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ON THE MEAN PASSAGE TIME CONCERNING SOME QUEUING PROBLEMS OF THE TENDAM TYPE

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INTRODUCTION

In the study of queuing systems in which queues are in tandem, it is somewhat difficult to calculate in a common way mean waiting time, mean number of customers in queues for these systems, since there occur situations that some service stations are being blocked.

However, if it is possible to obtain conditions that these systems be in equilibrium, we shall have an advantage in providing information of characteristics of the systems concerned.

There are a few papers with this viewpoint, obtaining the maximum possible utilization $\rho_{\max}$ about the simple case of the system.[2][7][8]

In every case of these papers, first of all, they provide balance equations for steady state probabilities, and then, by considering the generating function of these balance equations, they evaluate some numerical values of $\rho_{\max}$.

Unfortunately, except in the case of simple systems, this method, in practice, will not be applicable to general systems, because of tremendous amount of calculations.

Moreover, this is because the characteristics of $\rho_{\max}$ is not always clear in the mathematical meaning.

Then, we introduce the concept of quantity named "mean passage time".

At first, when we think of the service station in the first stage, the time, when the customer begins to receive the service in the station, and go out from the station after receiving service is the summation of the time when the customer received the service in the station and the waiting time at the service station where the customer can not move forward to the next service station because of, so called, blocking state.

We call this as the passage time. If there were no influence of blocking in the other service station in the system, the mean passage time in the first stage service station should coincide with the mean service time in the first stage service station.

Consequently, to know the value of the mean passage time becomes the measure of knowing the influence of blocking conditions in the system.

Now, we will consider the following system in order to make the problem treat easily.

This is the system where there are infinite waiting line in front of the first stage service station, and that, next customer occupies the service station as soon as the first stage service station becomes empty. (In reality, it is letter to consider the ware-house in front of the first stage service station.)

So, it is not necessary to consider the empty state in the first stage service station.

In this case, we examine only the mean passage time in the first stage service station.

From now on, when we refer to the mean passage time, it only indicates the mean passage time in the first stage service station abbribiated as [M.P.T.].

The reasons why we analyze only the first stage service station with the infinite input are as follows.

We consider the prolongation of the mean passage time by the blocking effects is most significant in the first stage service station with the infinite input.

The main part in this paper is considered from two themas. One of the themas is to calculate the mean passage time in some systems of tandem type.

The other thema is to find the interdeparture time distributions, so called the output distributions, in the system of same types.

Specially, we refer to the problem how the mean passage time is varried by the rearrangement of the order of the service stations in the system of tandem type with the different service rate. We think it very important because of the considerations to have the mean passage time minimize upon the construction of such a systems.

[I] CALCULATION OF THE MEAN PASSAGE TIME

§ 1.1 SINGLE 2-STAGE SERVICE

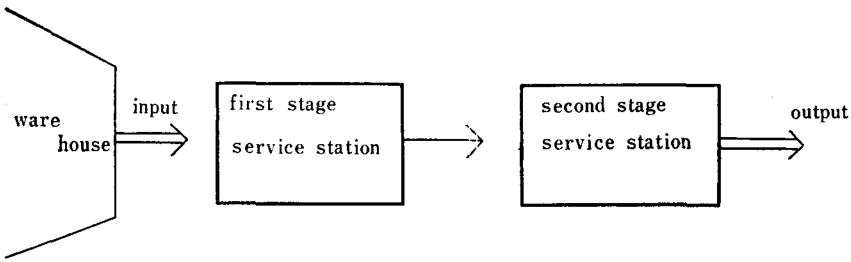

Fig. 1

At first, a customer is served in the first stage station, then move on the second stage. Even if after service to this customer in the first stage has finished, when the second stage station is occupied by the preceding customer, he must wait occupying the first station. We call this state as the first stage station is in blocking.

We shall calculate the [M.P.T.], supposing that the first stage station is always occupied in all time, so that we set a supposition on the system that if when the first stage station becomes empty, next customer comes in immediately and receives service in the first stage.

Let t_1 and t_2 are service times in the first stage and the second stage, respectively.

In order to find the value of the mean passage time, we may calculate the expectation of max (t_1, t_2), since

$$[\text{M.P.T.}]=E\{\max(t_1, t_2)\}.$$

Suppose that $f_1(t_1)$ and $f_2(t_2)$ are the density functions of service time distributions in the first stage and in the second stage respectively.

Then

$$P\{t_1>t_2\}\cdot E\{t_1 \mid t_1>t_2\}=\int_0^\infty\int_0^{t_1} t_1\cdot f_1(t_1)\cdot f_2(t_2)dt_2dt_1 \quad \cdots\cdots(1)$$

$$P\{t_2>t_1\}\cdot E\{t_2 \mid t_2>t_1\}=\int_0^\infty\int_0^{t_2} t_2\cdot f_1(t_1)\cdot f_2(t_2)dt_1dt_2. \quad \cdots\cdots(2)$$

It follows that the sum of both expressions

$$P\{t_1>t_2\}\cdot E\{t_1 \mid t_1>t_2\}+P\{t_2>t_1\}\cdot E\{t_2 \mid t_2>t_1\} \quad \cdots\cdots(3)$$

is equal to the mean passage time in the first stage.

In the following examples, we shall find results for some special service time distributions.

(Example 1.1) Exponential services at both stoges.

Let

$$f_1(t_1)=\mu_1 e^{-\mu_1 t_1}, \qquad f_2(t_2)=\mu_2 e^{-\mu_2 t_2},$$

then

$$P\{t_1>t_2\}=\iint_{t_1>t_2} f_1(t_1)\cdot f_2(t_2)dt_1dt_2=\frac{\mu_2}{\mu_1+\mu_2}, \quad \cdots\cdots(4)$$

$$P\{t_2>t_1\}=\iint_{t_2>t_1} f_1(t_1)\cdot f_2(t_2)dt_1dt_2=\frac{\mu_1}{\mu_1+\mu_2}, \quad \cdots\cdots(5)$$

and

$$P\{t_1>t_2\}\cdot E\{t_1 \mid t_1>t_2\}=\frac{\mu_2\cdot(2\mu_1+\mu_2)}{\mu_1\cdot(\mu_1+\mu_2)^2}, \quad \cdots\cdots(6)$$

$$P\{t_2>t_1\}\cdot E\{t_2 \mid t_2>t_1\}=\frac{\mu_1\cdot(2\mu_2+\mu_1)}{\mu_2\cdot(\mu_1+\mu_2)^2}, \quad \cdots\cdots(7)$$

Therefore, the mean passage time for the first stage is

$$[\text{M.P.T.}]=\frac{\mu_2\cdot(2\mu_1+\mu_2)}{\mu_1\cdot(\mu_1+\mu_2)^2}+\frac{\mu_1\cdot(2\mu_2+\mu_1)}{\mu_2\cdot(\mu_1+\mu_2)^2}$$

$$=\frac{1}{\mu_1}+\frac{1}{\mu_2}-\frac{1}{\mu_1+\mu_2} \quad \cdots\cdots(8)$$

In the case when $\mu_1=\mu_2=\mu$, this becomes

$$[\text{M.P.T.}]=\frac{3}{2}\cdot\frac{1}{\mu}. \quad \cdots\cdots(9)$$

(Example 1.2) Exponential service time at the first stage and 2-Erlang service at the second stage.

Suppose

$$f_1(t_1)=\mu e^{-\mu t_1}, \qquad f_2(t_2)=(2\mu)^2 t_2\cdot e^{-2\mu t_2} \quad \cdots\cdots(10)$$

In this case

$$[\text{M.P.T.}]=\int_0^\infty\int_0^{t_1} t_1(\mu e^{-\mu t_1})(4\mu^2\cdot t_2\cdot e^{-2\mu t_2})dt_2 dt_1$$

$$+\int_0^\infty\int_0^{t_2} t_2\cdot(\mu e^{-\mu t_1})(4\mu^2 t_2\cdot e^{-2\mu t_2})dt_1 dt_2$$

$$=\frac{13}{9}\cdot\frac{1}{\mu} \quad \cdots\cdots(11)$$

(Example 1.3) 2-Erlang services at both stages.

Suppose

$$f_1(t_1)=(2\mu)^2 t_1\cdot e^{-2\mu t}, \qquad f_2(t_2)=(2\mu)^2 t_2\cdot e^{-2\mu t_2}. \quad \cdots\cdots(12)$$

In this case

$$[\text{M.P.T.}]=\frac{11}{8}\cdot\frac{1}{\mu} \quad \cdots\cdots(13)$$

These results show the tendency that the decrease in randomness of service time distribution produce the decrease of the value of [M.P.T.] also.

Sometimes we face problems to determine the order of two service stations, in which the initially specified order must be preserved. In this case, it will be more advantageous to choose the order which gives smaller [M.P.T.].

In the case discussed above, we can easily know that the [M.P.T.] does not depend on the order of services in these two stations, because

$$[\text{M.P.T.}]=E\{\max(t_1, t_2)\}. \quad \cdots\cdots (14)$$

We call this property the reversibility of the [M.P.T.].

§ 1.2 SINGLE 3-STAGE SERVICE

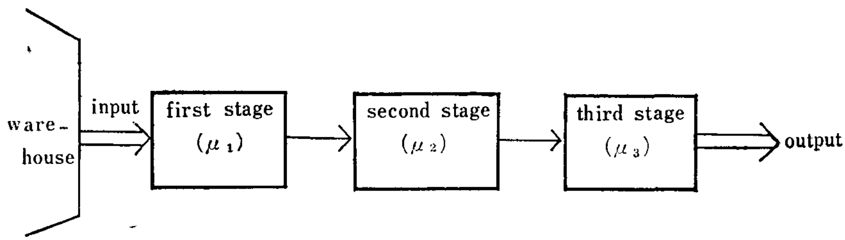

Fig. 2

A customer is served sequentially by these three service stations. A queue always exists in front of first stage, but no queues are allowed in front of other stages. Service time distributions of all stages are assumed to be exponential with mean $1/\mu_i$ $(i=1, 2, 3)$.

Assuming that a queue exist in front of the first stage station, states which may occur at the instant of the renewal of the first stage are as follows;

Ⓐ ………… the second stage is occupied and the third stage is empty.

Ⓑ ………… the second stage and the third stage are occupied.

In the case of Ⓐ state, mean passage time is obtained by the method of last paragraph, that is,

$$\frac{1}{\mu_1}+\frac{1}{\mu_2}-\frac{1}{\mu_1+\mu_2}\ . \qquad \cdots\cdots(15)$$

On the other hand, denoting service times after the renewal by t_i $(i=1, 2, 3)$, we have

$$P\{t_1>t_2, t_3\}\cdot E\{t_1 \mid t_1>t_2, t_3\}$$

$$=\iiint_{t_1>t_2, t_3} t_1\cdot f_1(t_1)\cdot f_2(t_2)\cdot f_3(t_3)dt_1dt_2dt_3$$

$$=\mu_1\cdot\left\{\frac{1}{\mu_1^2}-\frac{1}{(\mu_1+\mu_2)^2}-\frac{1}{(\mu_1+\mu_3)^2}+\frac{1}{(\mu_1+\mu_2+\mu_3)^2}\right\}.$$

Similarly,

$$P\{t_2>t_1, t_3\}\cdot E\{t_2 \mid t_2>t_1, t_3\}=\mu_2\cdot\left[\frac{1}{\mu_2^2}-\frac{1}{(\mu_1+\mu_2)^2}-\frac{1}{(\mu_2+\mu_3)^2}+\frac{1}{(\mu_1+\mu_2+\mu_3)^2}\right]$$

$$P\{t_3>t_1, t_2\}\cdot E\{t_3 \mid t_3>t_1, t_2\}=\mu_3\cdot\left[\frac{1}{\mu_3^2}-\frac{1}{(\mu_2+\mu_3)^2}-\frac{1}{(\mu_1+\mu_3)^2}+\frac{1}{(\mu_1+\mu_2+\mu_3)^2}\right].$$

By summing these expressions we have the mean passage time in the case of Ⓑ;

$$\frac{1}{\mu_1}+\frac{1}{\mu_2}+\frac{1}{\mu_3}-\frac{1}{\mu_1+\mu_2}-\frac{1}{\mu_2+\mu_3}-\frac{1}{\mu_3+\mu_1}-\frac{1}{\mu_1+\mu_2+\mu_3} \qquad \cdots\cdots(16)$$

Now, we have to calculate probabilities of the occurrence of Ⓐ and Ⓑ. For this purpose we calculate state probabilities under the condition of the first stage is in service.

The states showed in the (table 1) is considered to be state immediately before the customer in the service station at the first stage is going to be renewal.

Accordingly, it means that state Ⓐ arrises from the state 0 and the state Ⓑ arrises from other state 1, 2, 3, and 4.

Table 1.

State	First Stage	Second Stage	Third Stage
0	1	0	0
1	1	0	1
2	1	1	0
3	1	1	1
4	1	B	1

(0, 1 and B denote the state of the service station is in empty, busy and block, respectively.)

Ballance eqations are as follows:

$$\mu_3 \cdot P_1 - \mu_1 \cdot P_0 = 0$$

$$\mu_2 \cdot P_2 + \mu_3 P_4 - (\mu_1 + \mu_3) P_1 = 0$$

$$\mu_3 \cdot P_3 + \mu_1 P_0 - (\mu_1 + \mu_2) P_2 = 0$$

$$\mu_2 \cdot P_3 - (\mu_1 + \mu_3) P_4 = 0$$

Solutions of these equations are

$$P_1 = \frac{\mu_1}{\mu_3} P_0$$

$$P_2 = \frac{1}{\mu_1 + \mu_2} \cdot \left\{ \frac{\mu_1^2}{\mu_2 \cdot \mu_3} \cdot \frac{(\mu_1 + \mu_3)(\mu_1 + \mu_2 + \mu_3)}{(2\mu_1 + \mu_2 + \mu_3)} + \mu_1 \right\} \cdot P_0$$

$$P_3 = \frac{\mu_1^2}{\mu_2 \cdot \mu_3^2} \cdot \left\{ \frac{(\mu_1 + \mu_3)(_1 + \mu_2 + \mu_3)\mu}{2\mu_1 + \mu_2 + \mu_3} \right\} \cdot P_0$$

$$P_4 = \frac{\mu_1^2 \cdot (\mu_1 + \mu_2 + \mu_3)}{\mu_3^2 \cdot (2\mu_1 + \mu_2 + \mu_3)} \cdot P_0$$

Considering that $\sum_{i=0}^{4} P_i = 1$, we can obtain the values of $P_i (i = 0, 1, \cdots 4)$, Completely.

Since from (15) and (16), mean passage time is equal to

$$[\text{M.P.T.}]=\left(\frac{1}{\mu_1}+\frac{1}{\mu_2}-\frac{1}{\mu_1+\mu_2}\right)\cdot P_0$$
$$+\left(\frac{1}{\mu_1}+\frac{1}{\mu_2}+\frac{1}{\mu_3}-\frac{1}{\mu_1+\mu_2}-\frac{1}{\mu_2+\mu_3}-\frac{1}{\mu_3+\mu_1}+\frac{1}{\mu_1+\mu_2+\mu_3}\right)(1-P_0). \quad \cdots\cdots(17)$$

By some elementary calculations the above expression reduces to

$$[\text{M.P.T.}]=\frac{1}{\mu_1}+\frac{1}{\mu_2}+\frac{1}{\mu_3}-\frac{(\mu_1+\mu_3)(\mu_1+\mu_2+\mu_3)^3}{(\mu_1+\mu_2)(\mu_2+\mu_3)\{(\mu_1+\mu_3)^2(\mu_1+\mu_2+\mu_3)-\mu_1\mu_2\mu_3\}}. \quad \cdots\cdots(18)$$

Since the right hand side of (18) is symmetric with respect to μ_1 and μ_3, we find the reversibility of [M.P.T.] holds.

We rewrite (18) as follows

$$[\text{M.P.T.}]=\frac{1}{\mu_1}+\frac{1}{\mu_2}+\frac{1}{\mu_3}-\frac{(\mu_1+\mu_2+\mu_3)^3}{(\mu_1+\mu_2)(\mu_2+\mu_3)(\mu_3+\mu_1)\left\{(\mu_1+\mu_3+\mu_3)-\frac{\mu_1\mu_2\mu_3}{(\mu_1+\mu_3)^2}\right\}}.$$

From this expression we can conclude that the minimum value of [M.P.T.] is obtained by assigning the service which has maximum $\mu_i (i=1, 2, 3)$ to the second stage.

(Example 1. 4)

(1) In the case of $\mu_1=\mu_2=\mu_2=\mu$,

$$[\text{M.P.T.}]=\frac{39}{22}\cdot\frac{1}{\mu}.$$

This result resembles to the value of $\rho_{\max}$ by Hunt[2] and Morse[1].

(2) In the case of $\mu_1=3\mu,\ \mu_2=2\mu,\ \mu_3=\mu$,

$$[\text{M.P.T.}]=\frac{179}{150}\cdot\frac{1}{\mu}.$$

(3) In the case of $\mu_1=2\mu,\ \mu_2=\mu,\ \mu_3=3\mu$,

$$[\text{M.P.T.}]=\frac{29}{24}\cdot\frac{1}{\mu}\,.$$

(4) In the case of $\mu_1=\mu,\ \mu_2=3\mu,\ \mu_3=2\mu,$

$$[\text{M.P.T.}]=\frac{139}{120}\cdot\frac{1}{\mu}\,.$$

Now to obtain [M.P.T.] by another method, we consider the following additional states.

Table 2

Stage	First Stage	Second Stage	Third Stage
5	B	1	0
6	B	1	1
7	B	B	1

Balance epuations are

$$\begin{cases} \mu_3P_1'-\mu_1P_0'=0 \\ \mu_2P_2'+\mu_3P_4'-(\mu_1+\mu_3)P_1'=0 \\ \mu_3P_3'+\mu_1P_0'-(\mu_1+\mu_2)P_2'=0 \\ \mu_2P_3'-(\mu_1+\mu_3)P_4'=0 \end{cases}$$

$$\begin{cases} \mu_1P_1'+\mu_2P_5'+\mu_3P_7'-(\mu_1+\mu_2+\mu_3)P_3'=0 \\ \mu_1P_2'+\mu_3P_6'-\mu_2P_5'=0 \\ \mu_1P_3'-(\mu_2+\mu_3)P_6'=0 \\ \mu_2P_6'+\mu_1P_4'-\mu_3P_7'=0 \end{cases}$$

Where we denoted the state probabilities by P_i' $(i=0,\ 1,\ \cdots,\ 7)$, in order to avoid the confusion.

These equations are solved to obtain P_i' and P_B where P_B is the probability of blocking of the first stage station and is equal to $P_5'+P_6'+P_7'$.

Using the relation

[Passage Time]=[blocking time]+[service time], we have the relation between [M.P.T.] and [Mean Service Time], that is

$$[\text{M.P.T.}]=\frac{[\text{Mean Service Time}]}{1-P_B}.$$

Therefore, the value of [M.P.T.] can be obtained by using the value of the blocking probability P_B, also.

§ 1.3 MULTIPLE 2-STAGE SERVICE

1.3.1 Structure model

In this paragraph we evaluate [M.P.T.] in the service system with the structure shown in the figure.

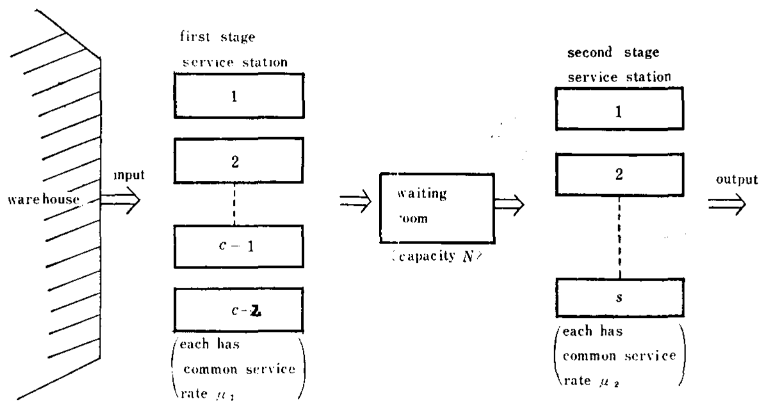

Fig. 3.

At first, the customer is served at the first service station, after finishing being served, the customer moved on to the second service station.

However in the state when the second service station is occupied, we suppose that customer must wait in the waiting room.

As there is limitation N in the number of accomodation of the customers in the waiting room, the customers out of limitation N, are in the state of occupying the first service station, that is to say, there occurs the state of blocking.

The first stage consists of C parallel stations. The distributions of service times in these service stations are exponential with service rate μ_1. The second stage consists of S parallel service stations, and the distributions of service times in these service stations are exponential with service rate μ_2.

1.3.2 Calculation of [M.P.T.]

Just like as preceding paragraphs, we shall restrict ourselves to the states that all of the first stage stations are in service or in blocking (that is, we discard the states that some of stations are empty).

We classify these states as follows;

Table 3

State	First Stage (No. of Stations in blocking)	No. in waiting room	Second Stage (No. of Stations in Service)
(0, 0)	0	0	0
(0, 1)	0	0	1
⋮	⋮	⋮	⋮
$(0, S)$	0	0	S
$(1, S)$	0	1	S
$(2, S)$	0	2	S
⋮	⋮	⋮	⋮
(N, S)	0	N	S
$(B_1 S)$	1	N	S
$(B_2 S)$	2	N	S
⋮	⋮	⋮	⋮
(B_C, S)	C	N	S

Let $P_{i,j}$ be probabilities that the system be in the state (i, j), then we have balance equations for steady state.

$$\mu_2 \cdot P_{0,1} - C\mu_1 \cdot P_{0,0} = 0$$

$$C\mu_1 \cdot P_{0,h-2} + h\mu_2 \cdot P_{0,h} - \{C\mu_1 + (h-1)\mu_2\} P_{0,h-1} = 0 \qquad (\text{for} \quad 2 \leq h \leq S)$$

$$C\mu_1 \cdot P_{0,S-1} + S\mu_2 \cdot P_{1,S} - (C\mu_1 + S\mu_2) \cdot P_{0,S} = 0$$

$$C\mu_1 \cdot P_{k-2,S} + S\mu_2 \cdot P_{k,S} - (C\mu_1 + S\mu_2) P_{k-1,S} = 0 \qquad (\text{for} \quad 2 \leq k \leq N.)$$

$$C\mu_1 \cdot P_{N-1,S} + S\mu_2 \cdot P_{B_1,S} - (C\mu_1 + S\mu_2) \cdot P_{N,S} = 0$$

$$C\mu_1 \cdot P_{N,S} + S\mu_2 \cdot P_{B_2,S} - \{(C-1)\mu_1 + S\mu_2\} \cdot P_{B_1,S} = 0$$

$$\{C-(l-1)\}\mu_1 \cdot P_{B_{l-1},S} + S\mu_2 \cdot P_{B_{l+1},S} - \{(C-l)\mu_1 + S\mu_2\} \cdot P_{B_l,S} = 0$$
$$(\text{for} \quad 2 \leq l \leq C-1)$$

$$\mu_1 \cdot P_{B_{C-1},S} - S\mu_2 \cdot P_{B_C,S} = 0.$$

From these equations we have relations

$$P_{0,h} = \frac{1}{h!}\left(\frac{C\mu_1}{\mu_2}\right)^h \cdot P_{00} \qquad (\text{for} \quad 0 \leq h \leq S), \cdots\cdots\cdots(19)$$

$$P_{k,S} = \frac{1}{S!}\left(\frac{C\mu_1}{\mu_2}\right)^S \cdot \left(\frac{C\mu_1}{S\mu_2}\right)^k \cdot P_{0,0} \qquad (\text{for} \quad 0 \leq k \leq N), \cdots\cdots\cdots(20)$$

$$P_{B_l,S} = \left\{\frac{1}{S!}\left(\frac{C\mu_1}{\mu_2}\right)^S \cdot \left(\frac{C\mu_1}{S\mu_2}\right)^N \cdot P_{00}\right\}\left(\frac{\mu_1}{S\mu_2}\right)^l \left\{C(C-1)\cdots(C-l+1)\right\}.$$
$$(\text{for} \quad 1 \leq l \leq C) \cdots\cdots\cdots(21)$$

Considering that

$$\sum_{h=0}^{S} P_{0,h} + \sum_{k=1}^{N} P_{k,S} + \sum_{l=1}^{C} P_{B_l,S} = 1,$$

we obtain

$$P_{00}=\frac{1}{\sum_{h=0}^{S}\frac{1}{h!}\left(\frac{C\mu_1}{\mu_2}\right)^h+\sum_{k=1}^{N}\left\{\frac{1}{S!}\left(\frac{C\mu_1}{\mu_2}\right)^S\cdot\left(\frac{C\mu_1}{S\mu_2}\right)^k\right\}+\sum_{l=1}^{C}\left[\frac{1}{S!}\left(\frac{C\mu_1}{\mu_2}\right)^S\left(\frac{C\mu_1}{S\mu_2}\right)^N\left(\frac{\mu_1}{S\mu_2}\right)^l\left\{C(C-1)\cdots(C-l+1)\right\}\right]}\cdot \qquad \cdots\cdots\cdots(22)$$

If we define the mean blocking rate by

$$P_B=\frac{1}{C}\sum_{l=1}^{C}l\cdot P_{B_l,S}, \qquad \cdots\cdots\cdots(23)$$

then the ratio of the mean service time to the mean passage time per one station in the first stage must be equal to $1-P_B$.

And the [M.P.T.] for the entire system will be equal to

$$\frac{[\text{mean service time}]}{C(1-P_B)}\cdot$$

Here we put

$$\alpha_C=\frac{C!}{C^C}\ ,\ \varphi_C(x)=\sum_{\nu=0}^{C-1}\frac{x^\nu}{\nu!}\ ,\ (\varphi_0(x)=0),\quad x=\frac{C\mu_1}{S\mu_2}\cdot$$

Then we have

$$P_{0,h}=\frac{(Sx)^h}{h!}P_{00},\quad P_{k,S}=\frac{S^S}{S!}x^{k+S}P_{00},$$

$$P_{B_l,S}=\frac{1}{\alpha_S}x^{S+C+N}\cdot\frac{\left(\frac{C}{x}\right)^{C-l}}{(x-l)!}\cdot\frac{C!}{C^C}P_{00},$$

and

$$\frac{1}{P_{00}}=\varphi_S(Sx)+\frac{1}{\alpha_S}x^S\cdot\sum_{\nu=0}^{N}x^\nu+\frac{\alpha_C}{\alpha_S}x^{S+C+N}\cdot\varphi_C\left(\frac{C}{x}\right)\cdot$$

Therefore

$$P_B=\frac{1}{C}\sum_{l=1}^{C} l\cdot P_{B_l,S}=P_{00}\cdot\frac{\alpha_C}{\alpha_S}\cdot x^{S+C+N}\cdot\left\{\varphi_C\left(\frac{C}{x}\right)-\frac{1}{x}\varphi_{C-1}\left(\frac{C}{x}\right)\right\}$$

$$=\frac{\alpha_C\cdot x^{S+C+N}\left\{\varphi_C\left(\frac{C}{x}\right)-\frac{1}{x}\varphi_{C-1}\left(\frac{C}{x}\right)\right\}}{\alpha_S\varphi_S(Sx)+x^S\sum_{\nu=0}^{N}x^\nu+\alpha_C x^{S+C+N}\varphi_C\left(\frac{C}{x}\right)},$$

it follows that

$$1-P_B=\frac{\alpha_S\varphi_S(Sx)+x^S\sum_{\nu=0}^{N}x^\nu+\frac{1}{x}\alpha_C x^{S+C+N}\varphi_{C-1}\left(\frac{C}{x}\right)}{\alpha_S\varphi_S(Sx)+x^S\sum_{\nu=0}^{N}x^\nu+\alpha_C x^{S+C+N}\varphi_C\left(\frac{C}{x}\right)}. \quad\cdots\cdots(24)$$

Especially when $C\mu_1=S\mu_2$, we have $x=1$ and

$$1-P_B=\frac{\alpha_S\varphi_S(S)+N+\alpha_C\varphi_C(C)}{\alpha_S\varphi_S(S)+N+1+\alpha_C\varphi_C(C)}. \quad\cdots\cdots(25)$$

(Note; This is symmetric with respect to S and C.)

Therefore we obtain the mean passage time;

$$[\text{M.P.T.}]=\frac{\frac{1}{\mu_1}}{C(1-P_B)}=\frac{\alpha_S\varphi_S(S)+N+1+\alpha_C\varphi_C(C)}{\alpha_S\varphi_S(S)+N+\alpha_C\varphi_C(C)}\cdot\frac{1}{C\mu_1}, \quad\cdots\cdots(26)$$

and then we see that the reversibility of [M.P.T] holds.

(Example 1.5)

(1) $S=1$, $\mu_2=C\mu_1$.

In this case, we get

$$[\text{M.P.T.}]=\frac{(N+1)\cdot\frac{C^C}{C!}+\sum_{\nu=0}^{C}\frac{C^\nu}{\nu!}}{N\cdot\frac{C^C}{C!}+\sum_{\nu=0}^{C}\frac{C^\nu}{\nu!}}\cdot\frac{1}{C\mu_1}.$$

This result is already given in Sience College Report.[8]

(2) $C=1,\quad S=1$.

In this case, we get

$$[\text{M.P.T.}]=\frac{\sum_{j=0}^{N+2}\left(\frac{\mu_1}{\mu_2}\right)^j}{\sum_{i=0}^{N+1}\left(\frac{\mu_1}{\mu_2}\right)^i}\cdot\frac{1}{\mu_1}$$

This result is already known as the value of $\rho_{\max}$[2][7].

(3) $C=2,\quad S=2,\quad \mu_1=\mu_2=\mu$.

In this case, we have

$$[\text{M.P.T.}]=\frac{N+4}{N+3}\cdot\frac{1}{2\mu}.$$

[II] THE OUTPUT DISTRIBUTIONS

§ 2.1 THE OUTPUT DISTRIBUTION IN THE SINGLE 2-STAGE SERVICE SYSTEM

(the case excluding the waiting room).

This model is the one treated in § 1.1 (see Fig. 1).

If we observe the state of the whole system on the time immediately before the customer leave the second service station, we can conceive the two different states.

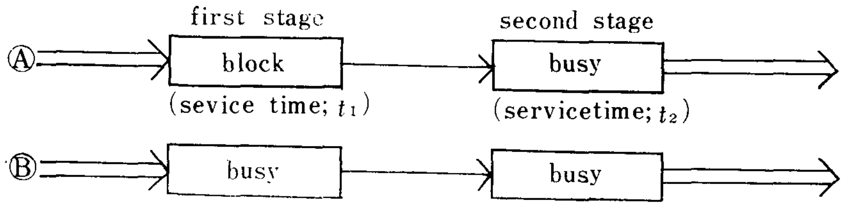

Fig. 4

In the case of Ⓐ state, the interdeparture time distribution, that is the time interval distribution from the departure of a customer to the departure of the next customer is the distribution of t_2 itself.

Suppose that this generating function is denoted by $M_1(\theta)$.

In the case of Ⓑ state, we think as follows;

On the time soon after the customer left the system, the state of the system is shown as Fig. 5.

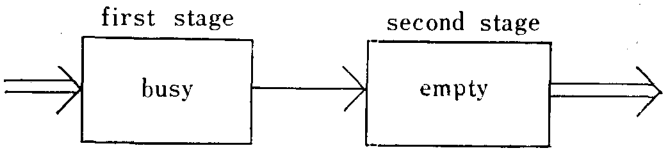

Fig. 5

In this state, the distribution of the output time interval is the distribution of the sum of two independent random variables, one of which has the distribution of (t_1-t_2) under the condition $t_1>t_2$, and another has the distribution of t_2.

We shall write the moment generating function of this distribution by $M_2(\theta)$.

It is required to calculate the state probabilities of Ⓐ and Ⓑ.

In this system, since the first stage service station and the second station are to begin their services simultaneously, the state probability of Ⓐ; $P\{A\}$ is

$$P\{A\}=P\{t_1<t_2\},$$

and the state probability of Ⓑ is

$$P\{B\}=P\{t_1>t_2\}=1-P\{A\}$$

Consequently, the generating function $M(\theta)$ of the distribution of the output from the second stage service station is

$$M(\theta)=P\{A\}\cdot M_1(\theta)+P\{B\}\cdot M_2(\theta).$$

As the special case, consider the case where the first and second station have the exponentially distributed service time.

Suppose the density function of the service time distribution in the first and second service station as

$$f(t_1)=\mu_1 e^{-\mu_1 t_1}, \quad g(t_2)=\mu_2 \cdot e^{-\mu_2 t_2},$$

respectively.

In this case we get

$$P\{A\}=\frac{\mu_1}{\mu_1+\mu_2}, \quad P\{B\}=\frac{\mu_2}{\mu_1+\mu_2}.$$

Now, the moment generating function $M_1(\theta)$ can be easily obtained as follows.

$$M_1(\theta)=E(e^{\theta t_2})=\int_0^\infty e^{\theta t_2}\mu_2 e^{-\mu_2 t_2}dt_2=\frac{\mu_2}{\mu_2-\theta}.$$

And then, we are going to get $M_2(\theta)$. To do so, we are interested in the density function

$$\varphi(z=t_1-t_2 \,|\, t_1>t_2).$$

On the other hand, the density function $h(z)$ of the distribution with the random variable

$$z=t_1-t_2, \quad (t_1>t_2)$$

is expressed as follows.

$$h(z)=\int_0^\infty g(t_2)f(z+t_2)dt_2.$$

In general,

$$P\{z\geq t_1-t_2\geq -\infty \,|\, t_1>t_2\}$$
$$=\frac{P(z\geq t_1-t_2\geq 0)}{P(t_1>t_2)}.$$

Consequently,

$$\varphi(z=t_1-t_2\,|\,t_1>t_2)=\frac{\dfrac{\mu_1\mu_2}{\mu_1+\mu_2}\cdot e^{-\mu_1 z}}{\dfrac{\mu_2}{\mu_1+\mu_2}}=\mu_1\cdot e^{-\mu_1 z}.$$

Therefore, $M_2(\theta)$ can be obtained as follows.

$$M_2(\theta)=\left(\frac{\mu_1}{\mu_1-\theta}\right)\cdot\left(\frac{\mu_2}{\mu_2-\theta}\right).$$

So the moment generating function $M(\theta)$ is

$$M(\theta)=\frac{\mu_1}{\mu_1+\mu_2}\cdot\left(\frac{\mu_2}{\mu_2-\theta}\right)+\frac{\mu_2}{\mu_1+\mu_2}\left(\frac{\mu_1}{\mu_1-\theta}\right)\left(\frac{\mu_2}{\mu_2-\theta}\right). \quad\cdots\cdots(27)$$

Thus, for example, the average interdeparture time interval is

$$\left.\frac{dM(\theta)}{d\theta}\right]_{\theta=0}=\frac{1}{\mu_1}+\frac{1}{\mu_2}-\frac{1}{\mu_1+\mu_2}. \quad\cdots\cdots(28)$$

This result is the same as that obtained from the value of ρ_{max}, intuitively.

§ 2.2 SINGLE 3-STAGE SERVICE

The model treated here is the same as that was treated in §1.2 (refer Fig. 2).

In the following we shall observe the system on the time when the customer are going to leave the third service station. As the state that the third service station is vacant is excluded, we shall get the state probabilities under the condition that the third service station is busy. We can easily obtain these probabilities using the result of §1.2. So we set newly these state probabilities P_1, P_3, P_4, P_6 and P_7. $(P_1+P_3+P_4+P_6+P_7=1)$

Now when we are going to get the output distribution, we shall consider the following three types of states at the time immediately after the departure of a customer;

Type [1]·········the third station is busy, Type [2]·········the third station is empty and the second station is busy,

Type [3]·········the third station and the second station are empty, the first station is busy.

Under these conditions the moment generating functions of distributions are

Type [1]·········$M_1(\theta)=\frac{\mu_3}{\mu_3-\theta}$

Type [2]·········$M_2(\theta)=\left(\frac{\mu_2}{\mu_2-\theta}\right)\left(\frac{\mu_3}{\mu_3-\theta}\right)$

Type [3]·········$M_3(\theta)=\left(\frac{\mu_1}{\mu_1-\theta}\right)\left(\frac{\mu_2}{\mu_2-\theta}\right)\left(\frac{\mu_3}{\mu_3-\theta}\right)$,

and the probabilities of occurrence of these three types are

Type [1]·········$P_{[1]}=P_4+P_7$

Type [2]·········$P_{[2]}=P_3+P_6$

Type [3]·········$P_{[3]}=P_1$

respectively. It follows that the moment generating function of the output distribution are

$$M(\theta)=P_{[1]}\cdot M_1(\theta)+P_{[2]}\cdot M_2(\theta)+P_{[3]}\cdot M_3(\theta)$$
$$=(P_4+P_7)\left\{\frac{\mu_3}{\mu_3-\theta}\right\}+(P_3+P_6)\cdot\left\{\left(\frac{\mu_2}{\mu_2-\theta}\right)\left(\frac{\mu_3}{\mu_3-\theta}\right)\right\}$$
$$+P_1\cdot\left\{\left(\frac{\mu_1}{\mu_1-\theta}\right)\left(\frac{\mu_2}{\mu_2-\theta}\right)\left(\frac{\mu_3}{\mu_3-\theta}\right)\right\}. \quad\cdots\cdots(29)$$

From this expression we can calculate, for example, the average output time interval as follows.

$$\left.\frac{dM(\theta)}{d\theta}\right]_{\theta=0}=P_1\cdot\left(\frac{1}{\mu_1}+\frac{1}{\mu_2}+\frac{1}{\mu_3}\right)$$
$$+(P_3+P_6)\cdot\left(\frac{1}{\mu_2}+\frac{1}{\mu_3}\right)+(P_4+P_7)\cdot\frac{1}{\mu_3}. \quad\cdots\cdots(30)$$

Here we calculate the numerical value of the average output time interval for the system shown in the last part of § 1.2.

The state probabilities are as follows;

Table 4

	i) $\mu_1=\mu_2=\mu_3=\mu$	ii) $\mu_1=3\mu$, $\mu_2=2\mu$, $\mu_3=\mu$	iii) $\mu_1=2\mu$, $\mu_2=\mu$, $\mu_3=3\mu$	iv) $\mu_1=\mu$, $\mu_2=3\mu$, $\mu_3=2\mu$
P_1	4/22	6/150	4/24	35/80
P_3	6/22	24/150	10/24	15/80
P_4	3/22	12/150	2/24	15/80
P_6	3/22	24/150	5/24	3/80
P_7	6/22	84/150	3/24	12/80

and the average output time intervals are

$$\text{Case i)}\cdots\cdots\frac{4}{22}\left(\frac{1}{\mu}+\frac{1}{\mu}+\frac{1}{\mu}\right)+\left(\frac{6}{22}+\frac{3}{22}\right)\cdot\left(\frac{1}{\mu}+\frac{1}{\mu}\right)$$

$$+\left(\frac{3}{22}+\frac{6}{22}\right)\cdot\frac{1}{\mu}=\frac{39}{22}\cdot\frac{1}{\mu}$$

$$\text{Case ii)}\cdots\cdots\frac{6}{150}\left(\frac{1}{3\mu}+\frac{1}{2\mu}+\frac{1}{\mu}\right)+\left(\frac{24}{150}+\frac{24}{150}\right)\left(\frac{1}{2\mu}+\frac{1}{\mu}\right)$$

$$+\left(\frac{12}{150}+\frac{84}{150}\right)\cdot\frac{1}{\mu}=\frac{179}{150}\cdot\frac{1}{\mu}$$

$$\text{Case iii)}\cdots\cdots\frac{4}{22}\left(\frac{1}{2\mu}+\frac{1}{\mu}+\frac{1}{3\mu}\right)+\left(\frac{10}{24}+\frac{5}{24}\right)\left(\frac{1}{\mu}+\frac{1}{3\mu}\right)+\left(\frac{2}{24}+\frac{3}{24}\right)\cdot\frac{1}{3\mu}$$

$$=\frac{29}{24}\cdot\frac{1}{\mu},$$

$$\text{Case iv)}\cdots\cdots\frac{35}{80}\left(\frac{1}{\mu}+\frac{1}{3\mu}+\frac{1}{2\mu}\right)+\left(\frac{15}{80}+\frac{3}{80}\right)\left(\frac{1}{3\mu}+\frac{1}{2\mu}\right)+\left(\frac{15}{80}+\frac{12}{80}\right)\cdot\frac{1}{2\mu}$$

$$=\frac{139}{120}\cdot\frac{1}{\mu}.$$

These results coincide with the values obtained in (Example 1.4).

§ 2.3 MULTIPLE 2-STAGE SERVICE

In this paragraph we shall have the output interval distribution for the model treated in § 1.3. (refer Fig. 3).

As we want to find the time interval of the customer's departure distribution from a certain system, so like a preceding paragraph, we are going to check the three types of the states.

Type [1]·········all stations of the second stage are busy,
Type [2]·········all stations of the second stage are empty,
Type [3]·········$h(1 \leq h \leq S)$ stations of the second stage are busy and the remaining $(S-h)$ stations are empty.

Now we are going to investigate the time interval distribution under each condition. For convenience we set the departure time of a customer from a system on the origin.

Type [1]·········In this case, the distribution of time interval is;

$$f(t_2)=S\mu_2 \cdot e^{-S\mu_2 t_2}$$

and its moment generating function is

$$M_1(\theta)=\frac{S\mu_2}{S\mu_2-\theta} \cdot$$

Type [2]·········Immediately at the time after the customer leaves a system, that is to say, the second stage service stations are all empty. The customer succeeding to this customer is one of the customers being served in the first stage service stations.

Now, the earliest customer who moves to the second service station at the time τ_1 from the first stage service station is denoted by customer ①. And the service time of this customer at the second stage service station is denoted by v_1. Then this customer will leave the system at the time τ_1+v_1. The second earliest customer who moves from the first stage service station to the second stage service station is denoted by customer ②, and let this customer ② move to the second stage service station later than the customer ① by the time τ_2, and is being served at the second stage station for time v_2, then the departure time of this customer from this system is $\tau_1+\tau_2+v_2$. The third, the fourth and so forth customers are treated similarly. The point here is to obtain the time when the customer who is being served in the second stage service station leaves the earliest.

Consider;

the departure time of customer ①$\cdots\cdots\cdots\tau_1+v_1$

the departure time of customer ②$\cdots\cdots\cdots\tau_1+\tau_2+v_2$

the departure time of customer ③$\cdots\cdots\cdots\tau_1+\tau_2+\tau_3+v_3$

$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$

$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$

the departure time of customer Ⓢ$\cdots\cdots\cdots\tau_1+\tau_2+\tau_3+\cdots+\tau_S+v_S$,

all of τ_i's have the density function

$$f(\tau)=C\mu_1\cdot e^{-C\mu_1\cdot\tau}$$

and all of v_i's have the density function

$$f(v)=\mu_2\cdot e^{-\mu_2 v}$$

moreover they are mutually indepenpent.

Setting

$$\xi_1=\tau+v,$$

we have the distribution of ξ_1:

$$f(\xi_1)=\int_0^{\xi_1}C\mu_1\cdot e^{-C\mu_1 t}\cdot\mu_2\cdot e^{-\mu_2(\xi_1-t)}dt$$

$$=\frac{C\mu_1\mu_2}{C\mu_1-\mu_2}\cdot\left(e^{-\mu_2\xi_1}-e^{-C\mu_1\xi_1}\right)$$

$$(\text{for}\quad C\mu_1\neq\mu_2)$$

This represents the output distribution in case of one empty service station.

In case of two emptly stations this distribution is obtained as follows; the distribution of

$$\eta_1=\min(\xi_1, v)=\min(\tau_2+v_2, v_1)$$

is

$$f(\eta_1)=\frac{\mu_2}{C\mu_1-\mu_2}\cdot\left\{2C\mu_1 e^{-2\mu_2\eta_1}-(C\mu_1+\mu_2)\cdot e^{-(C\mu_1+\mu_2)\eta_1}\right\}$$

and the probability density function of $\xi_2=\tau+\eta_1$ is

$$f(\xi_2)=\int_0^{\xi_2}\frac{\mu_2}{C\mu_1-\mu_2}\cdot\left\{2C\mu_1{}^{e}{}^{-2\mu_2 t}-(C\mu_1+\mu_2)\cdot e^{-(C\mu_1+\mu_2)t}.(C\mu_1 e^{-C\mu_1(\xi_2-t)})\right\}dt$$

$$=\frac{C\mu_1}{(C\mu_1-\mu_2)(C\mu_1-2\mu_2)}\cdot\Big[2C\mu_1\mu_2 e^{-2\mu_2\xi_2}-(C^2\mu_1{}^2+C\mu_1\mu_2-2\mu_2{}^2)\cdot e^{-C\mu_1\xi_2}$$

$$+(C\mu_1+\mu_2)(C\mu_1-2\mu_2)\cdot e^{-(C\mu_1+\mu_2)\xi_2}\Big].$$

Thus the output distribution in case of two empty stations was obtained. Iterating this procedure we have the distribution function $f(\xi_S)$ of the output time interval ξ_S.

(where, $\min(\xi_{k-1}, v)=\eta_{k-1}$, $\xi_k=\tau+\eta_{k-1}$, (for $2\leq k\leq S$))

We denote the moment generating function of this distribution by $M_2(\theta)$.

Type [3]·········Immediately after the departure of the preceding customer, we suppose that h second stage stations are busy ($h=1, 2,\cdots\cdots\cdots, S-1$), and remaining $(S-h)$ stations are empty. The distribution of the time interval between the departure of the earliest customer who leaves the system and the departure of the preceding customer is

$$f(t)=h\mu_2 e^{-h\mu_2}.$$

On the other hand, the distribution $f(\xi_{S-k})$ of the time interval between the departure of the earliest customer who leaves the system through the first stage and the departure of the preceding customer is obtained by the procedure mentioned in Type [2].

Thus, by finding the distribution of two random variables, each of which has the density function $f(t)$ and $f(\xi_{S-h})$ respectively, we can obtain the distribution of the time interval between the departure of the earliest customer and that of the preceding customer. We denote the moment generating function of this distribution by $M_{3h}(\theta)$.

Using the above computational results we can have the moment generating function of the output distribution in this system;

$$M(\theta)=\{P_{[1]}\times M_1(\theta)\}+\{P_{[2]}\times M_2(\theta)\}+\sum_{h=1}^{S-1}\{P_{[3h]}\times M_{3h}(\theta)\},$$

where $P_{[1]}$, $P_{[2]}$, $P_{[3h]}$ are the state probabilities of]1], [2] and [3*h*] types respectively. These can be obtained by the following method.

Considering the condition that a customer is being served in the second stage service station, state (0, 0) is excluded, and we only consider the remaining states. Now, immeditaely after the customer leaves the system, for instance, state (0, 1) moves to the state (0, 0). In this case the time interval distribution till when the next customer leaves the system is of [2] type.

Similarly, as state (0, 2) moves to the state (0, 1), this output distribution is of [3, 1] type. [3, 2], ..., [3, *S*–1] and [1] type are considered similarly.

Now, the probability $P_{[2]}$ of the occurrence of [2] type is the probability under the condition that a customer is being served in the second stage service station. So let us denote this by $a\cdot P_{01}$, then the probability of [3, 1], [3, 2], ..., [3, *S*-1], that is to say, $P_{[31]}$, $P_{[32]}$, ..., $P_{[3.S-1]}$ are $2a\cdot P_{0,2}$, $3a\cdot P_{0,3}$, ..., $Sa\cdot P_{0,S}$.

The probability of the occurrence of [1] type is

$$Sa\cdot\{(P_{1S}+P_{2S}+\cdots\cdots+P_{NS})+(P_{B_1,S}+P_{B_2,S}+\cdots\cdots+P_{B_C,S})\}.$$

Setting the sum to be unity, we have

$$a=\frac{1}{P_{01}+(2\cdot P_{02}+\cdots\cdots+S\cdot P_{0S})+S\cdot\{(P_{1S}+\cdots\cdots+P_{N,S})+(P_{B_1,S}+\cdots\cdots+P_{B_C,S})\}}.$$

On the other hand as P_{ij} are already obtained, we can determine the value of a.

Therefore the probabilities $P_{[1]}$, $P_{[2]}$, $P_{[3h]}$ of [1], [2], [3h] can be obtained.

To explain this procedure, we shall show simple examples.

<Example 2.1>

In the case of $C=2$, $S=1$, $N=0$, $\mu_1=\frac{1}{2}\mu_2=\mu$.

Since

$$P_{[1]}=(P_{B_1,1}+P_{B_2,1})\cdot a,$$

$$P_{[2]}=a\cdot P_{0,1}$$

and

$$P_{[1]}+P_{[2]}=1,$$

we have

$$a=\frac{1}{P_{B_1,1}+P_{B_2,1}+P_{0,1}}.$$

Meanwhile we obtain from the preceding result that

$$P_{0,1}=P_{00},\quad P_{B_1,1}=P_{00},\quad P_{B_2,1}=\frac{1}{2}\cdot P_{00},$$

so we have

$$P_{[1]}=\frac{3}{5},\quad P_{[2]}=\frac{2}{5}.$$

The moment generating function $M(\theta)$ of the output distribution for this system is

$$M(\theta)=\left\{P_{[1]}\times\left(\frac{2\mu}{2\mu-\theta}\right)\right\}+\left\{P_{[2]}\times\left(\frac{2\mu}{2\mu-\theta}\right)^2\right\},$$

that is,

$$M(\theta)=\frac{3}{5}\left(\frac{2\mu}{2\mu-\theta}\right)+\frac{2}{5}\left(\frac{2\mu}{2\mu-\theta}\right)^2$$

The value of the mean output time interval is given by

$$\left.\frac{dM(\theta)}{d\theta}\right]_{\theta=0}=\frac{7}{10}\cdot\frac{1}{\mu},$$

and thus we have the same result as that is anticipated from the value of [M.P.T.] which can be obtained by the procedure in the paragraph 1.3.

(Example 2.2)

In the case of $C=1$, $S=2$, $N=0$, $\mu_2=\frac{1}{2}\mu_1=\mu$.

The moment generating functions of the output distributions of the type [1] and [3] are

$$M_1(\theta)=\frac{2\mu}{2\mu-\theta}$$

and

$$M_3(\theta)=\frac{4\mu}{2\mu-\theta}-\frac{3\mu}{3\mu-\theta},$$

respectively. The moment generating function of the output distributions of the type [2] is obtained as follows; since

$$f(\xi_2)=8\mu^2\xi\cdot e^{-2\mu\xi}+6\mu\cdot e^{-3\mu\xi}-6\mu\cdot e^{-2\mu\xi}$$

we have

$$M_2(\theta)=\int_0^\infty e^{\theta\xi_2}\cdot f(\xi_2)d\xi_2=\frac{8\mu^2}{(2\mu-\theta)^2}+\frac{6\mu}{3\mu-\theta}-\frac{6\mu}{2\mu-\theta}.$$

On the other hand the probabilities $P_{[1]}$, $P_{[2]}$ and $P_{[3]}$ of the occurrence of these three types are calculated as follows;

By the relations

$$P_{[1]}=2a\cdot P_{B_1,2},\quad P_{[2]}=a\cdot P_{0,1},\quad P_{[3]}=2a\cdot P_{0,2}$$

and

$$P_{[1]}+P_{[2]}+P_{[3]}=1,$$

we have

$$a=\frac{1}{2\cdot P_{B_1,2}+P_{0,1}+2\cdot P_{0,2}}.$$

And from the results in the paragraph 1. 3. 2, we have

$$P_{0,1}=2\cdot P_{00},\quad P_{0,2}=2\cdot P_{00},\quad P_{B_1,2}=2\cdot P_{00}$$

Hence

$$a=\frac{1}{10\cdot P_{00}}$$

and therefore,

$$P_{[1]}=\frac{2}{5},\quad P_{[2]}=\frac{1}{5},\quad P_{[3]}=\frac{2}{5}.$$

Thus, finally we can write the moment generating function of the output distribution as

$$M(\theta)=\left\{\frac{2}{5}\left(\frac{2\mu}{2\mu-\theta}\right)\right\}+\left\{\frac{1}{5}\left(\frac{8\mu^2}{(2\mu-\theta)^2}+\frac{6\mu}{3\mu-\theta}-\frac{6\mu}{2\mu-\theta}\right)\right\}$$
$$+\left\{\frac{2}{5}\left(\frac{4\mu}{2\mu-\theta}-\frac{3\mu}{3\mu-\theta}\right)\right\}.$$

The value of the mean output time interval

$$\left.\frac{dM(\theta)}{d\theta}\right]_{\theta=0}=\frac{7}{10}\cdot\frac{1}{\mu}$$

calculated from our formula is identical with that of the (Example 2. 1).

(Note) In the (Example 2. 1) and (2. 2) above, we treated the case that the capacity of the waiting room N is equal to 0.

However, in the case of $N \geq 1$, we can also easily obtain the output distribution with different values of $P_{[1]}$, $P_{[2]}$ and $P_{[3]}$.

<Example 2. 3>

In the case of $C=2$, $S=2$, $\mu_1=\mu_2=\mu$.

At first, we calculate the moment generating function of the output distribution of each types.

Type [1]$\cdots\cdots\cdots M_1(\theta)=\dfrac{2\mu}{2\mu-\theta}$.

Type [2]$\cdots\cdots\cdots M_2(\theta)=\dfrac{8\mu^2}{(2\mu-\theta)^2}+\dfrac{6\mu}{3\mu-\theta}-\dfrac{6\mu}{2\mu-\theta}$.

Type [3]$\cdots\cdots\cdots$Since the density function of

$$\eta_1=\min(\xi_1, v)$$

is
$$f(\eta_1)=4\mu e^{-2\mu\eta_1}-3\mu e^{-3\mu\eta_1},$$

the moment generating function is

$$M_3(\theta)=\int_0^\infty f(\eta_1)e^{\theta\eta_1}d\eta_1=\frac{4\mu}{2\mu-\theta}-\frac{3\mu}{3\mu-\theta}.$$

Therefore, the moment generating function $M(\theta)$ of the output distribution for the entire system has the form

$$M(\theta)=\left\{P_{[1]}\times\left(\frac{2\mu}{2\mu-\theta}\right)\right\}+\left\{P_{[2]}\times\left(\frac{8\mu^2}{(2\mu-\theta)^2}+\frac{6\mu}{3\mu-\theta}-\frac{6\mu}{2\mu-\theta}\right)\right\}$$
$$+\left\{P_{[3]}\times\left(\frac{4\mu}{2\mu-\theta}-\frac{3\mu}{3\mu-\theta}\right)\right\}.$$

The calculation of the values of $P_{[1]}$, $P_{[2]}$ and $P_{[3]}$ proceeds as follows. Using the relations

$$P_{[1]}=2a\cdot\{(P_{1,2}+P_{2,2}+\cdots\cdots+P_{N,2})+(P_{B_1,2}+P_{B_2,2})\}$$

$$P_{[2]}=a\cdot P_{01}$$

$$P_{[3]}=2a\cdot P_{02},$$

$$P_{[1]}+P_{[2]}+P_{[3]}=1.$$

and using the relations

$$P_{01}=P_{02}=P_{12}=P_{22}=\cdots\cdots=P_{N,2}=P_{B_1,2}=2\cdot P_{00}$$

$$P_{B_2,2}=P_{00}$$

obtained from the results in the paragraph 1.3.2, we have

$$P_{[1]}=\frac{2N+3}{2N+6},\quad P_{[2]}=\frac{1}{2N+6},\quad P_{[3]}=\frac{2}{2N+6}.$$

Thus, in conclusion, we have the moment generating function of the output distribution

$$M(\theta)=\left(\frac{2N+3}{2N+6}\right)\left\{\frac{2\mu}{2\mu-\theta}\right\}+\left(\frac{1}{2N+6}\right)\left\{\frac{8\mu^2}{(2\mu-\theta)^2}+\frac{6\mu}{3\mu-\theta}-\frac{6\mu}{2\mu-\theta}\right\}$$

$$+\left(\frac{2}{2N+6}\right)\cdot\left\{\frac{4\mu}{2\mu-\theta}-\frac{3\mu}{3\mu-\theta}\right\}.$$

The value of the mean output time interval is given by

$$\left.\frac{dM(\theta)}{d\theta}\right]_{\theta=0}=\frac{N+4}{2(N+3)}\cdot\frac{1}{\mu},$$

and thus we get the same result as that anticipated from the value of [M.P.T.].

In conclusion, the author express his sincerest thanks to Prof. Y. Tsumura, Dr. H. Morimura and Dr. H. Hatori, who have given valuable advice to him.

REFERENCES

[1] P.M. Morse, "Queues, Inventories and Maintenance," John Wiley & Sons, New York (1958)

[2] G.C. Hunt, "Sequential Arrays of Waiting Lines," JORSA (1956)

[3] P.J. Burke, "The output of a queueing system," JORSA (1956)
[4] E. Reich, "Waiting times when queues are in tandem," A.M.S. (1956)
[5] J. Sacks, "Ergodicity of queues in series," A.M.S. (1960)
[6] P.D. Finch, "The output process of the quequeing system M/G/1," J. R. S. S. (1960)
[7] T. Makino, "Some considerations of queues in tandem," Keiei Kagaku (1963)
[8] T. Makino, "Study of the ρ_{max} concerning some queuing systems," Tokyo Sience College Report (1964)