PRUDENCE—MEASURES OF EFFECTIVENESS FOR FATEFUL DECISIONS

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(*This paper does not necessarily represent the opinion of The Chief of Naval Operations*)

The research which I shall report in this paper was conducted by Mr. John Coyle of the Operations Evaluation Group. It is an extension of an earlier descriptive paper on the same subject and an attempt to provide a quantitative measure of effectiveness for decisions which modify a portfolio of risks.

In this attempt some generality relative to the descriptive treatment necessarily has been lost. It is recognized that the present paper represents only the first step along the way to a comprehensive measure of risk.

I shall describe the model which leads to the derivation of the measure of effectiveness; I shall give some examples of its application and I shall point out areas (in which generality has been sacrificed) for further investigation and extension of the measure beyond its current limitations.

The measure of effectiveness described provides what seems to be a potentially useful criterion. There are difficulties to be encountered in making the necessary quantitative evaluations but its advantages in permitting evaluations of measures such as insurance and mixtures of alternative policies which are recognized as prudent seem to outweigh the difficulties of the quantitative evaluations. It is known that the cri-

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terion of maximum expected value does not permit detailed consideration of prudent measures in the sence in which this paper cosiders them. The measure differs from the game theoretical approach in that it does not require the provision of a comprehensive scale of preference. Instead, it requires a quantitative statement of one's general optimism for the future and a quantitative estimate of the scale of the risks involved in terms of the amount of payoff and the cost of the individual decision, in the context of other commitments and other possible decisions. While these statements may be as difficult to treat quantitatively as those quantities required in game theory or statistical decision theory, the statements need not be so precise or so comprehensive.

To be more concrete: in numerous cases a decision maker has an opportunity to undertake a commitment in which the odds for success are favorable and in which payoff is rarge relative to costs. These would be unlikely perhaps at the gambling table but are or should be more commonplace in business or in national strategies. In many instances where the scale of the commitment is not large in relation to the total resources it suffices to choose among such possibilities on the basis of maximum expected value of the return. But in those instances where a large portion of the total available resources must be risked, even though the odds are favorable and the payoff is large, the decision maker will be prudent to give consideration to the risk of loss and to the means of safeguarding his future. For example, if you are offered an opportunity to match your quarter against a silver dollar you are likely to accept and you would not thereby demonstrate your imprudence, because the loss of a quarter is not likely to be catastrophic. On the other hand, if you were offered an opportunity to put up all of your possessions on the same advantageous basis of a payoff of 4—1 on an even chance, yon might accept, but if you did, it would suggest to an on-looker that you were either highly imprudent, or had no hope for the future.

The objective will be to describe a measure whereby one can investigate the transition between the first case and the second. We shall name the criterion or measure of effectiveness, the specific jeopardy. It can supplement other measures of effectiveness when the resources at stake are so large as to give pause to the decision maker. The two parameters required to describe this measure are, as mentioned above, the estimate of optimism for the future and the scale of the resources at risk. Beyond these two factors we need be concerned only with the usual estimate of probability of success or failure of the proposed venture, and the size of the cost and the payoff.

Now let us examine the model. The decision maker is seen at the moment just prior to undertaking a new venture. At this time his past decisions have led to a group of commitments, not yet completed, which constitute a portfolio of risks. Since they have not yet run their course, their output in terms of net worth is not known, but we shall name their possible outcomes w, and describe the probability distribution in terms of the density $p(w)$. Note that not all of these risks will be completed at the same moment and that in the course of time during which they do become complete, additional decisions will be made and new commitments will be undertaken. Indeed, the decision we are about to investigate will be undertaken prior to the time that the present commitments are completely investigated.

For the moment we shall restrict our attention only to the commitments already undertaken and our value w will refer only to those present commitments. We shall describe the entire future as the series of commitments undertaken beyond the present ones. We shall be concerned with indefinite survival into the future as if we had started with a net worth w. We shall examine first the probability of bankruptcy in the long run starting with a net worth w, characterizing all future ventures by typical prospective odds for success or failure, and by a scale of operations. Let Q_w represent the conditional probability of ultimate bankruptcy starting with a net worth w. Then the a priori probability of ruin Q_0 with the decision maker's present set of commitments is given by

$$Q_0 = \int_w p(w) Q_w dw. \tag{1}$$

Faced now with a new decision the decision maker is concerned with the effect of his decision on the ultimate probability of ruin, and we shall call the new resultant probability of ruin Q_1, the probability of ruin given his present commitments plus one new decision to undertake a risk. Call the ratio of Q_1 to Q_0 the specific jeopardy of the decision. Prior to evaluating Q_1, first examine Q_w, the conditional probability of ruin given an outcome of w of the present set of commitments.

As indicated before we need an assumption of "promise." In quantitative terms, we judge what the profitability of future ventures might be visualizing the future in terms of typical undertakings having probabilities π of success of winning a given amount and $1-\pi$ of failure resulting in loss of the same amount. Note that these probabilities are not those which characterize the decision soon to be made, which as a result of more detailed examination, can be specified in more precise terms. Rather, π and $1-\pi$ are estimates of the probabilities for decisons not yet faced, nor even defined, but which we anticipate will occur in the future. We probably will estimate π and $1-\pi$ in terms of our recent history and the expectation that similar decisions will arise in the future. Then the dimensionless parameter $S=\frac{1-\pi}{\pi}$ is sufficient description of our future prospects for our purposes. It will be observed that this parameter is the quantity that enters into the gambler's ruin problem of Feller. In addition to the quantity S we must also posess some scale factor ω which relates w to the size of initiatives. ω likewise is an estimate of the probable size of future (not yet defined) undertakings. With these two quantities we can in fact use the gambler's ruin model for our estimate of the future. If we take ω as the size of the stake in one individual play (the assumption is that a play is analagous to a future undertaking) then the probability Q_w that the decision maker starting with w will be ruined before he breaks a bank of assets b is given by

$$Q_w = \frac{S^{\frac{w}{\omega}} - S^{\frac{w+b}{\omega}}}{1 - S^{\frac{w+b}{\omega}}}.$$

Since we are concerned with indefinite survival, we consider the bank assets b as indefinitely large. If we further make the assumption that our future prospects are optimistic (which results when the value of π is greater than $^1/_2$ so that S will be less than 1) the equation reduces to

$$\begin{aligned} Q_w &= S^{\frac{w}{\omega}} \quad \text{where } S<1 \\ & \qquad\qquad\quad\; w \geq 0 \\ &= 1 \qquad\;\; \text{where } w<0. \end{aligned}$$

This provides the information needed to make equation (1) a sufficient description of the decision maker's state just prior to undertaking a new decision. Now examine his next decision, which can modify the result of his present investments by an amount x with probability $p_1(x)$. The new probability of eventual ruin from equation (1) is

$$Q_1 = \iint\limits_{x\,w} p_1(x) p(w) Q_{w+x} dw dx \tag{2}$$

(assuming that the outcomes of x and w are independent).

Since $Q_{w+x} = S^{\frac{w}{\omega}} \cdot S^{\frac{x}{\omega}}$ when $w+x \geq 0$ $(=1 \text{ if } w+x<0)$

and neglecting the probability of bankruptcy in the course of this new commitment:

$$Q_1 \cong \int\limits_w p(w) Q_w dw \int\limits_x p_1(x)\, S^{\frac{x}{\omega}} dx$$

$$Q_1 \cong Q_0 \int\limits_x p_1(x) S^{\frac{w}{x}} dx \quad \text{where} \quad p(w+x<0) \ll 1.$$

We call the expression $J = \int\limits_x p_1(x) S^{\frac{x}{\omega}} dx$ as "specific jeopardy." It is possible also to combine the two basic parameters of the ruin model S and ω into a single quantity

$$v=\frac{\omega}{\ln\frac{1}{S}}$$

Then the specific jeopardy takes the form of

$$J=\int_x p(x)e^{-\frac{x}{v}}\,dx.$$

By analogy with these equations if a number of decisions are to be made at the same time so as to constitute a policy, the context of prior decisions is subject to change only by these, and the probability of ruin resulting from the policy is

$$Q_p=Q_0\prod^{i}J_i.$$

To examine the behavior of the specific jeopardy in a particular case, suppose that the decision maker is asked to decide whether or not to commit himself to an undertaking which has only two outcomes, either a gain (g) or a loss ($-a$). Alternatively, he might be asked to determine the size of the commitment he would undertake under these conditions. Associate a probability p with the gain (g) and a corresponding to probability $q=1-p$ with the loss ($-a$). In this simple case of only two outcomes, the integral representating the specific jeopardy reduces to a two term sum:

$$J=pS^{\frac{q}{\omega}}+qS^{\frac{-a}{\omega}}.$$

I have plotted this case for the condition where p is equal to .75, $1-p=.25$; g and a are equal and S, the estimate of optimism for the future, is .8. I have plotted J as a function of a quantity $\frac{(g+a)}{\omega}.$ It will be noted from Fig. 1 that J approaches 1 when the stakes are sufficiently small and decreases with increasing stake going through a minimum at the point where $S^{\frac{g+a}{\omega}}$ is equal to $\frac{qa}{pg}.$ Thereafter as the scale of the stakes is increased, J increases radically as the "eggs in one basket" aspect dominates the situation, and approaches the value $q\ S^{-\frac{a}{\omega}}.$ At some point, of course, the size of the risk will reach a point where the condition that

$$p(w+x<0)\ll 1$$

no longer holds. At this point, where the venture threatens to bankrupt the portfolio in one event, the specific jeopardy is an underestimate of adopting the venture. This simple example illustrates that the expected value is an adequate criterion for small ventures and that specific jeopardy provides a measure which places warning signals when the scale of venture, though very favorable, becomes too large for the decision maker's purse.

We may examine one more example. In this instance we shall be concerned with prudent allocation of a given amount of resources between two alternatives having different probabilities of success.

Two alternatives affording each an outcome proportional to allocation, if successful, but with differing probabilities of success are postulated. The ploblem is how most prudently to allocate a fixed budget of resources between them.

Definitions:

	Alternatives	
	1	2
Allocation at risk	ka	$(1-k)a$
Value of outcome if successful	kg_1	$(1-k)g_2$
Expected payoff if successful	$p_1(kg_1)$	$p_2((1-k)g_2)$
Expected cost of failure	$q_1(-ka)$	$q_2(-(1-k)a)$

$$\overset{i}{\Pi} J_i=J_1J_2=[p_1S^{\frac{kg_1}{\omega}}+q_1S^{\frac{-ka}{\omega}}][p_2S^{\frac{(1-k)g_2}{\omega}}+q_2S^{\frac{-(1-k)a}{\omega}}]$$

The choice of k to minimize the specific jeopardy has not been solved analytically except in the special case where the ratio of gain or loss to allocation is the same for both alternatives, i.e., the alternatives differ only in probalility of success:

Let $\qquad g_1=g_2=g$

Then

$$\frac{dJ_1J_2}{dk}=0 \qquad \text{where}$$

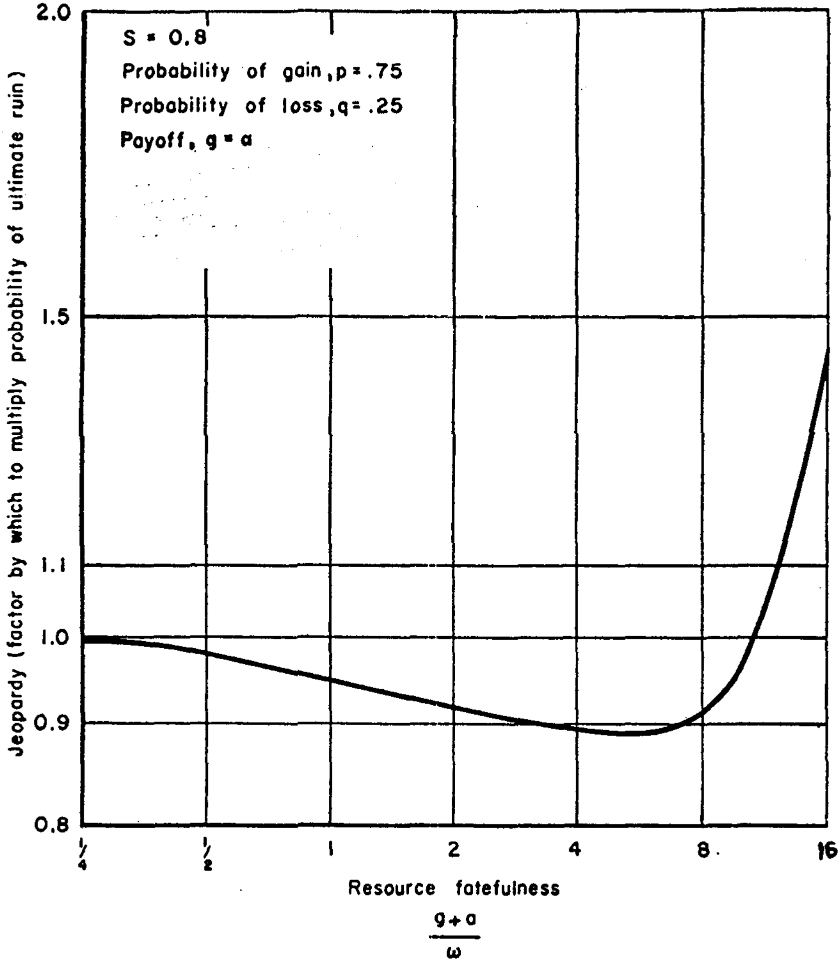

Fig. 1 Jeopardy as a function of resource fatefulness $\frac{g+a}{\omega}$

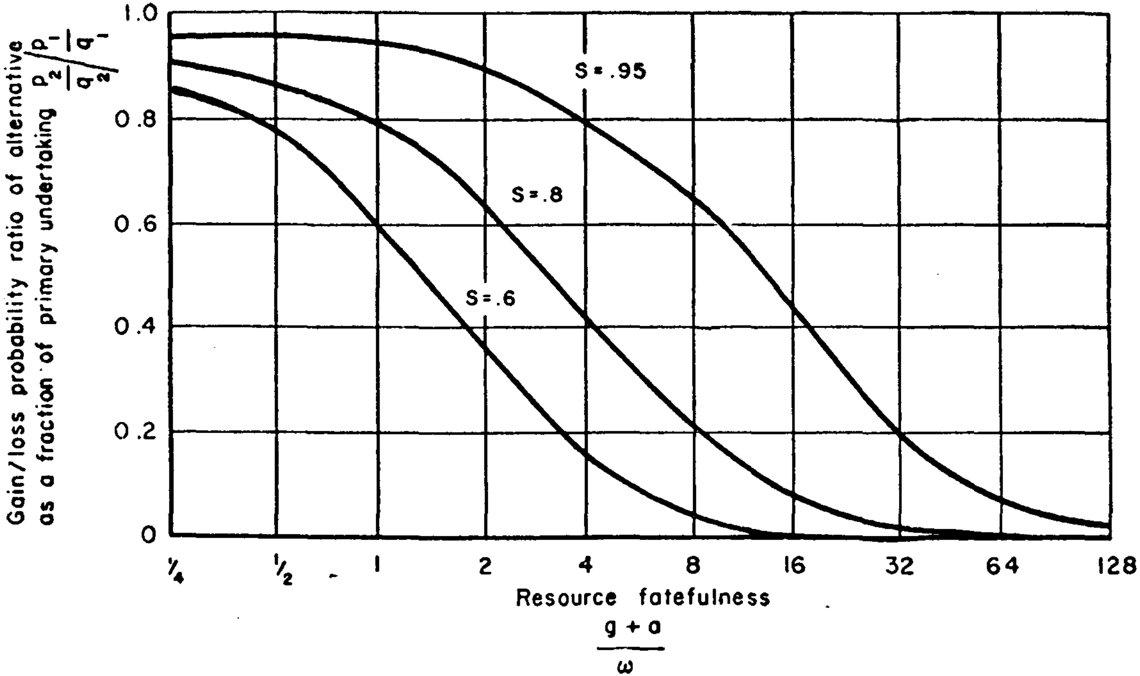

Fig. 2 Break-even gain/loss probability ratio at which it is prudent to divert resources to asecond less advantageous alternative (which differs only in probability of success). Above curve, divert some resource to second alternative; below curve, allocate all to first alternative.

$$k=\frac{1}{2}+\frac{\omega}{2(g+a)}\cdot\frac{\ln\frac{p_1q_2}{q_1p_2}}{\ln S^{-1}}$$

This is a minimum.

Where the probabilities are equal, $k={}^1/_2$.

For k to be less than one (i.e., to justify any allocation to the less advantageous alternative) the second term must be less than ${}^1/_2$.

Fig. 2 illustrates the break-even point as a function of resource fatefulness. An alternative would need to have better than the indicated relative loss/gain probability ratio to qualify for diversion of funds to it.

These two examples have indicated something about the kind of measure we seek and its general behavior. There are several other examples which we might have presented in a more extensive discussion; perhaps other examples will occur to you.

I would like now to emphasize the preliminary nature of this investigation, to point out some of its limitations in applicability, and to encourage further work leading to a more general, less limited formulation.

First, I call your attention to the basically optimistic nature of the situations thus far considered; the value of S has been restricted to quantities less than 1, implying a generally favorable future. Second, the decisions considered are those which are fataful but not catastrophic, in the sense that we exclude ruin by the end of current commitments, and also exclude immediate ruin as a direct result of the decision or policy to be undertaken. It is clear that in some instances decisions more fateful than the ones thus far discussed must occasionally be faced. It will be desirable to extend the current formulation to treat such decisions. However, I should note here that the current formulation will generally result in an overestimate of jeopardy for these very fateful decisions, at the expence of a strict interpretation of the quantities involved. In this sence the measure remains useful for the purpose of providing warning signals against too large a commitment in a very serious decision.

There will also be occasions when one's view of the future will not be optimistic and the examination of criteria for these situations will take on importance. In these instances, of course, survival will no longer be the primary goal since ultimate failure to survive is assured. One then would be concerned perhaps with the expected duration of survival in an unfavorable situation or perhaps in other goals. It is possible that an extension of the current formulation into these areas can be accomplished by extension of the gamblers ruin model.

It is not so clear that the extension into other cases of interest can use the same basic model. In particular, some decision even though optimistic, may be of such a nature as to affect our estimate of the future; that is, change the value of S depending upon the outcome of current decisions. In the present formulation it is assumed that the value of S does not change as a result of any single decision. Changes in the value of S may have particular significance in those instances where one's initial view of the future is not optimistic, as we have just discussed. Evidentily if S can be changed from a pessimistic to an optimistic value by the favorable outcome of a decision, survival may remain a proper goal even where one's initial estimate is pessimistic.

It has also been assumed that the outcome of each decision is essentially independent of other decisions—this too is a condition which may not occur in practice, but it will require further development of this model for proper accounting.

Despite these limitations it is believed that criteria of this type are a proper matter for study in operations research and management science and that it would be desirable to carry on this investigation to cover the limitations I have mentioned and others which may appear as further applications are sought.