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TRANSIENT BEHAVIOR OF POISSON QUEUE

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1. INTRODUCTION

We consider single Poisson queuing system which has the inter-arrival and holding time distributed by exponential distributions with means $1/\lambda$ and $1/\mu$ respectively. In this paper we shall discuss the transient behavior of this queuing system and we shall get the results on how fast the state probabilities and the mean value of the units in the system tend to their equilibrium limits when these exist. We shall easily see how small the absolute values of the differences between above mentioned quantities and their limits are for time t large enough whatever the initial state of the system may be. Moreover, we shall discuss how fast the queue length approaches to an asymptotic linear function in non-equilibrium case.

To get these results we introduce some notations; E_n represents the state of the system that there are n units waiting in the queue and being served at the counter, $P_n(t)$ $(n=0, 1,\cdots\cdots)$ are the state probabilities that the system is in the state E_n at time t, (p_n) is the equilibrium distribution of the states E_n when it exists and let ρ be the relative intensity which is equal to the ratio $\lambda:\mu$.

It is well-known fact that for any initial state of the system the state probabilities $P_n(t)$ $(n=0, 1,\cdots\cdots)$ satisfy the difference differential equations

$$(1)\quad \begin{aligned} &\frac{dP_0(t)}{dt}=-\lambda P_0(t)+\mu P_1(t),\\ &\frac{dP_n(t)}{dt}=-(\lambda+\mu)P_n(t)+\lambda P_{n-1}(t)+\mu P_{n+1}(t) \quad (n\geqq 1)\end{aligned}$$

at any time t. To make our calculations slightly simple we put $\mu=1$ into these equations. Then we obtain the equations

$$(1')\quad \begin{aligned} \frac{dP_0(t)}{dt} &= -\rho P_0(t)+P_1(t), \\ \frac{dP_n(t)}{dt} &= -(1+\rho)P_n(t)+\rho P_{n-1}(t)+P_{n+1}(t) \quad (n\geqq 1). \end{aligned}$$

It is well-known* that if $P_i(0)=1$ the solutions of (1′) are given by the formula

$$(2)\quad P_n(t)=e^{-(1+\rho)t}\Big\{(\sqrt{\rho})^{n-i}I_{n-i}(2\sqrt{\rho}\,t)+(\sqrt{\rho})^{n-i-1}I_{n+i+1}(2\sqrt{\rho}\,t) + (1-\rho)\rho^n \sum_{k=n+i+2}^{\infty}(\sqrt{\rho})^{-k}I_k(2\sqrt{\rho}\,t)\Big\}$$

and in the case when $n=0$ we have

$$(2')\quad P_0(t)=e^{-(1+\rho)t}\Big\{(\sqrt{\rho})^{-i}I_i(2\sqrt{\rho}\,t)+(\sqrt{\rho})^{-i-1}I_{i+1}(2\sqrt{\rho}\,t) + (1-\rho)\sum_{k=i+2}^{\infty}(\sqrt{\rho})^{-k}I_k(2\sqrt{\rho}\,t)\Big\},$$

where

$$I_m(x)=(\tfrac{1}{2}x)^m\sum_{n=0}^{\infty}\frac{(\frac{1}{2}x)^{2n}}{n!\,(m+n)!}$$

which we call modified Bessel function. The formula (2) is very important, from which we shall start to discuss.

If we want to have the solutions of (1) we can easily get them by only changing t on the right side of (2) into μt.

2. PROPERTIES OF MODIFIED BESSEL FUNCTION

To continue the discussion we must use some properties of modified Bessel function :

$$(3)\quad I_0(0)=1,\quad I_n(0)=0 \quad (n=1,\ 2,\cdots\cdots),$$

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*) See, for example, the recent publication [1], pp. 88—96.

(4) $I_n(x)=I_{-n}(x)$,

(5) $x(I_{n-1}(x)-I_{n+1}(x))=2nI_n(x)$,

(6) $I_n(x)<I_m(x) \quad (n>m\geqq 0)$

for positive x,

(7) $I_n(x)\sim \frac{e^x}{\sqrt{2\pi x}} \quad (x\to+\infty)$,

(8) $e^{\frac{x}{2}\left(y+\frac{1}{y}\right)}=\sum_{n=-\infty}^{\infty} y^n I_n(x)$

which is one of the formulas of Neumann's expansion of elentary functions in terms of Bessel function.

3. BEHAVIORS OF THE STATE PROBABILITIES $P_n(t)$

We can conclude from (2) that if $\rho\geqq 1$, then $P_n(t)\to 0$ $(t\to+\infty)$, because, using the properties of modified Bessel function (6) and (7), we can easily see

$$e^{-(1+\rho)t}I_n(2\sqrt{\rho}\,t)\sim e^{-(1+\rho)t}\frac{e^{2\sqrt{\rho}\,t}}{\sqrt{2\pi(2\sqrt{\rho}\,t)}}=\frac{e^{-(\sqrt{\rho}-1)^2}}{\sqrt{2\pi(2\sqrt{\rho}\,t)}}\to 0 \quad (t\to+\infty)$$

for any n, and

$$e^{-(1+\rho)t}\sum_{k=n}^{\infty}(\sqrt{\rho})^{-k}I_k(2\sqrt{\rho}\,t)<\frac{(\sqrt{\rho})^{-n}e^{-(1+\rho)t}}{1-\frac{1}{\sqrt{\rho}}}I_0(2\sqrt{\rho}\,t)\to 0(\quad t\to+\infty).$$

for any n and $\rho(>1)$.

If $\rho<1$, then we can not immediately conclude even that the summation on the right side of (2) or (2′) converges.

However, from (2) we can easily prove that

(9) $P_n(t)>0$

for positive ρ and t, and

(10) $\sum_{n=0}^{\infty} P_n(t)=1.$*

Using the property of Bessel function (8), we get

$$P_n(t)=e^{-(1+\rho)t}\Big[(\sqrt{\rho})^{n-i}I_{n-i}(2\sqrt{\rho}\,t)+(\sqrt{\rho})^{n-i-1}I_{n+i+1}(2\sqrt{\rho}\,t)$$
$$-(1-\rho)\rho^n\Big\{e^{(1+\rho)t}-\sum_{k=-n-i-1}^{\infty}(\sqrt{\rho})^k I_k(2\sqrt{\rho}\,t)\Big\}\Big].$$

Then we have the formula

(11) $$P_n(t)=(1-\rho)\rho^n+e^{-(1+\rho)t}\Big\{(\sqrt{\rho})^{n-i}I_{n-i}(2\sqrt{\rho}\,t)$$
$$+(\sqrt{\rho})^{n-i-1}I_{n+i+1}(2\sqrt{\rho}\,t)-(1-\rho)\rho^n\sum_{k=-n-i-1}^{\infty}(\sqrt{\rho})^k I_k(2\sqrt{\rho}\,t)\Big\},$$

where the first term on the right side represents the equilibrium limit p_n of $P_n(t)$ when ρ is less than unity.

Indeed, if $\rho<1$, considering the properties of Bessel function (6) and (7), we can easily see that the summation on the right side of (11) converges and that other terms than the first on the right side of (11) tend to 0 when $t\rightarrow+\infty$. Then we find immediately that

(12) $$P_n(t)\rightarrow p_n=(1-\rho)\rho^n \qquad (t\rightarrow+\infty).$$

Moreover, after some calculations, we shall have the formula (13) mentioned below. Let P_1 and $-N_1$ be the sum of the positive or negative terms in the brace on the right side of (11) respectively, then, using (6), we get

$$P_1=(\sqrt{\rho})^{n-i}I_{n-i}(2\sqrt{\rho}\,t)+(\sqrt{\rho})^{n-i-1}I_{n+i+1}(2\sqrt{\rho}\,t)$$
$$<(1+\sqrt{\rho})(\sqrt{\rho})^{n-i-1}I_0(2\sqrt{\rho}\,t)$$

and

$$N_1=(1-\rho)\rho^n\sum_{k=-n-i-1}^{\infty}(\sqrt{\rho})^k I_k(2\sqrt{\rho}\,t)$$

*) See also [1], pp. 93—95.

$$<(1-\rho)\rho^n\frac{(\sqrt{\rho})^{-n-i-1}}{1-\sqrt{\rho}}I_0(2\sqrt{\rho}\,t)$$

$$=(1+\sqrt{\rho})(\sqrt{\rho})^{n-i-1}I_0(2\sqrt{\rho}\,t).$$

Then we have the formulas

$$(13)\qquad |P_n(t)-p_n|<(1+\sqrt{\rho})(\sqrt{\rho})^{n-i-1}e^{-(1+\rho)t}I_0(2\sqrt{\rho}\,t)\to 0\qquad (t\to+\infty)$$

and

$$|P_n(t)-p_n|:p_n<(1-\sqrt{\rho})^{-1}(\sqrt{\rho})^{-n-i-1}e^{-(1+\rho)t}I_0(2\sqrt{\rho}\,t)$$

$$\to 0\qquad (t\to+\infty).$$

Finally we can change the formula (11) into other expression. The expression in the brace on the right side of (11) can be changed as follows:

$$(\sqrt{\rho})^{n-i}I_{n-i}(2\sqrt{\rho}\,t)+(\sqrt{\rho})^{n-i-1}I_{n+i+1}(2\sqrt{\rho}\,t)$$

$$-\rho^n\left\{\sum_{k=-n-i+1}^{\infty}(\sqrt{\rho})^kI_k(2\sqrt{\rho}\,t)-\sum_{k=-n-i-1}^{\infty}(\sqrt{\rho})^{k+2}I_k(2\sqrt{\rho}\,t)\right\}$$

$$=(\sqrt{\rho})^{n-i}I_{n-i}(2\sqrt{\rho}\,t)-(\sqrt{\rho})^{n-i}I_{-n-i}(2\sqrt{\rho}\,t)$$

$$-\rho^n\left\{\sum_{k=-n-i+1}^{\infty}(\sqrt{\rho})^kI_k(2\sqrt{\rho}\,t)-\sum_{k=-n-i-1}^{\infty}(\sqrt{\rho})^{k+2}I_k(2\sqrt{\rho}\,t)\right\}$$

$$=(\sqrt{\rho})^{n-i}I_{n-i}(2\sqrt{\rho}\,t)-(\sqrt{\rho})^{n-i}I_{n+i}(2\sqrt{\rho}\,t)$$

$$-\rho^n\sum_{k=-n-i+1}^{\infty}(\sqrt{\rho})^k\left\{I_k(2\sqrt{\rho}\,t)-I_{k-2}(2\sqrt{\rho}\,t)\right\}$$

$$=(\sqrt{\rho})^{n-i}I_{n-i}(2\sqrt{\rho}\,t)-(\sqrt{\rho})^{n-i}I_{n+i}(2\sqrt{\rho}\,t)$$

$$-\rho^n\sum_{k=-n-i+1}^{\infty}\frac{2(\sqrt{\rho})^k(k-1)I_{k-1}(2\sqrt{\rho}\,t)}{2\sqrt{\rho}\,t}.$$

Then, using the property (5), we obtain the following results;

(14) $$P_n(t)=(1-\rho)\rho^n+e^{-(1+\rho)t}\Big[\frac{\rho^n}{t}\sum_{k=-n-i}^{\infty}k(\sqrt{\rho})^k I_k(2\sqrt{\rho}\,t)$$
$$+(\sqrt{\rho})^{n-i}\Big\{I_{n-i}(2\sqrt{\rho}\,t)-I_{n+i}(2\sqrt{\rho}\,t)\Big\}\Big]$$

and in the case $n=0$

(14′) $$P_0(t)=1-\rho+\frac{e^{-(1+\rho)t}}{t}\sum_{k=-i}^{\infty}k(\sqrt{\rho})^k I_k(2\sqrt{\rho}\,t).$$

In the case when $i=0$ we have very simple formula

(15) $$P_n(t)=(1-\rho)\rho^n+\frac{e^{-(1+\rho)t}}{t}\rho^n\sum_{k=-n}^{\infty}k(\sqrt{\rho})^k I_k(2\sqrt{\rho}\,t).$$

4. BEHAVIOR OF THE MEAN VALUE $L(t)$ OF THE UNITS IN THE SYSTEM WHEN ρ IS LESS THAN UNITY

In this section we shall obtain the formulas on the mean value $L(t)$ of the units in the system at finite t and observe the behaviors of $L(t)$ at large t.

We can calculate $L(t)$ in the following way. Using the formula (2) of $P_n(t)$, we get

$$L(t)=\sum_{n=1}^{\infty}nP_n(t)=-1+\sum_{n=0}^{\infty}(n+1)P_n(t)$$
$$=-1+e^{-(1+\rho)t}\Big\{\sum_{n=0}^{\infty}(n+1)(\sqrt{\rho})^{n-i}I_{n-i}(2\sqrt{\rho}\,t)$$
$$+\sum_{n=0}^{\infty}(n+1)(\sqrt{\rho})^{n-i-1}I_{n+i+1}(2\sqrt{\rho}\,t)$$
$$+(1-\rho)\sum_{n=0}^{\infty}(n+1)\rho^n\sum_{k=n+i+2}^{\infty}(\sqrt{\rho})^{-k}I_k(2\sqrt{\rho}\,t)\Big\}.$$

The first term in the brace on the right side is equal to

$$\sum_{k=1}^{\infty} k(\sqrt{\rho}\,)^{k-i-1} I_{k-i-1}(2\sqrt{\rho}\,t).$$

The second term can be expressed as follows;

$$\sum_{k=i+1}^{\infty}(k-i)(\sqrt{\rho}\,)^{k-2i-2} I_k(2\sqrt{\rho}\,t)$$

$$=\frac{1}{\rho^{i+1}}\sum_{k=i+1}^{\infty} k(\sqrt{\rho}\,)^{k} I_k(2\sqrt{\rho}\,t)-\frac{i}{\rho^{i+1}}\sum_{k=i+1}^{\infty}(\sqrt{\rho}\,)^{k} I_k(2\sqrt{\rho}\,t).$$

When $\rho\neq 1$, by interchanging the summation, the last term becomes

$$(1-\rho)\sum_{k=i+2}^{\infty}\sum_{n=0}^{k-i-2}(n+1)\rho^{n}(\sqrt{\rho}\,)^{-k} I_k(2\sqrt{\rho}\,t)$$

$$=\sum_{k=i+2}^{\infty}\left\{\frac{1-\rho^{k-i-1}}{1-\rho}-(k-i-1)\rho^{k-i-1}\right\}(\sqrt{\rho}\,)^{-k} I_k(2\sqrt{\rho}\,t)$$

$$=\frac{1}{1-\rho}\sum_{k=i+2}^{\infty}(\sqrt{\rho}\,)^{-k} I_k(2\sqrt{\rho}\,t)-\frac{1}{(1-\rho)\rho^{i+1}}\sum_{k=i+2}^{\infty}(\sqrt{\rho}\,)^{k} I_k(2\sqrt{\rho}\,t)$$

$$-\frac{1}{\rho^{i+1}}\sum_{k=i+2}^{\infty} k(\sqrt{\rho}\,)^{k} I_k(2\sqrt{\rho}\,t)+\frac{i+1}{\rho^{i+1}}\sum_{k=i+2}^{\infty}(\sqrt{\rho}\,)^{k} I_k(2\sqrt{\rho}\,t).$$

Then we get

$$L(t)=-1+e^{-(1+\rho)t}\Bigg[\sum_{k=1}^{\infty} k(\sqrt{\rho}\,)^{k-i-1} I_{k-i-1}(2\sqrt{\rho}\,t)$$

$$+\frac{1}{\rho^{i+1}}\sum_{k=i+1}^{\infty}(\sqrt{\rho}\,)^{k} I_k(2\sqrt{\rho}\,t)+\frac{1}{1-\rho}\sum_{k=i+2}^{\infty}(\sqrt{\rho}\,)^{-k} I_k(2\sqrt{\rho}\,t)$$

$$-\frac{1}{(1-\rho)\rho^{i+1}}\sum_{k=i+2}^{\infty}(\sqrt{\rho}\,)^{k} I_k(2\sqrt{\rho}\,t)\Bigg].$$

The sum of the last three terms in the brace becomes

$$\frac{1}{1-\rho}\left\{\sum_{k=i+1}^{\infty}(\sqrt{\rho})^{-k}I_k(2\sqrt{\rho}\,t)-\frac{1}{\rho^i}\sum_{k=i+1}^{\infty}(\sqrt{\rho})^{k}I_k(2\sqrt{\rho}\,t)\right\}.$$

Then we obtain the equality

$$(16)\quad L(t)=-1+e^{-(1+\rho)t}\Bigg[\sum_{k=1}^{\infty}k(\sqrt{\rho})^{k-i-1}I_{k-i-1}(2\sqrt{\rho}\,t)$$

$$+\frac{1}{1-\rho}\left\{\sum_{k=i+1}^{\infty}(\sqrt{\rho})^{-k}I_k(2\sqrt{\rho}\,t)-\frac{1}{\rho^i}\sum_{k=i+1}^{\infty}(\sqrt{\rho})^{k}I_k(2\sqrt{\rho}\,t)\right\}\Bigg].$$

Using the property (8) of modified Bessel function, we can see

$$\sum_{k=i+1}^{\infty}(\sqrt{\rho})^{-k}I_k(2\sqrt{\rho}\,t)=e^{(1+\rho)t}-\sum_{k=-i}^{\infty}(\sqrt{\rho})^{k}I_k(2\sqrt{\rho}\,t).$$

Therefore we can change the form of $L(t)$ into the following:

$$L(t)=-1+e^{-(1+\rho)t}\Bigg[\sum_{k=1}^{\infty}k(\sqrt{\rho})^{k-i-1}I_{k-i-1}(2\sqrt{\rho}\,t)$$

$$+\frac{1}{1-\rho}\left\{e^{(1+\rho)t}-\sum_{k=-i}^{\infty}(\sqrt{\rho})^{k}I_k(2\sqrt{\rho}\,t)-\sum_{k=i+1}^{\infty}(\sqrt{\rho})^{k-2i}I_k(2\sqrt{\rho}\,t)\right\}\Bigg].$$

Then we can finally reach the result

$$(17)\quad L(t)=\frac{\rho}{1-\rho}+e^{-(1+\rho)t}\Bigg[\sum_{k=1}^{\infty}k(\sqrt{\rho})^{k-i-1}I_{k-i-1}(2\sqrt{\rho}\,t)$$

$$-\frac{1}{1-\rho}\left\{\sum_{k=0}^{\infty}(\sqrt{\rho})^{k-i}I_{k-i}(2\sqrt{\rho}\,t)+\sum_{k=1}^{\infty}(\sqrt{\rho})^{k-i}I_{k+i}(2\sqrt{\rho}\,t)\right\}\Bigg]$$

and in the case when $i=0$ we get

$$(17')\quad L(t)=\frac{\rho}{1-\rho}+e^{-(1+\rho)t}\left\{\sum_{k=1}^{\infty}k(\sqrt{\rho})^{k-1}I_{k-1}(2\sqrt{\rho}\,t)\right.$$

$$-\frac{1}{1-\rho}\sum_{k=1}^{\infty}\varepsilon_n(\sqrt{\rho})^k I_k(2\sqrt{\rho}\,t)\Big\},$$

where $\sum_{n=0}^{\infty}\varepsilon_n a_n = a_0 + \sum_{n=1}^{\infty} 2a_n$ for any sequence (a_n).

The first term on the right side of (17) or (17′) is equal to the equilibrium limit $L=\rho/(1-\rho)$ of $L(t)$ when $0<\rho<1$.

Depating from (17) we can reach the formula (18) mentioned below. Let P_2 and $-N_2$ be the sum of the positive or negative terms in the brace of (17) respectively, then

$$P_2=\sum_{k=1}^{\infty}k(\sqrt{\rho})^{k-i-1}I_{k-i-1}(2\sqrt{\rho}\,t)$$

$$<\frac{1}{(\sqrt{\rho})^i}\,\frac{1}{(1-\sqrt{\rho})^2}\,I_0(2\sqrt{\rho}\,t),$$

$$N_2=\frac{1}{1-\rho}\Big\{\sum_{k=0}^{\infty}(\sqrt{\rho})^{k-i}I_{k-i}(2\sqrt{\rho}\,t)+\sum_{k=1}^{\infty}(\sqrt{\rho})^{k-i}I_{k+1}(2\sqrt{\rho}\,t)$$

$$<\frac{1}{1-\rho}\Big\{\frac{1}{(\sqrt{\rho})^i(1-\sqrt{\rho})}+\frac{\sqrt{\rho}}{(\sqrt{\rho})^i(1-\sqrt{\rho})}\Big\}I_0(2\sqrt{\rho}\,t)$$

$$=\frac{1}{(\sqrt{\rho})^i(1-\sqrt{\rho})^2}I_0(2\sqrt{\rho}\,t).$$

Therefore, when the intensity is less than unity, we have

$$(18)\quad |L(t)-L|<\frac{1}{(\sqrt{\rho})^i(1-\sqrt{\rho})^2}\,e^{-(1+\rho)t}I_0(2\sqrt{\rho}\,t)\to 0 \qquad (t\to+\infty)$$

and

$$(18')\quad \frac{|L(t)-L|}{L}<\frac{1+\sqrt{\rho}}{\rho(\sqrt{\rho})^i(1-\sqrt{\rho})}\,e^{-(1+\rho)t}I_0(2\sqrt{\rho}\,t)\to 0 \qquad (t\to+\infty).$$

5. MEAN VALUE $L(t)$ OF THE UNITS IN THE SYSTEM WHEN ρ IS NOT LESS THAN UNITY

In this section we shall deal with the behavior of $L(t)$ when $\rho \geqq 1$. Of course, we know that in this case $L(t) \to +\infty$ $(t \to +\infty)$. To what function and how fast the function $L(t)$ becomes asymptotically equal when $t \to +\infty$?

First, we assume $\rho > 1$. We depart from the equation (16)

$$L(t) = -1 + e^{-(1+\rho)t}\Bigg[\sum_{k=1}^{\infty} k(\sqrt{\rho})^{k-i-1} I_{k-i-1}(2\sqrt{\rho}\, t)$$

$$-\frac{1}{\rho-1}\left\{\sum_{k=i+1}^{\infty}(\sqrt{\rho})^{-k} I_k(2\sqrt{\rho}\, t) - \frac{1}{\rho^i}\sum_{k=i+1}^{\infty}(\sqrt{\rho})^k I_k(2\sqrt{\rho}\, t)\right\}\Bigg],$$

which is valid for the case when $\rho \neq 1$. Under the consideration of the properties (5) and (8) of modified Bessel function, we can easily see that

$$\sum_{k=1}^{\infty} k(\sqrt{\rho})^{k-i-1} I_{k-i-1}(2\sqrt{\rho}\, t) = \sum_{k=-i}^{\infty}(k+i+1)(\sqrt{\rho})^k I_k(2\sqrt{\rho}\, t)$$

$$= \sum_{k=-i}^{\infty} k(\sqrt{\rho})^k I_k(2\sqrt{\rho}\, t) + (i+1)\sum_{k=-i}^{\infty}(\sqrt{\rho})^k I_k(2\sqrt{\rho}\, t)$$

$$= t\sum_{k=-i}^{\infty}(\sqrt{\rho})^{k+1}\left\{I_{k-1}(2\sqrt{\rho}\, t) - I_{k+1}(2\sqrt{\rho}\, t)\right\}$$

$$+(i+1)\left\{e^{(1+\rho)t} - \sum_{k=i+1}^{\infty}(\sqrt{\rho})^{-k} I_k(2\sqrt{\rho}\, t)\right\}$$

and

$$\sum_{k=i+1}^{\infty}(\sqrt{\rho})^k I_k(2\sqrt{\rho}\, t) = \sum_{k=-\infty}^{-i-1}(\sqrt{\rho})^{-k} I_k(2\sqrt{\rho}\, t)$$

$$= e^{(1+\rho)t} - \sum_{k=-i}^{\infty}(\sqrt{\rho})^{-k} I_k(2\sqrt{\rho}\, t).$$

Then we have an equality

$$L(t)=i+\frac{1}{\rho^{i}(\rho-1)}+te^{-(1+\rho)t}\sum_{k=-i}^{\infty}(\sqrt{\rho})^{k+1}\Big\{I_{k-1}(2\sqrt{\rho}\,t)$$

$$-I_{k+1}(2\sqrt{\rho}\,t)\Big\}-e^{-(1+\rho)t}\Big\{\Big(i+1+\frac{1}{\rho-1}\Big)\sum_{k=i+1}^{\infty}(\sqrt{\rho})^{-k}I_{k}(2\sqrt{\rho}\,t)$$

$$+\frac{1}{\rho^{i}(\rho-1)}\sum_{k=-i}^{\infty}(\sqrt{\rho})^{-k}I_{k}(2\sqrt{\rho}\,t)\Big\}.$$

The first summation on the right side of this equation can be changed as follows:

$$\sum_{k=-i}^{\infty}(\sqrt{\rho})^{k+1}\Big\{I_{k-1}(2\sqrt{\rho}\,t)-I_{k+1}(2\sqrt{\rho}\,t)\Big\}$$

$$=\rho\sum_{k=-i-1}^{\infty}(\sqrt{\rho})^{k}I_{k}(2\sqrt{\rho}\,t)-\sum_{k=-i+1}^{\infty}(\sqrt{\rho})^{k}I_{k}(2\sqrt{\rho}\,t)$$

$$=(\sqrt{\rho})^{-i+1}I_{i+1}(2\sqrt{\rho}\,t)+(\sqrt{\rho})^{-i+2}I_{i}(2\sqrt{\rho}\,t)$$

$$+(\rho-1)\sum_{k=-i+1}^{\infty}(\sqrt{\rho})^{k}I_{k}(2\sqrt{\rho}\,t)$$

$$=(\sqrt{\rho})^{-i+1}I_{i+1}(2\sqrt{\rho}\,t)+(\sqrt{\rho})^{-i+2}I_{i}(2\sqrt{\rho}\,t)$$

$$+(\rho-1)\Big\{e^{(1+\rho)t}-\sum_{k=i}^{\infty}(\sqrt{\rho})^{-k}I_{k}(2\sqrt{\rho}\,t)\Big\}.$$

Then we finally find the formula

$$(19)\quad L(t)=i+\frac{1}{\rho^{i}(\rho-1)}+(\rho-1)t$$

$$+e^{-(1+\rho)t}\Big[\Big\{(\sqrt{\rho})^{-i+1}I_{i+1}(2\sqrt{\rho}\,t)+(\sqrt{\rho})^{-i+2}I_{i}(2\sqrt{\rho}\,t)$$

$$-(\rho-1)\sum_{k=i}^{\infty}(\sqrt{\rho})^{-k}I_k(2\sqrt{\rho}\ t)\Big\}t$$

$$-\Big\{\Big(i+1+\frac{1}{\rho-1}\Big)\sum_{k=i+1}^{\infty}(\sqrt{\rho})^{-k}I_k(2\sqrt{\rho}\ t)$$

$$+\frac{1}{\rho^i(\rho-1)}\sum_{k=-i}^{\infty}(\sqrt{\rho})^{k}I_k(2\sqrt{\rho}\ t)\Big\}\Big].$$

Let P_3, $-N_3$ be the sum of positive or negative terms in the brace [ ] on the right side of this equation, then

$$P_3<\frac{1+\sqrt{\rho}}{(\sqrt{\rho})^{i-1}}tI_0(2\sqrt{\rho}\ t) \qquad (t>0)$$

$$N_3<(\rho-1)\frac{\frac{1}{(\sqrt{\rho})^i}}{1-\frac{1}{\sqrt{\rho}}}tI_0(2\sqrt{\rho}\ t)$$

$$+\left\{\Big(i+1+\frac{1}{1-\rho}\Big)\frac{\frac{1}{(\sqrt{\rho})^{i+1}}}{1-\frac{1}{\sqrt{\rho}}}+\frac{1}{\rho^i(\rho-1)}\frac{(\sqrt{\rho})^i}{1-\frac{1}{\sqrt{\rho}}}\right\}I_0(2\sqrt{\rho}\ t)$$

$$=\left\{\frac{1+\sqrt{\rho}}{(\sqrt{\rho})^{i-1}}t+\frac{i(\sqrt{\rho}-1)+\sqrt{\rho}}{(\sqrt{\rho}-1)^2(\sqrt{\rho})^i}\right\}I_0(2\sqrt{\rho}\ t).$$

Then we finally reach the result

$$\text{(20)}\qquad \left|L(t)-\Big\{i+\frac{1}{\rho^i(\rho-1)}+(\rho-1)t\Big\}\right|$$

$$<\frac{2(1+\sqrt{\rho})}{(\sqrt{\rho})^{i-1}}te^{-(1+\rho)t}I_0(2\sqrt{\rho}\ t) \quad \Big(t>\frac{1}{1-\rho}\Big(\frac{i}{\sqrt{\rho}}+\frac{1}{\sqrt{\rho}-1}\Big)\Big)$$

$$\sim \frac{2(1+\sqrt{\rho})}{(\sqrt{\rho})^{i-1}} t e^{-(1+\rho)t} \frac{e^{2\sqrt{\rho}t}}{\sqrt{2\pi \cdot 2\sqrt{\rho}t}} = \frac{1+\sqrt{\rho}}{(\sqrt{\rho})^{i-1}} \frac{e^{-(\sqrt{\rho}-1)^2 t}}{\sqrt{\pi}\sqrt{\rho}} \sqrt{t} \to 0$$

$$(t \to +\infty).$$

Next, we consider the case when $\rho=1$. In this case we obtain the state probabilities $P_n(t)$ from (2);

$$P_n(t)=e^{-2t}\left\{I_{n-i}(2t)+I_{n+i+1}(2t)\right\}.$$

Then we have an equality

$$L(t)=\sum_{n=1}^{\infty} nP_n(t)$$

$$=e^{-2t}\left\{\sum_{n=1}^{\infty} nI_{n-i}(2t)+\sum_{n=1}^{\infty} nI_{n+i+1}(2t)\right\}$$

$$=e^{-2t}\left\{\sum_{k=-i+1}^{\infty}(k+i)I_k(2t)+\sum_{k=i+2}^{\infty}(k-i-1)I_k(2t)\right\}$$

$$=e^{-2t}\left\{i\sum_{k=-i+1}^{\infty} I_k(2t)-(i+1)\sum_{k=i+2}^{\infty} I_k(2t)\right.$$

$$\left.+\sum_{k=-i+1}^{\infty} kI_k(2t)+\sum_{k=i+2}^{\infty} kI_k(2t)\right\}.$$

Using the properties (8) and (5) of modified Bessel function, we get

$$i\sum_{k=-i+1}^{\infty} I_k(2t)-(i+1)\sum_{k=i+2}^{\infty} I_k(2t)$$

$$=i\sum_{k=-i+1}^{i+1} I_k(2t)-\sum_{k=i+2}^{\infty} I_k(2t)$$

$$=i\sum_{k=-i+1}^{i+1} I_k(2t)-\frac{1}{2}e^{2t}+\frac{1}{2} I_0(2t)+\sum_{k=1}^{i+1} I_k(2t)$$

and

$$\sum_{k=-i+1}^{\infty} kI_k(2t)+\sum_{k=i+2}^{\infty} kI_k(2t)$$

$$=t\left[\sum_{k=-i+1}^{\infty}\left\{I_{k-1}(2t)-I_{k+1}(2t)\right\}+\sum_{k=i+2}^{\infty}\left\{I_{k-1}(2t)-I_{k+1}(2t)\right\}\right]$$

$$=t\sum_{k=i-1}^{i+2} I_k(2t).$$

Then we have a formula

$$(21)\quad L(t)=-\frac{1}{2}+te^{-2t}\sum_{k=i-1}^{i+2} I_k(2t)$$

$$+e^{-2t}\left\{i\sum_{k=-i+1}^{i+1} I_k(2t)+\frac{1}{2}I_0(2t)+\sum_{k=1}^{i+1} I_k(2t)\right\},$$

from which we can obtain an inequality

$$(22)\quad 0<L(t)-\left\{te^{-2t}\sum_{k=i-1}^{i+2} I_k(2t)-\frac{1}{2}\right\}$$

$$<\left(2i^2+2i+\frac{3}{2}\right)e^{-2t}I_0(2t)$$

$$\sim\left(i^2+i+\frac{3}{4}\right)\frac{1}{\sqrt{\pi t}}\to 0 \qquad (t\to+\infty),$$

where

$$te^{-2t}\sum_{k=i-1}^{i+2} I_k(2t)\sim\frac{2\sqrt{t}}{\sqrt{\pi}} \qquad (t\to+\infty)$$

In the case where $i=0$ we have an inequality

$$(22') \quad 0<L(t)-\left[te^{-2t}\left\{I_0(2t)+2I_1(2t)+I_2(2t)\right\}-\frac{1}{2}\right]$$

$$<\frac{3}{2}e^{-2t}I_0(2t)\to 0 \qquad (t\to+\infty).$$

6. FURTHER RESULTS

Departing from (2) we can calculate the probability that at least m units are in the system at time t

$$\sum_{n=m}^{\infty} P_n(t)=e^{-(1+\rho)t}\Big[\sum_{n=m}^{\infty}(\sqrt{\rho})^{n-i}I_{n-i}(2\sqrt{\rho}\,t)$$

$$+\sum_{n=m}^{\infty}(\sqrt{\rho})^{n-i-1}I_{n+i+1}(2\sqrt{\rho}\,t)$$

$$+(1-\rho)\sum_{n=m}^{\infty}\rho^n\sum_{k=n+i+2}^{\infty}(\sqrt{\rho})^{-k}I_k(2\sqrt{\rho}\,t)\Big].$$

By interchanging the summation the last term in the brace on the right side becomes

$$(1-\rho)\sum_{k=m+i+2}^{\infty}\sum_{n=m}^{k-i-2}\rho^n(\sqrt{\rho})^{-k}I_k(2\sqrt{\rho}\,t)$$

$$=\sum_{k=m+i+2}^{\infty}\rho^m(1-\rho^{k-i-m-1})(\sqrt{\rho})^{-k}I_k(2\sqrt{\rho}\,t)$$

$$=\rho^m\sum_{k=m+i+2}^{\infty}(\sqrt{\rho})^{-k}I_k(2\sqrt{\rho}\,t)-\sum_{k=m+i+2}^{\infty}(\sqrt{\rho})^{k-2i-2}I_k(2\sqrt{\rho}\,t)$$

$$=\sum_{k=m+i+2}^{\infty}(\sqrt{\rho})^{2m-k}I_k(2\sqrt{\rho}\,t)-\sum_{k=m+1}^{\infty}(\sqrt{\rho})^{k-i-1}I_{k+i+1}(2\sqrt{\rho}\,t).$$

Then we have an equality

$$(23)\quad \sum_{n=m}^{\infty} P_n(t)=e^{-(1+\rho)t}\sum_{k=m-i}^{\infty}(\sqrt{\rho})^k I_k(2\sqrt{\rho}\,t)+\sum_{k=m+i+1}^{\infty}(\sqrt{\rho})^{2m-k}I_k(2\sqrt{\rho}\,t).$$

From the property (8) of modified Bessel function we have

$$\sum_{k=m+i+1}^{\infty}(\sqrt{\rho})^k I_k(2\sqrt{\rho}\,t)=e^{(1+\rho)t}-\sum_{k=-m-i}^{\infty}(\sqrt{\rho})^k I_k(2\sqrt{\rho}\,t).$$

Substituting this into (23), we can reach the result

$$(24)\quad \sum_{n=m}^{\infty} P_n(t)$$

$$=\rho^m+e^{-(1+\rho)t}\left\{\sum_{k=m-i}^{\infty}(\sqrt{\rho})^k I_k(2\sqrt{\rho}\,t)-\sum_{k=m-i}^{\infty}(\sqrt{\rho})^k I_{k-2m}(2\sqrt{\rho}\,t)\right\},$$

where the first term of the right side equals to the probability $p_{\geqq m}$ that at least m units are in the system in equilibrium. From (24) we can easily obtain the following inequalities;

$$(25)\quad \left|\sum_{n=m}^{\infty} P_n(t)-p_{\geqq m}\right|<\frac{(\sqrt{\rho})^{m-i}}{1-\sqrt{\rho}}e^{-(1+\rho)t}I_0(2\sqrt{\rho}\,t)\to 0\quad (t\to+\infty)$$

and

$$(25')\quad \left|\sum_{n=0}^{m-1} P_n(t)-p_{<m}\right|<\frac{(\sqrt{\rho})^{m-i}}{1-\sqrt{\rho}}e^{-(1+\rho)t}I_0(2\sqrt{\rho}\,t)\to 0\quad (t\to+\infty).$$

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