ON A TANDEM QUEUE WITH BLOCKING

TAKEJI SUZUKI

Hosei University

I. INTRODUCTION

In this paper we shall concerned with the following queueing system.

There are two counters in series. The output of the first counter comprises the input into the second, and no queue is allowed to form before the second counter, whereas an infinite queue is allowed before the first. This results in blocking of service at the first counter even though the service of a customer has just been completed and there is a queue. The counter opens for service when the customer can go to service in the second counter when the latter becomes free.

Customers arrive at the first counter at the instants $\tau_1, \tau_2, \ldots, \tau_n, \ldots$ The customers will be served in the order of their arrival. Let us denote by χ_n and η_n service times of the *nth* customer at the first and the second counter, respectively. It is supposed that the inter-arrival times $\{\tau_{n+1}-\tau_n, n\geqq 0;\ \tau_0\equiv 0\}$. the service times $\{\chi_n\}$ and $\{\eta_n\}$ are independent sequences of identically distributed, positive random variables with distributions $P(\tau_{n+1}-\tau_n\leqq x)=F(x)$, $P(\chi_n\leqq x)=G(x)$ and $P(\eta_n\leqq x)=H(x)$, respectively. For convenience, such a queueing system can be described by the triplet $[F(x), G(x), H(x)]$.

In what follows we shall be interested in the case where

$$F(x)=\begin{cases}1-e^{-\lambda x} & (x\geqq 0)\\ 0 & (x<0),\end{cases}$$

that is, input process is a Poisson process of density λ.

G.C. Hunt [1] studied utilization, ratio of mean arrival rate to mean service rate, for some tandem queues, among these the above mentioned model, in which both of $G(x)$ and $H(x)$ are exponential distributions, has been considered for the steady distribution of queue size.

In the same model, T. Kishi [2] and M. Makino [3] obtained the steady distribution of queue size by using a generating method.

Here we assume that the both of $G(x)$ and $H(x)$ are general distributions. In the section II we provide criteria for the ergodicity of the general system. Furthermore in the sections III and IV we obtain the steady distributions of queue size and the total elapsed time in the system of an arrival customer. Finally for some special cases we consider the relationship between mean queue length, mean arrival rate and mean service rate.

Calculations in te section *V* are carried out on an electronic computer by Dr. H. Morimura.

The author wishes to express his sincere thanks to Dr. H. Morimura for his encouragement and help.

II. THE ERGODICITY OF THE SYSTEM

Denote by $x(t)$ and $y(t)$ the queue sizes at the instant t at the first and the second counter, respectively. Let us denote by $\tau_1', \tau_2', \ldots, \tau_n', \ldots$ the instants of the successive departures at the first counter.

Define $x_n = x(\tau_n'+0)$ and $y_n = y(\tau_n'-0)$. Then $z_n = (y_n, x_n)$ process is a Markov process with stationary transition probabilities as follows,

$$\begin{aligned}
&P(z_1=(0,\ n)\,|\,z_0=(i,\ 0))=\alpha_n && (n\geqq 0,\ i=0,\ 1)\\
&P(z_1=(1,\ n)\,|\,z_0=(i,\ 0))=\beta_n && (n\geqq 0,\ i=0,\ 1)\\
&P(z_1=(0,\ n)\,|\,z_0=(i,\ m))=p_{n+1-m} && (n\geqq 0,\ m\geqq 1,\ i=0,\ 1)\\
&P(z_1=(1,\ n)\,|\,z_0=(i,\ m))=q_{n+1-m} && (n\geqq 0,\ m\geqq 1,\ i=0,\ 1)
\end{aligned} \tag{1}$$

where we put $p_\alpha=0$ if $\alpha<0$ and $q_\beta=0$ if $\beta<0$, and also

$$\alpha_n=\int_0^\infty dH(y)\int_0^y dt\int_{y-t}^\infty dG(x)\left[\lambda e^{-\lambda(t+x)}\frac{(\lambda x)^n}{n!}\right]$$

$$+\left(\int_0^\infty e^{-\lambda x}\frac{(\lambda x)^n}{n!}dG(x)\right)\left(\int_0^\infty e^{-\lambda x}dH(x)\right),$$

$$\beta_n=\int_0^\infty dH(x)\int_0^x dt\left[\lambda e^{-\lambda x}G(x-t)\frac{[\lambda(x-t)]^n}{n!}\right], \tag{2}$$

$$p_n=\int_0^\infty e^{-\lambda x}\frac{(\lambda x)^n}{n!}H(x)dG(x),$$

and $q_n=\int_0^\infty e^{-\lambda x}\frac{(\lambda x)^n}{n!}G(x)dH(x).$

Then clearly $\alpha^n>0$ for all n. However other quantities β_n, p_n and q_n may be zero for some n.

Define

$$\mu_1=\left[\int_0^\infty xdG(x)\right]^{-1} \text{ and } \mu_2=\left[\int_0^\infty xdH(x)\right]^{-1}.$$

If both of $G(x)$ and $H(x)$ are lattice and $\mu_1<\mu_2$, $\beta_n=0$ for all n and $q_n=0$ for all n. Then our system, $z_n=(0, x_n)$ process, is reduced to the well-known system $M/D/1$. Except the above case, our system, z_n process, forms an irre-ducible aperiodic Markov chain. In the sequel we shall treate such a Markov chain, however all results obtained in the system are also valied in the exception.

Now if we introduce the notation

$$\phi(s)=\int_0^\infty e^{-sx}d(G(x)H(x)) \qquad (R(s)\geqq 0),$$

be (2) we have

$$\sum_{n=0}^\infty(p_n+q_n)z^n=\phi(\lambda(1-z)) \qquad \text{for } |z|\leqq 1.$$

We define ρ by

$$\rho=\sum_{n=1}^\infty n(p_n+q_n)=\lambda\int_0^\infty xd(G(x)H(x)).$$

Theorem 1. The system is ergodic if

$$\rho<1.$$

Proof: We introduce the process $\tilde{z}=y_n+2x_n$ equivalent with $z_n=(y_n, x_n)$, then $\tilde{z}_n$ process is an irreducible aperiodic Markov chain with the following transition probabilities,

$$\begin{aligned} &\tilde{p}_{0,2n}=\tilde{p}_{1,2n}=\alpha_n && (n\geqq 0), \\ &\tilde{p}_{0,2n+1}=\tilde{p}_{1,2n+1}=\beta_n && (n\geqq 0), \\ &\tilde{p}_{2m,2n}=\tilde{p}_{2m+1,2n}=p_{n+1-m} && (m\geqq 1,\ n\geqq 0) \\ &\tilde{p}_{2m,2n+1}=\tilde{p}_{2m+1,2n+1}=q_{n+1-m} && (m\geqq 1,\ n\geqq 0). \end{aligned} \tag{3}$$

In order to prove the theorem we shall show the existence of y_i which satisfies the following inequalities

$$\sum_{j=0}^{\infty} \tilde{p}_{ij} y_j \leqq y_i - 1 \qquad (i \neq 0),$$

$$\sum_{j=0}^{\infty} \tilde{p}_{0j} y_j < \infty \quad \text{and} \quad y_i \geqq 0.$$

See Foster [4]. Now if we define the sequence $\{y_i\}$ by

$$y_{2n}=y_{2n+1}=\frac{n}{1-\rho} \qquad (n\geqq 0),$$

it is easily verified that the sequence possesses the required properties.

Next in order to prove the "only if" assertion we make some preparations.

Let x_n^* $(n\geqq 0)$ be a Markov chain having a stationary transition probability matrix $(p_{ij};\ i, j=1, 2, 3\ldots)$, with any initial distribution ω. Consider now a new process $y_n^*=f(x_n^*)$ (call herein the lumped process), where f is a given function on the states $i=1, 2, 3, \ldots$. The function f is a many-one function on the state space of x_n^* onto the state space of y_n^*. The state i of x_n^* on which f assumes the same value are collapsed into a single state of the y_n^* process. We lable the states of y_n^* A_α, $\alpha=1, 2, \ldots$. We shall say that a Markov chain x_n^* is lumpable with respect to a partition $A=\{A_\alpha,\ \alpha=1, 2, 3, \ldots\}$ if for every initial dfstribution ω of x_n^* the lumped process y_n^* is a Markov chain and the transition probabilities do not depend on the choice of ω.

Let $p_{iAj}=\sum_{k\epsilon Aj} p_{ik}$. Then p_{iAj} represents the probability of moving from state i into set A_j in one step for the original Markov chain x_n^*.

A Markovian function of a finite state Markov chain were considered by Burke and Rosenblatt [5] and Kemeny and Snell [6]. Here in order to prove Theorem 4 we extend their results to a denumerable state Markov chain.

Theorem 2. A necessary and sufficient condition for a Markov chain x_n^* to be lumpable with respect to a partition $A=\{A_\alpha, \alpha=1,2,\ldots\}$ is that for every pair of sets A_i and A_j, p_{kAj} have the save value for every k in A_i. These common values $\{\hat{p}_{ij}\}$ form the transition matrix for the lumped chain y_n^*.

Proof: For the chain x_n^* to be lumpable it is clearly necessary that

$$p_\omega(x_1^*\epsilon A_j\,|\,x_0^*\epsilon A_i)$$

be the same for every ω for which it is defined. Put this common value $\hat{p}_{ij}$. In particular this must be the same for ω having a 1 in its kth component, for state k in A_i. Hence

$$p_{kAj}=P_k(x_1^*\epsilon A_j)=\hat{p}_{ij} \quad \text{for every } k \text{ in } A_i.$$

Thus the condition is necessary.

To prove it is sufficient, we must show that if the condition is satisfied the probability

$$P_\omega(x_n^*\epsilon A_t\,|\,x_0^*\epsilon A_i,\ x_1^*\ \epsilon A_j,\ldots,\ x_{n-1}^*\epsilon A_s) \tag{4}$$

depends only on A_s and A_t. The probability may be written in the form

$$P_{\omega'}(x_1^*\epsilon A_t)$$

where ω' is a probability distribution with non-zero components only on the states of A_s. It depends on ω and on the first n outcomes. However since $P_k(x_1^*\epsilon A_t)=\hat{p}_{st}$ for all k in A_s, then it is clear also that

$$P_{\omega'}(x_1^*\epsilon A_t)=\hat{p}_{st}.$$

Thus the probability in (4) depends only on A_s and A_t.

Next we shall consider the special case in which each A_α contains only finite states of the chain x_n^*.

Let U be the matrix whose ith row is the probability vector having equal components for states in A_i and 0 for the remaining states. Let V be the matrix with the jth column a vector with 1's in the components corresponding to states in A_j and 0's other wise. Then the lumped transition matrix is given by

$$\hat{P}=UPV$$

where $\hat{P}=(\hat{p}_{st})$ and $P=(p_{ij})$.

Theorem 3. A necessary and sufficient condition for a Markov chain to be lumpable with respect to a partition $A=\{A_\alpha,\ \alpha=1,2,\dots\}$ where each A_α consists of finite states is

$$VUPV=PV \tag{5}$$

Proof: The matrix VU has the form

$$VU=\begin{pmatrix} W_1 & 0 & 0 & \dots \\ 0 & W_2 & 0 & \dots \\ 0 & 0 & W_3 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

where $W_n (n=1,2,\dots)$ are probability matrices. Condition (5) states that the columns of PV are fixed vectors of VU. But since the chain is lumpable, the probability of moving from a state of A_i to the set A_j is the same for all states in A_i, hence the components of a column of PV corresponding to A_j are all the same. There they form a fixed vector for W_j. This proves (5).

Conversely, let us suppose that (5) holds. Then the columns of PV are fixed vectors for VU. But each W_j is the transition matrix of an ergodic chain, hence its only fixed column vectors are of the form $c\xi$ (ξ is a column vector having all components equal to 1.). Hence all the components of a column of PV corresponding to one set A_j must be the same. That is, the chain is lumpable by Theorem 2.

Using Theorem 2 or Theorem 3, we shall prove the following theorem.

Theorem 4. The system, z_n process, is ergodic only if $\rho<1$.

Proof: We consider the lumped process $z_n^*=f(z_n)=x_n$, and we put $A_i=\{(0,i), (1,i)\}$ $(i=0,1,2,\ldots)$. Let P_0 be the transition matrix for the z_n process. If the matrices U_0 and V_0 are defined as above, by (1) we see easily that the condition (5) holds, that is,

$$V_0U_0P_0V_0=P_0V_0.$$

Therefore z_n process is lumpable with respect to the defined partition. Thus z_n^* process is a lumped Markov chain with the stationary transition matrix $\hat{P}=U_0P_0V_0$;

$$\begin{aligned} \hat{p}_{0,n}&=\alpha_n+\beta_n && (n\geqq 0) \\ \hat{p}_{m,n}&=p_{n+1-m}+q_{n+1-m} && (m\geqq 1,\ n\geqq 0). \end{aligned} \tag{6}$$

If z_n process is ergodic, then z_n^* process is also ergodic. Therefore in what follows we assume that z_n^* process is ergodic and we consider the z_n^* process. Let M_{ij} be the mean first-passage time from the state A_i to the state A_j. From the properties of the transition matrix $\hat{P}$ we see that

$$M_{i,\,i-1}=M_{1,\,0} \qquad (i\neq 0).$$

Since $M_{i,\,0}=M_{i,\,i-1}+M_{i-1,\,0}$, using the induction method we obtain

$$M_{i,\,0}=iM_{1,\,0} \qquad (i\neq 0).$$

Therefore by Foster's result [4] we have

$$M_{1,\,0}\sum_{j=1}^{\infty}\hat{p}_{1j}\cdot j=M_{1,\,0}-1,$$

that is

$$M_{1,\,0}\cdot\rho=M_{1,\,0}-1$$

then $\rho=1-1/M_{1,0}<1$.

Remark. The process $f(z_n)=y_n$ is not lumpable. Also in the system of tandem type treated in Suzuki [7] the lumped process $f(z_n)=x_n$ in which $z_n=(y_n, x_n)$ process be defined similarly as above is lumpable and x_n process is the well-known system $M/G/1$. In this case $f(z_n)=y_n$ process is not lumpable. Furthermore in the case where finite queue is allowed before the second counter at which the service time be exponential, the lumped processes $f(z_n)=x_n$ and $f(z_n)=y_n$ for the process z_n defined as above are not lumpable.

III. THE STEADY DISTRIBUTION OF QUEUE SIZE

We devote the section to obtain the steady distribution of $\tilde{z}_n$ process defined in Theorem 1.

Under the condition $\rho<1$, for every pair of states i and j

$$\lim_{n\to\infty} \tilde{p}_{ij}^{(n)}=\pi_j$$

exists and is independent of i. The π_j satisfy the equations

$$\pi_j=\sum_{j=0}^{\infty}\pi_i\tilde{p}_{ij}$$

with $\sum_{j=0}^{\infty}\pi_j=1$ and the distribution $\{\pi_j\}$ is uniquely determined. In order to obtain the distribution we introduce the following generating fuctions for $|z|\leqq 1$,

$$\pi(z)=\sum_{n=0}^{\infty}\pi_n z^n,\quad \pi_0(z)=\sum_{n=0}^{\infty}\pi_{2n}z^{2n},$$

$$\pi_1(z)=\sum_{n=0}^{\infty}\pi_{2n+1}z^{2n+1},\quad \alpha(z)=\sum_{n=0}^{\infty}\alpha_n z^n, \tag{7}$$

$$\beta(z)=\sum_{n=0}^{\infty}\beta_n z^n,\quad p(z)=\sum_{n=0}^{\infty}p_n z^n,$$

and $$q(z)=\sum_{n=0}^{\infty}q_n z^n.$$

Theorem 5. If $|z|\leqq 1$, then

$$\pi(z)=\left\{\frac{1-\rho}{1-\rho+\sum_{n=1}^{\infty}n(\alpha+\beta)}\right\}.$$

$$\{(1-z)[p(z^2)\beta(z^2)-q(z^2)\alpha(z^2)]+z^2[\alpha(z^2)+z\beta(z^2)]$$

$$-[p(z^2)+zq(z^2)]\}/\{z^2-\phi(\lambda(1-z^2))\}$$

Proof: Using transition probabilities in (3) we have

$$\pi(z)=(\pi_0+\pi_1)[\alpha(z^2)+z\beta(z^2)]$$
$$+[p(z^2)+zq(z^2)]\sum_{n=1}^{\infty}(\pi_{2n}+\pi_{2n+1})z^{2(n-1)}. \tag{8}$$

Comparing the order of z in the both members of the equation we have

$$z\pi_0(z)[z^2-p(z^2)]-\pi_1(z)p(z^2)$$
$$=(\pi_0+\pi_1)z[z^2\alpha(z^2)-p(z^2)], \tag{9}$$

$$-z\pi_0(z)q(z^2)+[z^2-q(z^2)]\pi_1(z)$$
$$=(\pi_0+\pi_1)z[z^2\beta(z^2)-q(z^2)]. \tag{10}$$

From (9) and (10) if we define Δ, Δ_0 and Δ_1 by

$$\Delta=z^3[z^2-(p(z^2)+q(z^2))],$$

$$\Delta_0=(\pi_0+\pi_1)z^3\begin{vmatrix}\alpha(z^2) & -p(z^2)\\ \beta(z^2)-1 & z^2-q(z^2)\end{vmatrix},$$

$$\Delta_1=(\pi_0+\pi_1)z^4\begin{vmatrix}z^2-p(z^2) & \alpha(z^2)-1\\ -q(z^2) & \beta(z^2)\end{vmatrix},$$

then

$$\pi(z)=(\Delta_q+\Delta_1)/\Delta. \tag{11}$$

In (7) we have as $z\to 1\text{-}0$ that

$$\pi_0+\pi_1=(1-\rho)\rho/\{1-\rho+\sum_{n=1}^{\infty}n(\alpha n+\beta_n)\}. \tag{12}$$

In particular π_0 is given by

$$\pi_0=\left[\frac{\Delta_0}{\Delta}\right]_{Z\to1-0}=\frac{1-\rho}{1-\rho+\sum_{n=1}^{\infty}n(\alpha_n+\beta_n)}\cdot\frac{p_0(1-\beta_0)+q_0\alpha_0}{p_0+q_0}. \tag{13}$$

Substituting (12) into (11), we have the required result (7). Except the the trivial case where both of $G(x)$ and $H(x)$ are lattice and $\mu_1\leqq\mu_2$,

P (as soon as a customer have just completed his service at the first counter, he can go to the second counter)

$$=\pi_0(1)=\frac{(1-\rho)\alpha(1)+p(1)\sum_{n=1}^{\infty}n(\alpha_n+\beta_n)}{(1-\rho)+\sum_{n=1}^{\infty}n(\alpha_n+\beta_n)},$$

P (a customer completed his service at the first counter cannot go at once to the second counter)

$$=\pi_1(1)=1-\pi_0(1)=\frac{(1-\rho)\beta(1)+q(1)\sum_{n=1}^{\infty}n(\alpha_n+\beta_n)}{1-\rho+\sum_{n=1}^{\infty}n(\alpha_n+\beta_n)}.$$

In the above trivial case,

$$\pi_0(1)=1 \quad\text{and}\quad \pi_1(1)=0.$$

Next we consider the lumped process $z_n^*=f(z_n)=x_n$ process defined in Theorem 4. Under the condition $\rho<1$, for every pair of states A_i and A_j.

$$\lim_{n\to\infty}\hat{p}_{ij}^{(n)}=\hat{\pi}_j$$

exists and is independent of i. We introduce the generating function of the probability distribution $\{\hat{\pi}_j\}$,

$$\hat{\pi}(z)=\sum_{n=0}^{\infty}\hat{\pi}_n z^n \quad\text{for}\quad |z|\leqq1.$$

From (6) we obtain the following theorem.

Theorem 6. For $|z|\leqq1$,

$$\hat{\pi}(z)=\frac{1-\rho}{1-\rho+\sum_{n=1}^{\infty}n(\alpha_n+\beta_n)}\quad\frac{z(\alpha(z)+\beta(z))-\phi(\lambda(1-z))}{z-\phi(\lambda(1-z))}.$$

If $\int_0^\infty x^2dG(x)<\infty$ and $\int_0^\infty x^2dH(x)<\infty$, the mean queue length L at the first counter is given by

$$L=\left[\frac{d\hat{\pi}(z)}{dz}\right]_{Z\to 1-0}$$

$$=\Big\{(1-\rho)\sum_{n=1}^{\infty}n(\alpha_n+\beta_n)+(1-\rho)\sum_{n=1}^{\infty}n^2(\alpha_n+\beta_u)$$

$$+\lambda^2\sum_{n=1}^{\infty}n(\alpha_n+\beta_n)\int_0^\infty x^2d(G(x)H(x))\Big\}/2(1-\rho)\Big[1-\rho+\sum_{n=1}^{\infty}n(\alpha_n+\beta_n)\Big].$$

We define two independent random variables X and Y with the same probability distributions as those of χ_n and η_n respectively, then the random variable max (X, Y) has the distributiou $G(x)H(x)$. Therefore we have

$$\rho=\lambda E(\max(X, Y))$$
$$\geqq\max(\lambda E(X),\ \lambda E(Y))=\max(\rho_1,\ \rho_2),$$

wher $\rho_1=\lambda/\mu_1$ and $\rho_2=\lambda/\mu_2$.

If we suppose

$$G(x)=\begin{cases}1 & (x\geqq 1/\mu_1)\\ 0 & (x<1/\mu_1)\end{cases}$$

and $$H(x)=\begin{cases}1 & (x\geqq 1/\mu_2)\\ 0 & (x<1/\mu_2),\end{cases}$$

then it holds that $\rho=\max(\rho_1,\ \rho_2)$, that is, among these model considered in this paper the above mentioned model has minimum value for ρ.

Put $x_n^*=x(\tau_n-0)$, x_n^* is the queue size at the first counter immediately before the nth arrival. If $x(0)=0$ then we have

$$P(x_n\leqq k)=P(x^*_{n+k+1}\leqq k)\qquad (k=0, 1, 2, \ldots)$$

because the event $(x_n \leqq k)$ and also the event $(x^*_{n+k+1} \leqq k)$ occures if and only if $\tau_n' < \tau_{n+k+1}$. If $x(0)$ is arbitrary then a similar relation holds between $\{x_n^*\}$ and $\{x_n\}$. Then it follows that

$$\lim_{n\to\infty} P(x_n^* = k) = \lim_{n\to\infty} P(x_n = k) = \hat{\pi}_k \qquad (k \geqq 0).$$

IV. THE ELAPSED TIME IN THE SYSTEM

Let T_n denote the nth customer's elapsed time at the first counter, including three durations of the customer's waiting time, service time and blocking time. Let $x(0)=i$. Then we have

$$P(x_{n+i}=j \mid x_0=i) = \int_0^\infty e^{-\lambda x}\frac{(\lambda x)^j}{j!} dT_n(x).$$

Define

$$\psi_n(s) = E\{e^{-sT_n}\} = \int_0^\infty e^{-sx} dT_n(x) \qquad (R(s) \geqq 0),$$

then we have

$$E\{z^{x_{n+i}} \mid x_0 = i\} = \psi_n(\lambda(1-z)) \quad \text{for} \quad |z| \leqq 1.$$

Under the condition $\rho < 1$, we have from Theorem 6,

$$\lim_{n\to\infty} \psi_n(s) = \hat{\pi}\left(1 - \frac{s}{\lambda}\right) \quad \text{for} \quad |s-\lambda| \leqq \lambda,$$

that is, $T(x) = \lim_{n\to\infty} T_n(x)$ exists and its Laplace-Stieltjes transform is given by the right member of the above. The expectation T of $T(x)$ is obtained from the above formula and also is given easily from the formula

$$L = \lambda T.$$

V. THE EXPECTED LENGTH OF QUEUE SIZE AT THE FIRST COUNTER

In this section we consider some special cases and compare the expected length L for each case.

Case 1. We suppose

$$G(x)=1-e^{-\mu_1 x}(x\geqq 0) \quad \text{and} \quad H(x)=1-e^{-\mu_2 x} \qquad (x\geqq 0).$$

Then

$$L=\frac{2(1-\rho)A+(1-\rho)B+AC}{2(1-\rho)[1-\rho+A]}, \tag{14}$$

where

$$A=\rho_1+\rho_2\left(\frac{\lambda}{\lambda+\mu_2}\right)-\left(\frac{\lambda}{\mu_1+\mu_2}\right)\left(\frac{\lambda}{\lambda+\mu_2}\right),$$

$$B=2\left[\rho_1^2+\rho_2^2\left(\frac{\lambda}{\lambda+\mu_2}\right)-\left(\frac{\lambda}{\lambda+\mu_2}\right)\left(\frac{\lambda}{\mu_1+\mu_2}\right)^2\right],$$

$$C=2\left[\rho_1^2+\rho_2^2-\left(\frac{\lambda}{\mu_1+\mu_2}\right)^2\right],$$

$$\rho=\rho_1+\rho_2-\frac{\lambda}{\mu_1+\mu_2}.$$

Note that this result is somewhat different from Kishi's result [2] obtained at arbitrary time.

Case 2. We suppose

$$G(x)=1-e^{-\mu_1 x} \quad (x\geqq 0) \text{ and } H(x)=\begin{cases} 1 & \left(x\geqq \frac{1}{\mu_2}\right) \\ 0 & \left(x<\frac{1}{\mu_2}\right) \end{cases}$$

Then in (14)

$$A=\rho_2+\rho_1\left(\frac{\lambda}{\lambda-\mu_1}\right)e^{-\mu_1/\mu_2}-\left(\frac{\mu_1}{\lambda-\mu_1}\right)e^{-\rho_2}-1,$$

$$B=1+(1-\rho_2)^2+2\rho_1^2\rho_2\frac{\lambda(\mu_1+\mu_2)-\mu_1(2\mu_2+\mu_1)}{(\lambda-\mu_1)^2}e^{-\mu_1/\mu_2},$$

$$C=\rho_2^2+\rho_2(1-\rho_2)e^{-\mu_1/\mu_2}+2\rho_1^2e^{-\mu_1/\mu_2}+2\rho_1\rho_2e^{-\mu_1/\mu_2},$$

$$\rho=\rho_2+\rho_1e^{-\mu_1/\mu_2}.$$

Case 3. We suppose

$$G(x)=\begin{cases} 1 & (x\geqq 1/\mu_1) \\ 0 & (x<1/\mu_1) \end{cases} \quad \text{and } H(x)=1-e^{-\mu_2 x} \quad (x\geqq 0).$$

Then in (14),

$$A=\rho_1+\rho_2\left(\frac{\lambda}{\lambda+\mu_2}\right)e^{-\mu_2/\mu_1},$$

$$B=\rho_1^2+2\rho_1\rho_2^2\left(\frac{\mu_1+\mu_2}{\lambda+\mu_2}\right)e^{-\mu_2/\mu_1},$$

$$C=\rho_1^2+\rho_1(1-\rho_1)e^{-\mu_2/\mu_1}+2\rho_2^2e^{-\mu_2/\mu_1}+2\rho_1\rho_2e^{-\mu_2/\mu_1},$$

$$\rho=\rho_1+\rho_2e^{-\mu_2/\mu_1}.$$

Case 4. We suppose

$$G(x)=\begin{cases}1 & (x\geqq 1/\mu_1)\\ 0 & (x<1/\mu_1)\end{cases} \quad\text{and}\quad H(x)=\begin{cases}1 & (x\geqq 1/\mu_2)\\ 0 & (x<1/\mu_2).\end{cases}$$

(1) For $\mu_1\leqq\mu_2$,

$$A=\rho_1,\ B=\rho_1^2,\ C=\rho_1^2 \text{ and } \rho=\rho_1.$$

(2) For $\mu_1>\mu_2$,

$$A=\rho_2+e^{\rho_1-\rho_2}-1,$$

$$B=1+(1-\rho_2)^2-2(1-\rho_1)e^{\rho_1-\rho_2},$$

$$C=\rho_2^2 \text{ and } \rho=\rho_2.$$

Now we shall calculate the value of L for each case for fixed $\mu_1=1$.

Table I—Case 1

μ_2	λ	L	μ_2	λ	L
0.3	0.1	0.284	1.0	0.6	3.366
0.3	0.2	1.842	1.0	0.7	15.938
0.5	0.1	0.163	1.2	0.1	0.119
0.5	0.2	0.557	1.2	0.2	0.289
0.5	0.3	1.699	1.2	0.3	0.544
0.5	0.4	11.389	1.2	0.4	0.958
			1.2	0.5	1.739
0.8	0.1	0.130	1.2	0.6	3.759
0.8	0.2	0.347	1.2	0.7	21.697
0.8	0.3	0.736			
0.8	0.4	1.559	1.5	0.1	0.116
0.8	0.5	4.248	1.5	0.2	0.274
			1.5	0.3	0.499
0.9	0.1	0.125	1.5	0.4	0.838
0.9	0.2	0.324	1.5	0.5	1.409
0.9	0.3	0.657	1.5	0.6	2.573
0.9	0.4	1.292	1.5	0.7	6.314
0.9	0.5	2.912			
0.9	0.6	15.021	2.0	0.1	0.113
			2.0	0.2	0.263
1.0	0.1	0.118	2.0	0.3	0.466
1.0	0.2	0.287	2.0	0.4	0.757
1.0	0.3	0.534	2.0	0.5	1.207
1.0	0.4	0.925	2.0	0.6	1.999
1.0	0.5	1.633			

μ_2	λ	L	μ_2	λ	L
2.0	0.7	3.788	3.0	0.9	34.916
2.0	0.8	11.789			
			∞	0.1	0.111
3.0	0.1	0.112	∞	0.2	0.250
3.0	0.2	0.255	∞	0.3	0.428
3.0	0.3	0.445	∞	0.4	0.666
3.0	0.4	0.705	∞	0.5	1
3.0	0.5	1.085	∞	0.6	1.500
3.0	0.6	1.698	∞	0.7	2.333
3.0	0.7	2.852	∞	0.8	4.000
3.0	0.8	5.866	∞	0.9	9.000

In the case $\mu_2=\infty$ our system is the well-known system $M/M/1$ and $L=\frac{\lambda}{1-\lambda}$.

Table II—Case 2

μ_2	λ	L	μ_2	λ	L
0.3	0.1	0.196	0.8	0.1	0.122
0.3	0.2	0.943	0.8	0.2	0.305
			0.8	0.3	0.593
0.5	0.1	0.138	0.8	0.4	1.098
0.5	0.2	0.400	0.8	0.5	2.229
0.5	0.3	0.981	0.8	0.6	7.608
0.6	0.4	3.199			

μ_2	λ	L	μ_2	λ	L
0.9	0.1	0.120	1.5	0.4	0.790
0.9	0.2	0.294	1.5	0.5	1.269
0.9	0.3	0.558	1.5	0.6	2.121
0.9	0.4	0.991	1.5	0.7	4.099
0.9	0.5	1.843	1.5	0.8	14.290
0.9	0.6	4.409			
			2.0	0.1	0.114
1.0	0.1	0.118	2.0	0.2	0.263
1.0	0.2	0.287	2.0	0.3	0.465
1.0	0.3	0.534	2.0	0.4	0.747
1.0	0.4	0.925	2.0	0.5	1.168
1.0	0.5	1.633	2.0	0.6	1.861
1.0	0.6	3.366	2.0	0.7	3.223
1.0	0.7	15.938	2.0	0.8	7.179
			2.0	0.9	222.417
1.2	0.1	0.117			
1.2	0.2	0.277	3.0	0.1	0.113
1.2	0.3	0.505	3.0	0.2	0.258
1.2	0.4	0.847	3.0	0.3	0.451
1.2	0.5	1.415	3.0	0.4	0.714
1.2	0.6	2.556	3.0	0.5	1.096
1.2	0.7	6.187	3.0	0.6	1.694
			3.0	0.7	2.764
1.5	0.1	0.115	3.0	0.8	5.232
1.5	0.2	0.270	3.0	0.9	17.075
1.5	0.3	0.483			

Table III—Case 3

μ_2	λ	L
0.3	0.1	0.281
0.3	0.2	1.772
0.5	0.1	0.157
0.5	0.2	0.525
0.5	0.3	1.515
0.5	0.4	6.704
0.8	0.1	0.123
0.8	0.2	0.315
0.8	0.3	0.633
0.8	0.4	1.232
0.8	0.5	2.700
0.8	0.6	11.467
0.9	0.1	0.119
0.9	0.2	0.292
0.9	0.3	0.559
0.9	0.4	1.015
0.9	0.5	1.951
0.9	0.6	4.978
1.0	0.1	0.116
1.0	0.2	0.277
1.0	0.3	0.511
1.0	0.4	0.884
1.0	0.5	1.570
1.0	0.6	3.276
1.0	0.7	15.816
1.2	0.1	0.112
1.2	0.2	0.258
1.2	0.3	0.455
1.2	0.4	0.740
1.2	0.5	1.200
1.2	0.6	2.090
1.2	0.7	4.703
1.5	0.1	0.169
1.5	0.2	0.243
1.5	0.3	0.414
1.5	0.4	0.642
1.5	0.5	0.975
1.5	0.6	1.524
1.5	0.7	2.665
1.5	0.8	6.910
2.0	0.1	0.107
2.0	0.2	0.233
2.0	0.3	0.386
2.0	0.4	0.579

μ_2	λ	L	μ_2	λ	L
2.0	0.5	0.841	3.0	0.7	1.600
2.0	0.6	1.230	3.0	0.8	2.606
2.0	0.7	1.908	3.0	0.9	5.895
2.0	0.8	3.502			
2.0	0.9	13.510	∞	0.1	0.105
			∞	0.2	0.225
3.0	0.1	0.106	∞	0.3	0.364
3.0	0.2	0.227	∞	0.4	0.533
3.0	0.3	0.369	∞	0.5	0.750
3.0	0.4	0.544	∞	0.6	1.050
3.0	0.5	0.771	∞	0.7	1.516
3.0	0.6	1.091	∞	0.8	2.400
			∞	0.9	4.950

In the case $\mu_2=\infty$ our system is the well-knowu system $M/D/1$ and $L=\lambda(2-\lambda)/2(1-\lambda)$.

Table IV—Case 4

μ_2	λ	L
0.3	0.1	0.183
0.3	0.2	0.866
0.5	0.1	0.125
0.5	0.2	0.333
0.5	0.3	0.750
0.5	0.4	2.000

The table of L in the case $\mu_2 \geqq 1$ is the same as the case $\mu_2 = \infty$ in Table III.

In Table I we see that for $\mu_2 < 2$ arrival rate λ has remarkably influence of blocking on the expected queue length, but for $\mu_2 \geqq 2$ and $\lambda \leqq 0.6$ the latter seems almost no influence of λ. Similarly in Tables II and III it will be seen that for $\mu_2 < 2$, λ has influence of blocking on the expected length, but for $\mu_2 \geqq 2$ and $\lambda \leqq 0.7$ the length appears no influence of λ. In Table IV, for $\mu_2 \geqq 1$ λ has no influence of blocking on the length, but for $\mu_2 < 1$ λ has clearly an effect on it.

Let us write the expected length in each case by L_1, L_2, L_3 and L_4 respectively. Then it holds that

$$L_1 > L_2 > L_3 > L_4 \text{ for } \mu_2 \geqq 1,$$
$$L_1 > L_3 > L_2 > L_4 \text{ for } \mu_2 \leqq 0.8,$$
$$L_1 > L_2 > L_3 > L_4 \text{ for } \mu_2 = 0.9 \text{ and } 0 \leqq \lambda \leqq 0.2,$$

and
$$L_1 > L_3 > L_2 >_4 \text{ for } \mu_2 = 0.9 \text{ and } \lambda \geqq 0.3.$$

Then L_2 is larger or smaller than L_3 according to the value of λ and μ_2.

Also we may be interested to consider the case exchanging the first counter for the second one, that is, the case in distributions of the first service time and the second are interchanged. In order to discussion the results of interchange we will need to prepare the similar tables as above for various values of μ_1.

Note. Recently the auther proved the ergodicity of the system in which input process be recurrent. This will be appear in near future.

REFERENCES

1. G.C. Hunt: Sequential arrays of waiting lines, Opns Res., vol. 4, no. 6, pp. 674–683 (1956)
2. T. Kishi: Queueing system with contemporaneous service, Keiei Kagaku (Japan), vol. 3, no. 3 (1960)
3. M. Makino: Some problems on tandem queues, Keiei Kagaku (Japan), vol. 6, no. 4 (1963)
4. F.G. Foster: On stochastic matrices associated with certain queueing processes, Ann. Math. Stat., vol. 24, pp. 355–360 (1953)

5. C.K. Burke and M. Rosenblatt: A Markovian function of a Markov chain, Ann. Math. Stat., vol. 29, pp. 1112–1122 (1958)
6. J.G. Kemeny and J.L. Snell: Finite Markov chains, D. Van Norstrand Co. Inc. (1960)
7. T. Suzuki: Two queues in series, J.O.R.S.J. (Japan), vol. 5, no. 4, pp. 149-155 (1963)