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ERGODICITY OF A TANDEM QUEUE WITH BLOCKING

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I. INTRODUCTION

Continued from the previous paper [1] we consider the following system.

There are two counters in series. Customers arrive at the first counter at the instants $\tau_0, \tau_1, \tau_2, \ldots, \tau_n, \ldots$ The coustomers will be served in order of their arrival.

The output of the first counter comprises the input into the second, and no queue is allowed to form before the second counter, whereas an infinite queue is allowed before the first. This results in blocking of service at the first counter even though the service of a customer has just been completed and there is a queue. The first counter opens for service when the customer can go to service in the second counter when the latter becomes free.

Let $R_n^{(1)}$ denote the service time in the first counter of the n-th customer and let $R_n^{(2)}$ be the service time of the n-th customer in the second counter. Let $g_n=\tau_{n+1}-\tau_n \quad (n\geqq 0)$.

We assume that each of the three sequences $\{R_n^{(1)}, n\geqq 0\}$, $\{g_n, n\geqq 0\}$, $\{R_n^{(2)}, n\geqq 0\}$ is a sequence of independent and identieally distributed random variables and that the three sequences are mutually independent. Furthermore we assume that the $R_n^{(1)}$'s, g_n's and $R_n^{(2)}$'s are non-negative random variables and that $ER^{(2)}<\infty$, $Eg<\infty$, $ER^{(2)}<\infty$.

Let W_n be the time of completion of service of the $(n-1)$-th customer at the first counter minus the time τ_n of arrivel of the n-th customer at the first counter. Then max $(0, W_n)$ is the waiting time for the

n-th customer.

In Theorem 1 and 2 below, we shall find criteria for the existence of a limiting probability distribution of $\{W_n\}$. The criteria include the ones obtained in [1].

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II. ERGODICITY

Let $a\vee b$ denote the maximum of a and b, and let $a\wedge b$ the minimum of a and b. Then we have

$$W_{n+1}=\begin{cases} W_n+R_n^{(1)}\vee R_{n-1}^{(2)}-g_n & (W_n\geqq 0)\,, \\ W_n+(-W_n+R_n^{(1)})\vee R_{n-1}^{(2)}-g_n & (W_n<0)\,. \end{cases}$$

Therefore we may immediately write the following equation:

$$W_{n+1}=W_n+R_{n-1}^{(2)}\vee(R_n^{(1)}-0\wedge W_n)-g_n \qquad (n\geqq 1)\,. \tag{1}$$

In addition, let $P_n=R_n^{(1)}-g_n$ and $Q_n=R_{n-1}^{(2)}-g_n$ for all n; then equation (1) may be written

$$\begin{aligned} W_{n+1}&=[R_n^{(1)}-g_n+W_n-0\wedge W_n]\vee[R_{n-1}^{(2)}-g_n+W_n] \\ &=[R_n^{(1)}-g_n+0\vee W_n]\vee[R_{n-1}^{(2)}-g_n+W_n] \\ &=[R_n^{(1)}-g_n]\vee[R_n^{(1)}-g_n+W_n]\vee[R_{n-1}^{(2)}-g_n+W_n] \\ &=P_n\vee[P_n\vee Q_n+W_n]\,. \end{aligned} \tag{2}$$

Let $Z_n=(W_n,\ R_{n-1}^{(2)})$. The sequence $\{Z_n\}$ is a Markov process with stationary transition probabilities.

Let t, x, y and ρ be real numbers.

Lemma 1. $P(W_n\leqslant t\,|\,W_1=x,\ R_0^{(2)}=\rho)\leqq P(W_n\leqslant t\,|\ W_1=y,\ R_0^{(2)}=\rho)$
$\leqq P(W_n\leqslant t\,|\ W_1=y,\ R_\rho^{(2)}=0)$

for all n, ρ, x and y where $x\geqq y$.

Proof: Fix a point ω in the sample space of $R_1^{(1)},\ldots,R_n^{(1)}$, g_1, $\ldots$, g_n, $R_1^{(2)},\ldots,R_{n-1}^{(2)}$ and let

$$W_1(\omega,\ x,\ \rho)=x\,,\qquad W_1(\omega,\ y,\ \rho)=y\,,$$
$$\Omega_k=P_k\vee Q_k+W_k \qquad \text{for} \qquad 1\leqq k\leqq n-1\,.$$

Then

$$\begin{aligned}\Omega_1(\omega,\ x,\ \rho)&=P_1(\omega)\vee Q_1(\omega,\ \rho)+W_1(\omega,\ x)\\ &=P_1(\omega)\vee Q_1(\omega,\ \rho)+x\geqq P_1(\omega)\vee Q_1(\omega,\ \rho)+y\\ &=\Omega_1(\omega,\ y,\ \rho)\geqq\Omega_1(\omega,\ y,\ 0).\end{aligned}$$

Thus

$$\begin{aligned}W_2(\omega,\ x,\ \rho)&=P_1(\omega)\vee\Omega_1(\omega,\ x,\ \rho)\geqq P_1(\omega)\vee\Omega_1(\omega,\ y,\ \rho)\\ &=W_2(\omega,\ y,\ \rho)\geqq W_2(\omega,\ y,\ 0).\end{aligned}$$

From this

$$\begin{aligned}\Omega_2(\omega,\ x,\ \rho)&=P_2(\omega)\vee Q_2(\omega,\ \rho)+W_2(\omega,\ x,\ \rho)\\ &\geqq P_2(\omega)\vee Q_2(\omega)+W_2(\omega,\ y,\ \rho)=\Omega_2(\omega,\ y,\ \rho)\\ &\geqq\Omega_2(\omega,\ y,\ 0).\end{aligned}$$

Therefore

$$W_3(\omega,\ x,\ \rho)\geqq W_3(\omega,\ y,\ \rho)\geqq W_3(\omega,\ y,\ 0).$$

In the same way we have

$$W_k(\omega,\ x,\ \rho)\geqq W_k(\omega,\ y,\ \rho)\geqq W_k(\omega,\ y,\ 0) \quad \text{for all} \quad 1\leqq k\leqq n.$$

This concludes the proof of Lemma 1.

Lemma 2. $P(W_n\leqslant t\,|\,W_1=0,\ R_0^{(2)}=0)\to F_0(t)$ as $n\to\infty$ where F_0 may not be a probability distribution function.

Proof: Let $H(x,\ \rho)=P(W_2\leqq x,\ R_1^{(2)}\leqq\rho\,|\,W_1=0,\ R_0^{(2)}=0)$.

Then

$$\begin{aligned}&P(W_{n+1}\leqslant t\,|\,W_1=0,\quad R_0^{(2)}=0)\\ &=\int P(W_{n+1}\leqslant t\,|\,W_2=x,\ \ R_1^{(2)}=\rho,\ \ W_1=0,\ R_0^{(2)}=0)dH(x,\ \rho). \qquad (3)\end{aligned}$$

Since $\{Z_n\}$ is a stationary Markov process and because of Lemma 1, for $x\geqq0$

$$\begin{aligned}&P(W_{n+1}\leqslant t\,|\,W_2=x,\quad R_1^{(2)}=\rho,\quad W_1=0,\ \ R_0^{(2)}=0)\\ &=P(W_n\leqslant t\,|\,W_1=x,\quad R_0^{(2)}=\rho)\\ &\leqq P(W_n\leqslant t\,|\,W=0,\quad R_0^{(2)}=0) \qquad (4)\end{aligned}$$

and also for $x<0$

$$\begin{aligned} &P(W_n \leqslant t \mid W_1=x, \quad R_0^{(2)}=\rho) \\ &\qquad =P(W \leqslant t \mid W_1=0, \quad R_0^{(2)}=(\rho+x)\vee 0) \\ &\qquad \leqq P(W_n \leqslant t \mid W_1=0, \quad R_0^{(2)}=0). \end{aligned} \tag{5}$$

Then, using (4) and (5) in (3) we have

$$\begin{aligned} &P(W_{n+1} \leqslant t \mid W_1=0, \quad R_0^{(2)}=0) \\ &\qquad \leqq \int P(W_n \leqslant t \mid W_1=0, \quad R_0^{(2)}=0)dH(x, \rho) \\ &\qquad =P(W_n \leqslant t \mid W_1=0, \quad R_0^{(2)}=0). \end{aligned}$$

Thus $P(W_n \leqslant t \mid W_1=0,\ R_0^{(2)}=0)$ is a monotone sequence and therefore converges to a limit which we call $F_0(t)$.

Theorem 1. If $E \max (R^{(1)}, R^{(2)})<Eg$, then F_0 defined in Lemma 2 is a probability distribution.

Proof: Because of Lemma 2 we need only show that, under the assumptions in the theorem, $\{W_n\}$ is bounded in probability, that is for all n

$$P(W_n \leqslant t \mid W_1=0, \quad R_0^{(2)}=0) \geqq 1-\varepsilon(t) \tag{6}$$

where $\varepsilon(t) \to 0$ as $t \to \infty$.

Iterating (2) we have

$$\begin{aligned} W_{n+1} &= P_n \vee [P_n \vee Q_n + W_n] \\ &= P_n \vee [P_n \vee Q_n + P_{n-1} \vee (P_{n-1} \vee Q_{n-1} + W_{n-1})] \\ &= P_n \vee [P_n \vee Q_n + P_{n-1}] \vee [P_n \vee Q_n + P_{n-1} \vee Q_{n-1} + W_{n-1}] \\ &= P_n \vee [P_n \vee Q_n + P_{n-1}] \vee [P_n \vee Q_n + P_{n-1} \vee Q_{n-1} + P_{n-2}] \\ &\qquad \vee [P_{n-2} \vee Q_{n-2} + W_{n-2})] \\ &= P_n \vee [P_n \vee Q_n + P_{n-1}] \vee [P_n \vee Q_n + P_{n-1} \vee Q_{n-1} + P_{n-2} \vee Q_{n-2}] \\ &\qquad \vee [P_n \vee Q_n + P_{n-1} \vee Q_{n-1} + P_{n-2} \vee Q_{n-2} + W_{n-2}] = \text{etc}. \end{aligned} \tag{7}$$

The iteration is continued until the last term is reached and we note the conditions $W_1=0$, $R_0^{(2)}=0$, namely,

$$W_{n+1} = P_n \vee [P_n \vee Q_n + P_{n-1}] \vee [P_n \vee Q_n + P_{n-1} \vee Q_{n-1} + P_{n-2}]$$

$$\vee \ldots \vee [P_n \vee Q_n + P_{n-1} \vee Q_{n-1} + \ldots + P_2 \vee Q_2 + P_1] . \quad (8)$$

Replacing P_i in each term of the right hand by $P_i \vee Q_i$, we have

$$W_{n+1} \leqq \max_{1 \leqslant k \leqslant n} \left\{ \sum_{i=k}^{n} [\max (R_i^{(1)}, \quad R_{i-1}^{(2)}) - g_i] \right\} .$$

Define $\xi_i = \max (R_i^{(1)}, \quad R_{i-1}^{(2)}) - g_i$ and let $U_k = \sum_{i=1}^{k} \xi_i$. Thus U_k is the k-th partial sum of independent and identically distributed random variables. Then $\max_{1 \leqslant k \leqslant n} (U_n - U_{k-1})$ has the same distribution as $\max_{1 \leqslant k \leqslant n} U_k$. Using the law of large numbers, under the assumption $E \max (R^{(1)}, R^{(2)}) < Eg$, $\max_{1 \leqslant k \leqslant n} U_k$ converges to a finite random variable with probability one which implies that $\{W_n\}$ is bounded in probability. See Lindley [2] and Sacks [3].

The following theorem shows the necessity of the condition of Theorem 1.

Theorem 2. If $E \max (R^{(1)}, R^{(2)}) \geqq Eg$, then $F_0(t) \equiv 0$, where the case $\max (R^{(1)}, R^{(2)}) - g = 0$ identically is excluded.

Proof: Let $W_n{}^*$ be the departure time of the n-th customer at the first counter minus the arrival time of the n-th customer at the first counter. Then

$$W_n{}^* = \begin{cases} W_n + \max (R_n^{(1)}, \quad R_{n-1}^{(2)}) & (W_n \geqq 0) \\ \max (R_n^{(1)}, \quad R_{n-1}^{(2)} + W_n) & (W_n < 0) . \end{cases}$$

Therefore we may write the following equation:

$$\begin{aligned} W_n{}^* &= [R_n^{(1)} + 0 \vee W_n] \vee [R_{n-1}^{(2)} + W_n] \\ &= R_n^{(1)} \vee [R_n^{(1)} + W_n] \vee [R_{n-1}^{(2)} + W_n] \\ &= R_n^{(1)} \vee [R_n^{(1)} \vee R_{n-1}^{(2)} + W_n] . \end{aligned} \quad (9)$$

Also we have

$$W_{n+1}^* = \begin{cases} W_n{}^* - g_n + R_{n+1}^{(1)} \vee R_n^{(2)} & (W_n{}^* - g_n \geqq 0) \\ R_{n+1}^{(1)} \vee [W_n{}^* - g_n + R_n^{(2)}] & (W_n{}^* - g_n < 0, \quad W_n{}^* - g_n + R_n^{(2)} > 0) \\ R_{n+1}^{(1)} & (W_n{}^* - g_n + R_n^{(2)} < 0 . \end{cases}$$

Then the above equation may be written

$$\begin{aligned} W_{n+1}^{*} &= [R_{n+1}^{(1)} + (W_n^{*} - g_n) \vee 0] \vee [R_n^{(2)} + W_n^{*} - g_n] \\ &= R_{n+1}^{(1)} \vee [R_{n+1}^{(1)} + W_n^{*} - g_n] \vee [R_n^{(2)} + W_n^{*} - g_n] \\ &= R_{n+1}^{(1)} \vee [(R_{n+1}^{(1)} - g_n) \vee (R_n^{(2)} - g_n) + W_n^{*}] \,. \end{aligned} \tag{10}$$

Let $P_n^{*} = R_{n+1}^{(1)} - g_n$ and $Q_n^{*} = R_n^{(2)} - g_n$; then equation (10) may be written

$$W_{n+1}^{*} = R_{n+1}^{(1)} \vee [P_n^{*} \vee Q_n^{*} + W_n^{*}] \,. \tag{11}$$

Iterating (11), we have

$$\begin{aligned} W_{n+1}^{*} = R_{n+1}^{(1)} &\vee [P_n^{*} \vee Q_n^{*} + R_n^{(1)}] \\ &\vee [P_n^{*} \vee Q_n^{*} + P_{n-1}^{*} \vee Q_{n-1}^{*} + R_{n-1}^{(1)}] \\ &\vee \ldots \vee [P_n^{*} \vee Q_n^{*} + P_{n-1}^{*} \vee Q_{n-1}^{*} + \ldots + P_1^{*} \vee Q_1^{*} + R_1^{(1)}] \,. \end{aligned} \tag{12}$$

Neglecting $R_i^{(1)}$ in each term of the right hand, we have

$$W_{n+1}^{*} \geqq \max_{1 \leqslant k \leqslant n} \left\{ \sum_{i=k}^{n} [\max(R_{i+1}^{(2)},\ R_i^{(2)}) - g_n] \right\} . \tag{13}$$

Under the condition $E \max(R^{(1)}, R^{(2)}) \geqq Eg$ where $\max(R^{(1)}, R^{(2)}) \neq g$ certainly,

$$\max_{1 \leqslant k \leqslant n} \left\{ \sum_{i=n}^{n} [\max(R_{i+1}^{(1)},\quad R_i^{(2)}) - g_i] \right\} \rightarrow +\infty$$

in probablity. This fact shows that $W_{n+1}^{*} \rightarrow +\infty$ in probability and then (9) shows that $W_n \rightarrow +\infty$ in probability which proves the theorem.

Corollary. (a) If $ER^{(1)} \geqq Eg$ then $F_0(t) \equiv 0$, where the case $R^{(1)} - g = 0$ indentically is excluded.

(b) If $ER^{(2)} \geqq Eg$ then $F_0(t) \equiv 0$, where the case $R^{(2)} - g = 0$ identically is excluded.

Proof. For (a) we have from (8) that

$$W_{n+1} \geqq \max_{1 \leqslant k \leqslant n} \left[\sum_{i=k}^{n} (R_i^{(1)} - g_i) \right] .$$

Thus we see that, under the assumption $ER^{(1)} \geqq Eg$ where $R^{(2)} - g \neq 0$ certainly, $W_n \rightarrow +\infty$ in probability.

For (b) we have from (13) that that

$$W_{n+1}^{*} \geqq \max_{1 \leqslant k \leqslant n} \left[\sum_{i=k}^{n} (R_i^{(2)} - g_i) \right].$$

Thus we see that, under the condition $ER^{(2)} \geqq Eg$ where $R^{(2)} - g \neq 0$ certainly, $W_n \to +\infty$ in probability.

Remark 1. In the case where $\max(R^{(1)}, R^{(2)}) = g$, $R^{(1)} = g$ identically, then $W_n = 0$. Thus $F_0(t)$ is unit distribution with jump 1 at 0. Under the initial condition W_1,

$$W_n = \begin{cases} W_1 & (W_1 \geqq 0) \\ 0 & (W_1 < 0)\,. \end{cases}$$

Therefore
$$\lim_{n \to \infty} P(W_n \leqslant t \mid W_1 = x) = F_0(t - x \vee 0).$$

Now we shall show that, under the condition in Theorem 1 the limiting distribution of $\{W_n\}$ is independent of the initial conditions. From (7),

$$\begin{aligned} &P(W_{n+1} \leqslant t \mid W_1 = x, \quad R_0^{(2)} = \rho) \\ &\quad = P(P_n \leqslant t, \quad P_n \vee Q_n + P_{n-1} \leqslant t, \ldots, P_n \vee Q_n + \ldots + P_2 \vee Q_2 + P_1 \leqslant t, \\ &\quad P_n \vee Q_n + \ldots + P_2 \vee Q_2 + P_1 \vee Q_1 \leqslant t - x) \\ &\quad \geqq P(W_{n+1} \leqslant t \mid W_1 = 0, \ R_0^{(2)} = 0) - P(P_n \vee Q_n + \ldots + P_1 \vee Q_1 > t - x)\,. \end{aligned}$$

Hence

$$\liminf_{n \to \infty} P(W_n \leqslant t \mid W_1 = x, \quad R_0^{(2)} = \rho) \geqq F_0(t)\,, \tag{14}$$

since the second term in the right member tends to zero by the strong law of large numbers whenever $EP \vee Q < 0$. But also from (4) and (5),

$$P(W_n \leqslant t \mid W_1 = x, \ R_0^{(2)} = \rho) \leqq P(W_n \leqslant t \mid W_1 = 0, \ R_0^{(2)} = 0)\,,$$

then

$$\limsup_{n \to \infty} P(W_n \leqslant t \mid W_1 = x, \ R_0^{(2)} = \rho) \leqq F_0(t)\,. \tag{15}$$

(14) and (15) give us the desired result.

If $Z_1 = (W_1, R_0^{(2)})$ has a distribution function $Z(x, \rho)$,

$$\lim_{n\to\infty}\int P(W_n\leqslant t \mid W_1=x,\ R_0^{(2)}=\rho)dZ(x,\ \rho)$$
$$=\int\lim_{n\to\infty} P(W_n\leqslant t \mid W_1=x,\ R_0^{(2)}=\rho)dZ(x,\ \rho)$$
$$=F_0(t)$$

by Lebesgue's theorem on limits of integrals.

Clearly $F_0(t)$ satisfies the equation:

$$F_0(t)=\int K(t,\ u)dF_0(u) \tag{16}$$

where

$$K(t,\ u)=P(R^{(1)}-g+0\vee u\leqslant t,\quad R^{(2)}-g+u\leqslant t)\,. \tag{17}$$

The previous paragraph state that there exists one and only one solution satisfying the equation (16) whenever $E\max(R^{(1)},\ R^{(2)})<Eg$. To see it is unique, suppose there were another solution and that it was the distribution of W_1; then it would be the distribution of every customer and hence the limiting distribution, which is impossible.

Remark 2. In the case where Poisson arrival, exponential services, the solution of the equation (16) has not the following simple form:

$$F_0(t)=\begin{cases} 1-Ae^{-Bt} & (t\geqq 0) \\ (1-A)e^{Ct} & (t<0)\,, \end{cases}$$

where $\qquad 0<A<1,\quad B>0,\quad C>0.$

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