{k,\, i_p} - t_k^{(i_{p-1})}\}$$

$$= m_{m,\, i_p} - m_{1,\, i_p}$$

$$+ \sum_{k=1}^{m-1} \max\{\min(\underline{t}_{k-1}^{(i_{p-1})} - m_{k-1,\, i_p},\ 0),\ m_{k,\, i_p} - t_k^{(i_{p-1})}\} \tag{15}$$

By putting $p=1,\ 2,\ \cdots,\ n-1,\ n$ successively in (15) and adding, we have the next expression of $\sum_{k=1}^{m-1} t_k^{(i_n)}$.

$$\sum_{k=1}^{m-1} t_k^{(i_n)} = \sum_{p=1}^{n} m_{m,\, i_p} - \sum_{p=1}^{n} m_{1,\, i_p}$$

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$$+\sum_{p=1}^{n}\sum_{k=1}^{m-1}\max\{\min(\underline{t}_{k-1}^{(i_{p-1})}-m_{k-1,i_p},\ 0),$$

$$m_{k,i_p}-t_k^{(i_{p-1})}\}. \qquad (16)$$

where we put $t_k^{(i_0)}=0$, $\underline{t}_{k-1}^{(i_0)}=0$, $\underline{t}_0^{(i_{p-1})}=0$, $m_{0,i_p}=0$ and the term $\sum_{p=1}^{n} m_{m,i_p}-\sum_{p=1}^{n} m_{1,i_p}$ is constant for any sequence.

6.2. **The Order of the lest two items**

Next, for each sequence $(J_{i_1}, J_{i_2}, \cdots, J_{i_{n-2}})$ of $(n-2)$ items from the given n items, we determine the order of the remained two items J_i, J_j in the last two positions which give the smaller total elapsed time. Then, after processing the first $(n-2)$ items, let t_k $(k=1, 2, \cdots, m-1)$ be the time that the machine M_{k+1} is committed for the machine M_k ($t_k=t_k^{(i_{n-2})}$), then in the case when

$$T(i_1, i_2, \cdots, i_{n-2}, i, j)<T(i_1, i_2, \cdots, i_{n-2}, j, i)$$

holds, from (16) the next relation must holds by eliminating the same terms from the both sides;

$$\sum_{k=1}^{m-1}\Big[\max\{\min(\underline{t}_{k-1}-m_{k-1,i},\ 0),\ \ m_{k,i}-t_k\}$$

$$+\max\{\min(\underline{t}_{k-1}^{(i)}-m_{k-1,j},\ 0),\ \ m_{k,j}-t_k^{(i)}\}\Big]$$

$$<\sum_{k=1}^{m-1}\Big[\max\{\min(\underline{t}_{k-1}-m_{k-1,j},\ 0),\ \ m_{k,j}-t_k\}$$

$$+\max\{\min(\underline{t}_{k-1}^{(j)}-m_{k-1,i},\ 0),\ \ m_{k,i}-t_k^{(j)}\}\Big] \qquad (17)$$

where $\underline{t}_{k-1}=\underline{t}_{k-1}^{(i_{n-2})}$, $t_k=t_k^{(i_{n-2})}$ and $\underline{t}_0=0$, $\underline{t}_0^{(i)}=0$, $m_{0,i}=0$, $m_{0,j}=0$.

That is, if (17) holds for the remained two items J_i, J_j, then we may only take account of the sequence $(J_{i_1}, \cdots, J_{i_{n-2}}, J_i, J_j)$ for the forward consideration.

6.3. **Determination of the optimal solution.**

From the above statement we obtain the next algorithm for obtaining the optimal sequence and the corresponding minimal total elapsed

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time; that is,

(i) generate the sequences $(J_{i_1}, J_{i_2}, \cdots, J_{i_{n-2}})$ of $(n-2)$ items among the given n items,

(ii) next, for each such sequence determine the order of the remained two items J_i, J_j by using the expression (7), (8) and (17).

(iii) then, for the sequence $(J_{i_1}, J_{i_2}, \cdots, J_{i_{n-2}}, J_i, J_j)$ calculate the sum $\sum_{k=1}^{m-1} t_k^{(j)}$,

(iv) finally, for all sequences $(J_{i_1}, J_{i_2}, \cdots, J_{i_{n-2}})$, determine the minimum of $\sum_{k=1}^{m-1} t_k^{(j)}$,

Then we obtain the optimal sequence and the corresponding minimal total elapsed time as in the section 4.

Next, we shall summarize the number of the fundamental operations for this algorithm as in the section 4.

It becomes

$$\left[\{3(2m-3)n+21m-31\}\cdot\frac{n!}{2}\right]+\left(\frac{n!}{2}-1\right) \tag{18}$$

By comparering (18) with (10) in section 4, the percent ratio of the number by (18) to the number by (10) for $m=3\sim7$ and $n=7\sim10$ are about $85\sim73$ percent and for $n=20$ it's about 50 percent, which shows the efficiency of the algorithm in this section in some cases.

§7. SOME REMARKS

First, in the section 4, the calculations of the minimum of the $\sum_{k=1}^{m-1} t_k^{(i_n)}$ for all sequences may be diminished, if we compare the values of $\sum_{k=1}^{m-1} t_k^{(i_p)}$ for $p(p<n)$ of a new sequence with the values of $\sum_{k=1}^{m-1} t_k^{(i_n)}$ of the sequences already calculated and whenever $\sum_{k=1}^{m-1} t_k^{(i_p)} \geqq \sum_{k=1}^{m-1} t_k^{(i_n)}$ holds among them, the calculation of this new sequence be finished and proceeds to consider another new sequence in this way.

Next, by using the similar arguments described in 6.2 for the last

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h items (h=3, 4, $\cdots$) the number of the fundamental operations may be fewer as h becomes larger, though the formula similar to (17) may be complicated.

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