TABULATION OF THE BUILD-UP TIME OF WAITING LINE

MICHIKO SORIMACHI

The Defense Agency

§1. INTRODUCTION

An important problem that often arises in many applications of the queuing theory is how to estimate just how long it takes to make the transition from the initial state to the equilibrium state.

As one way, Davis [1] proposed a notion of the build-up time (T) of waiting lines, and calculated the exact form of T in the case $M/M/s$ ($s \geqq 1$).

Morimura [2] pointed out that there was a petty mistake of calculation in the derivation of the equation (29) in [1], and it took tedious calculation to correct it. Then he introduced an analogous indicator of the mean waiting time in the case $M/G/1$.

Davis worked of simple queuing system, but it has practical interests to get the table of the build-up time for multiple channels.

Thus we shall try here to get the correct expression and the numerical table of T.

The exact formula of T has shown is §2 and an example of table is in §3 and in §4 we discuss about a result of the table.

§2. A FORMULA TO TABULATE T

We consider the queuing system of $M/M/s$. Let s denote the number of servers or service channels available. Let $P_n(t)$ denote the probability that there are exactly n past arrivals still in the system at the epoch t. Then $P_n(t)$ must satisfy the following difference-differential equations.

$$(1)\qquad (1/\mu)P_0'(t)=P_1(t)-(\lambda/\mu)P_0(t)$$

$$(2)\qquad (1/\mu)P_n'(t)=(n+1)P_{n+1}(t)-nP_n(t)-(\lambda/\mu)(P_n(t)-P_{n-1}(t) \quad 1\leqq n\leqq s-1$$

$$(1/\mu)P_n'(t)=s(P_{n+1}(t)-P_n(t))-(\lambda/\mu)(P_n(t)-P_{n-1}(t)) \quad s\leqq n$$

Put $\lfloor n\rfloor=\min(s, n)$ and supporse $P_n(t)=0 \quad (n<0)$,

Furthermore, without loss of generality, we shall assume that $\mu=1$, throughout this paper.

With these conventions, Eq. (1) can be written as

$$(3)\qquad P_n'(t)=\lfloor n+1\rfloor P_{n+1}(t)-\lfloor n\rfloor P_n(t)-\lambda[P_n(t)-P_{n-1}(t)]$$

If $\lambda/\mu<s$, then the limit

$$\lim_{t\to\infty} P_n(t)=P_n \qquad \text{exist.}$$

And as well known we have

$$(4)\qquad P_n=P_0\lambda^n/\lfloor n\rfloor !\, s^{n-\lfloor n\rfloor} \qquad n\geqq 0$$

$$(5)\qquad P_0=[\sum_0^{s-2}\frac{\lambda^n}{n!}+s\lambda^{s-1}/(s-\lambda)(s-1)!]^{-1}$$

Also of interest,

$$(6)\qquad \sum_0^s(s-n)P_n=\sum_0^\infty(s-\lfloor n\rfloor)P_n=s-\sum_0^\infty\lambda P_{n-1}=s-\lambda$$

$$(7)\qquad \sum_0^s n(s-n)P_n=\sum_0^\infty n(s-\lfloor n\rfloor)P_n=(s-\lambda)m_1-\lambda$$

$$(8)\qquad \sum_0^s n^2(s-n)P_n=(s-\lambda)m_2-2\lambda m_1-\lambda$$

where m_1 and m_2 denote the first and second moments of the equibrium distribution of n.

The left-hand sides of these equations can be simplified such as

$$(9)\qquad \sum_0^s n(s-n)P_n=\lambda(s-\lambda)-\lambda\sum_0^{s-1}P_n$$

$$(10)\qquad \sum_0^s n^2(s-n)P_n=\lambda[(s-\lambda)m_1-\lambda^2+2\lambda(s-\lambda)-\lambda(s+1)\sum_0^{s-1}P_n]$$

These, together with (7), (8) imply that

$$m_1=\lambda+\lambda\frac{1-\sum_0^{s-1}P_n}{s-\lambda} \tag{11}$$

and

$$m_2=\frac{m_1(s+\lambda)(s+1-\lambda)-2\lambda^2}{(s-\lambda)}-s\lambda \tag{12}$$

By the way, putting $\Pi_n(t)=P_n(t)-P_n$, Eq (3) can be rewritten as

$$\Pi_n'(t)=\lfloor n+1\rfloor\Pi_{n+1}-\lfloor n\rfloor\Pi_n-\lambda(\Pi_n-\Pi_{n-1}) \tag{13}$$

Summing up both sides of Eq. (13) respect to n from zero to n, we obtain

$$\sum_0^n\Pi_k'=\lfloor n+1\rfloor\Pi_{n+1}-\lambda\Pi_n \tag{14}$$

In particular, if $k\leqq s$,

$$\sum_0^{k-1}\Pi_j'=k\Pi_k-\lambda\Pi_{k-1} \tag{15}$$

Now, in Davis [1], the nominal build-up time is

$$T=\left[\int_0^\infty t dM_1(t)\right]\Big/\left[\int_0^\infty dM_1(t)\right] \tag{16}$$

Let $M_n(t)$ represent the moments at epoch t as follows.

$$M_n(t)=\sum_0^\infty k^n P_k(t)$$

Now, we shall consider the derivative of the first moment. We have

$$M_1'(t)=(d/dt)M_1(t)=\sum_0^\infty kP_k'(t)=\sum_0^\infty k\Pi_k'=\sum_{k=0}^\infty\sum_{k+1}^\infty\Pi_j' \tag{17}$$

$$=\sum_{k=0}^\infty(-\sum_0^k\Pi_j')=-\sum_0^\infty\lfloor k\rfloor\Pi_k=\sum_0^s(s-k)\Pi_k$$

In the same way, for the second moment, we get

$$M_2'(t)=\sum_0^\infty k^2\Pi_k'=-\sum_0^\infty(2k+1)\sum_0^k\Pi_j' \tag{18}$$

$$=-2(s-\lambda)(M_1-m_1)-M_1'+2\sum_0^s k(s-k)\Pi_k$$

Thus, using Eq. (17), we have

(19)
$$T=-\frac{1}{2m_1(s-\lambda)}\int_0^\infty[2\sum_0^s k(s-k)\Pi_k-M_1'-M_2']dt$$
$$=\left[m_1+m_2-2\int_0^\infty\sum_0^s k(s-k)\Pi_k dt\right]\Big/ 2m_1(s-\lambda)$$

In order to calculate the integral term in the above equation, from Eq. (15) we have (in case $k<s$)

(20)
$$\Pi_k=\frac{1}{k}\sum_0^{k-1}\Pi_j'+\frac{\lambda}{k}\Pi_{k-1}=\frac{1}{k}\sum_0^{k-1}\Pi_j'+\frac{\lambda}{k}\left(\frac{1}{k-1}\sum_0^{k-2}\Pi_j'+\frac{\lambda}{k-1}\Pi_{k-2}\right)$$
$$=\frac{1}{k}\left[\Pi'_{k-1}+\left(1+\frac{\lambda}{k-1}\right)\Pi'_{k-2}+\cdots+\left(1+\frac{\lambda}{k-1}+\frac{\lambda^2}{(k-1)(k-2)}+\cdots\right.\right.$$
$$\left.\left.+\frac{\lambda^{k-1}}{(k-1)!}\right)\Pi_0'\right]+\frac{\lambda^k}{k!}\Pi_0$$

Again, with Eq. (4), (5), we have the representation

(21)
$$\Pi_k=P_k\Pi_0(t)/P_0+(P_k/\lambda)\sum_{j=0}^{k-1}\Pi_j'\left(\sum_{l=j}^{k-1}\frac{1}{P_l}\right)$$

However, in Davis [1], the corresponding formula is expressed by

$$\Pi_k=P_k\Pi_0(t)/P_0+(P_k/\lambda)\sum_0^{k-1}\Pi_j'/P_j$$

This is an obvious mistake.
Here we can write that

(22)
$$\int_0^\infty\left[\sum_0^s k(s-k)\Pi_k\right]dt=\left[\sum_0^s k(s-k)P_k/P_0\right]\int_0^\infty\Pi_0(t)dt$$
$$+\frac{1}{\lambda}\sum_0^s k(s-k)P_k\sum_0^{k-1}\int_0^\infty\Pi_j'\sum_{l=j}^{k-1}\frac{1}{P_l}dt$$

From Eq. (7), the first term on the right-hand side of (22) is equal to

(23)
$$P_0^{-1}[(s-\lambda)m_1-\lambda]\int_0^\infty\Pi_0(t)dt$$

The second term can be written as follows.

(24)
$$\lambda^{-1}\sum_0^s k(s-k)P_k\sum_{j=0}^{k-1}(P_j-P_j(0))\left(\frac{1}{P_j}+\frac{1}{P_{j+1}}+\cdots+\frac{1}{P_{k-1}}\right)$$
$$=\lambda^{-1}\sum_0^s k(s-k)A(k)$$

where $A(k)$ is defined as follows.

(25)
$$A(k)=P_k\left[\sum_{j=0}^{k-1}P_j\left(\frac{1}{P_j}+\cdots+\frac{1}{P_{k-1}}\right)-\left(\frac{1}{P_0}+\frac{1}{P_1}+\cdots+\frac{1}{P_{k-1}}\right)\right]$$

We have yet to calculate the integral of $\Pi_0(t)$.
From Eq. (17), we have

(17)′
$$M_1'(t)=\sum_1^s(s-k)\Pi_k+s\Pi_0$$

Integrating both sides of Eq. (17)′, we are led to

(26)
$$m_1=\sum_1^s(s-k)[P_k/P_0\int_0^\infty\Pi_0(t)dt+\lambda^{-1}A(k)]+s\int_0^\infty\Pi_0(t)dt$$
$$=P_0^{-1}(\sum_1^s(s-k)P_k)\int_0^\infty\Pi_0(t)dt+\lambda^{-1}\sum_1^s(s-k)A(k)+s\int_0^\infty\Pi_0(t)dt$$

Using Eq. (6), we have

(27)
$$m_1=\frac{s-\lambda}{P_0}\int_0^\infty\Pi_0(t)dt+\lambda^{-1}\sum_0^s(s-k)A(k)$$

This gives us the relation

(28)
$$\int_0^\infty\Pi_0(t)dt=\frac{P_0}{s-\lambda}\cdot[m_1-\lambda^{-1}\sum_1^s(s-k)A(k)]$$

If we combine Eqs. (22), (23), (24) and (28), we obtain

(29)
$$\int_0^\infty\sum_0^s k(s-k)\Pi_k dt=[(s-\lambda)m_1-\lambda][m_1-\lambda^{-1}\sum_1^s(s-k)A(k)]/s-\lambda$$
$$+\lambda^{-1}\sum_0^s k(s-k)A(k)$$

hence, finally,

(30)
$$T=\Big[m_1+m_2-\frac{2[(s-\lambda)m_1-\lambda][m_1-\lambda^{-1}\sum_1^s(s-k)A(k)]}{s-\lambda}$$
$$+\lambda^{-1}\sum_0^s k(s-k)A(k)\Big]\Big/2m_1(s-\lambda)$$
$$=\Big[-2m_1{}^2+\frac{s+\lambda}{s-\lambda}m_1+m_2$$
$$-2\lambda^{-1}\sum_1^s A(k)(s-k)\Big(k-m_1+\frac{\lambda}{s-\lambda}\Big)\Big]\Big/2m_1(s-\lambda)$$

§3. A TABLE OF T

Using the above formula (30), we can tabulate the numerical values of T. Here, we shall show a table of T for $\lambda=0.1(0.1)$ 0.9 (0.01) 0.99 and $s=1$ (1) 5.

λ \ s	1	2	3	4	5
0.1	1.23	1.01	1.00	1.00	1.00
0.2	1.56	1.02	1.00	1.00	1.00
0.3	2.04	1.05	1.00	1.00	1.00
0.4	2.78	1.09	1.01	1.00	1.00
0.5	4.00	1.16	1.01	1.00	1.00
0.6	6.25	1.24	1.02	1.00	1.00
0.7	11.11	1.35	1.04	1.00	1.00
0.8	25.00	1.51	1.06	1.01	1.00
0.9	100.00	1.72	1.08	1.01	1.00
0.91	123.45	1.74	1.08	1.01	1.00
0.92	156.25	1.77	1.09	1.01	1.00
0.93	204.08	1.79	1.09	1.01	1.00
0.94	277.78	1.82	1.09	1.01	1.00
0.95	400.00	1.85	1.10	1.01	1.00
0.96	625.00	1.88	1.10	1.01	1.00
0.97	1111.11	1.90	1.10	1.01	1.00
0.98	2500.00	1.94	1.11	1.02	1.00
0.99	10000.00	1.97	1.11	1.02	1.00

§4. DISCUSSION

We, first, consider the case of infinite number of channels ($s=\infty$). In this case, the expected number waiting in the transient state if the initial state is 0 is:

$$M_1(t)=\frac{\lambda}{\mu}(1-e^{-\mu t})$$

Thus, if $s=\infty$ and $\mu=1$, we get

$$T=\frac{\int_0^\infty t dM_1(t)}{\int_0^\infty dM_1(t)}=\frac{\int_0^\infty \lambda t e^{-t}dt}{\lambda}=1$$

Now, refered to the table, build-up time T converges 1 rapidly. So it is found that we, in practice, can consider as $T=1$ for $s\geqq 3$ and any values of λ.

REFERENCES

1. Davis, H.: The build up time of waiting lines, Naval Res. Log Qaut. 7, (1960) 185–193
2. Morimura, H.: The build up time of equilibrium wating time, J. Operations Research Soc. Japan, Vol. 4, 76–86