A THEORY OF MULTIPLE PREDICTION

JIRO KONDO

Dept. of Aero/Astro Eng., University of Tokyo

1. Introduction

Only few problems in operations research are not related with the prediction technique of a time series. If the situation in the future, could be determined then not only the questions of long range planning and of demand analysis but also those of inventory control, production control, maintenance and supply would be turned out into simple problems.

So far, we have two ways of prediction, based upon quite different theories. The cross section analysis is well known among economists, which is a macroscopic method. The balance or the mechanism of an economic system being expressed in some relations among several variables, the predicted values of variables representing a status of the system can be obtained by a shift of equilibrium point due to the deterministic variations of other variables depending on the circumstance which should be determined by some policy from outside of the system. Also, the input-output analysis by means of an industrial relations table will be classified in this category. A cross ssection of multiple time series is being studied.

Although the macroscopic method has been succesful for a large scale or overall prediction of economic systems, but not very much suitable for a smaller system, and it would not be effective to a particular enterprise.

On the other hand, we have another method of prediction which is based on a time series analysis in theoretical statistics. This is the way to estimate the value in the future by detecting and utilizing, the

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characteristics inherited in the time series. Wiener's theory(1)(2) is the best known among those theories belonging to this category.

Applying the autocorrelation function to predict a time series, it is to extrapolate the time series by noticing the statistical attribute of time depending variables without trying to find out the mechanism of the system itself. This is the elegant method, established on a strict mathematical theory and well being recognized that the method is very powerful to predict natural phenomena, but it has no space to include theories in econometrics.

Nevertheless, since the variables expressing a status of an enterprise are related to each other, corresponding to the mechanism of the economic situation, we could have a better prediction if we were able to take into account of these correlations together with the time varying characteristics of figures representing the status. This is an effort to bridge the macroscopic theory of econometrics with the microscopic time series analysis of mathematical statistics.

The present writer will study the method of prediction of one variable by means of various information sources, as an extension of Wiener's theory. Then he will introduce the fundamental relation of prediction matrix, for a multiple prediction of m timeseries by means of n informations. The fundamental relation yields $m \times n$ simultaneous integral equations for $m \times n$ predictors. This is exactly an extension of the above method and hence of Wiener's theory. The formal solution of the equations could be obtained by means of Wiener-Hopf technique, but it seems hard to carry out in practice. Sometimes when the mechanism of the economic systems are known, and not complicated, then the predictor matrix could be simplified and hence the method will be feasible. If on the other hand an effective hysteresis duration time of a time series is limited, we could confine the interval of integrations in the fundamental equations and the solution can be carried out on a computor. But the determination of this effective period from the actual time series is not a simple problem, a method will be proposed and examined for a model case.

Still many problems are being remained unsolved, but efforts should be concentrated to get a feasible means of an effective multiple prediction of time series.

2. An Extension of Wiener's Prediction Theory for a Multiple Time Series.

The future value of a time series vector $\overrightarrow{y(t)}[y_1(t), y_2(t), \ldots, y_m(t)]$ is predicted from the information vector $\overrightarrow{x(t)}[x_1(t), x_2(t), \ldots, x_n(t)]$ through

$$\vec{y}(t+\alpha)=\vec{x}(t)[K] \tag{2-1}$$

where $[K]$ is a matrix composed by linear predictors $K_{ij}(\tau)$.

The equation (2-1) turns out to

$$y_j(t+\alpha)=\int_0^\infty x_1(t-\tau)dK_{1j}(\tau)+\int_0^\infty x_2(t-\tau)dK_{2j}(\tau)+ \cdots +\int_0^\infty x_n(t-\tau)dK_{nj}(\tau) \tag{2-2}$$

$$(j=1, 2, \ldots, m)$$

This equation indicates that we separate the matrix $[K_{ij}]$ by row and the mechanism of the multiple prediction shown on figure 2 can be decomposed into the one in figure 3.

The problem of the determination of K_{ij} reduced to a variational problem of a functional

$$I[K_{1j}(t), K_{2j}(t), \ldots, K_{nj}(t)] = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\left\{y_j(t+\alpha)-\sum_{i=1}^{n}\int_0^\infty x_i(t-\tau)dK_{ij}(\tau)\right\}^2 dt: \tag{2-3}$$

$$(j=1, 2, \ldots, m)$$

Introducing auto-correlation functions and cross-correlation functions

$$\left.\begin{aligned}
\varphi_{ij}(t)&=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}x_i(t+\tau)\overline{x_j(t)}dt, \quad \begin{matrix} i \\ j \end{matrix}=1,\ 2,\ \dots,\ n\ ; \\
\rho_{ij}(t)&=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}y_i(t+\tau)\overline{x_j(t)}dt, \quad \begin{matrix} i=1,\ 2,\ \dots,\ m; \\ j=1,\ 2,\ \dots,\ n\ ; \end{matrix} \\
\rho_{ii}(t)&=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}y_i(t+\tau)\overline{y_i(t)}dt, \quad i=1,\ 2,\ \dots,\ m;
\end{aligned}\right\} \tag{2-4}$$

we have

$$\begin{aligned}
I[K_{1j}(t),\ K_{2j}(t),\ \dots,\ K_{nj}(t)]&=\rho_{jj}(0)-2\sum_{i=1}^{n}\rho_{ij}(\alpha+\tau) \\
dK_{ij}(\tau)&+2\sum_{i=1}^{n}\sum_{h=1}^{m}\int_0^\infty K_{ij}(\tau)\int_0^\infty\varphi_{ij}(\tau-\sigma)dK_{hj}(\sigma)
\end{aligned} \tag{2-5}$$

And the necessary and sufficient condition for the variational problem yields to

$$\rho_{ij}(\alpha+\tau)-\sum_{h=1}^{n}\int_0^\infty\varphi_{ih}(\tau-\sigma)dK_{hj}(\sigma)=0, \quad \begin{pmatrix} i=1,\ 2,\ \dots,\ n \\ j=1,\ 2,\ \dots,\ m \end{pmatrix} \tag{2-6}$$

We shall introduce $f_i(\tau)$ as

$$\begin{aligned}
f_i(\tau)&\equiv-\rho_i(\tau+\alpha)+\sum_h\int_{-\infty}^{\infty}\varphi_{ih}(\tau-\sigma)dK_h, \\
&\qquad\qquad (i=1,\ 2,\ \dots,\ n) \\
f_i(\tau)&=0, \quad \text{for} \quad \tau\geqq 0.
\end{aligned} \tag{2-7}$$

and assume that there exist Fourier transforms such as

$$\left.\begin{aligned}
F_i(\omega)&=\int_{-\infty}^{\infty}f_i(t)e^{-j\omega t}dt, \\
\pi G_i(\omega)&=\int_{-\infty}^{\infty}\rho_i(t)e^{-j\omega t}dt, \\
G_{ih}(\omega)&=\int_{-\infty}^{\infty}\varphi_{ih}(t)e^{-j\omega t}dt, \\
Y_h(j\omega)&=\int_{-\infty}^{\infty}K_h(t)e^{-j\omega t}dt,
\end{aligned}\right\} \tag{2-8}$$

where, $j=\sqrt{-1}$,

then (2-7) yields

$$\frac{1}{\pi}F_i(\omega)=\sum_h G_{ih}(\omega)Y_h(j\omega)-e^{j\omega\alpha}G_i(\omega) \qquad (i=1,\ 2,\ \dots,\ n) \tag{2-9}$$

and $F_i(\omega)$ is analytic and bounded in the upper half of the ω-plane. This is a simultaneous equation for $Y_h(j\omega)$ and the solution may be obtained by the undetermined coefficients. [1]

3. Determination of the Influence Period

One of the difficulties involved in the Wiener's prediction theory is that we should take into account of the whole history of the information time series, that is to say, the regular part of the integral equation for the predictor has an integral from zero to infinity, instead of a finite range.

On the other hand, the period for which the information obtained, are usually limited in many practical cases, we can not extend the integration limit to infinity.

We shall call the period of hereditary or time series upon the other one.

It is not easy to find out this influence period for a perticular time series. It will be difined as

$$I[k(\tau),\ T]=\lim_{T\to\infty}\frac{1}{2T}\int_{-T'}^{T'}\left\{f(t+\alpha)-\int_0^T f(t-\sigma)K_\tau(\sigma)d\sigma\right\}^2 dt \tag{3-1}$$

where I is a functional of $K(t)$ and a function of T.

We have

$$\varphi(t+\alpha)=\int_0^T \varphi(\tau-\alpha)K_T(\sigma)d\sigma \tag{3-2}$$

by taking the first variation of I and the optimum length of the period

T^* might be determined by the condition

$$\frac{dI}{dT}=0 \tag{3-3}$$

The original conjecture of the present author was the curve A in figure 4, but he found that the curve $I(T)$ was like B for some cases, and there were no minimum point of $I(T)$ in the finite range of T.

For the most general case we have

$$\frac{dI}{dT}=0 \quad \text{for} \quad T=\infty \tag{3-4}$$

that seems to be reasonable in one sence, since, the better prediction should be obtained, if the more informations are available. However the risk of including unexplainable relation (or meaningless statistical correlations) shall be increased when a longer influence time is taken.

The author still believe that the optimum value T^* can be determined if we have a curve like C for some particular time series although it would be a tiresome computation to carry out $I(T)$ for several values of T since the predictor $K_T(\tau)$ will be influenced by the assumption of T.

For almost all practical cases the influence period will be limited by the length of the available statistical data or be determined by some transcendental consideration and sometimes it can be fixed.

For an extrapolation of the balance sheet of certain company, the influence range was fixed as 3 years and we have obtained some proper results.

4. Prediction theory for a Multiple Time Series when Periods of Influence are Limited.

When the influence periods are limited, we have the prediction formula for stationary time series

$$y(t+\alpha)=\sum_{i=1}^{n}\int_0^{T_i} x_i(t-\tau)dK_i(\tau) \tag{4-1}$$

instead of (2–1)° where the influence time of $x_i(t)$ on $y(t)$ are assumed to be known.

The equations for $K_i(\tau)$ are introduced by the same principle as one used in section 2, that is the principle of least squeres

$$I[K_1(t),\ K_2(t),\ \dots,\ K_n(t)]=\lim_{T'\to\infty}\frac{1}{2T'}\int_{-T'}^{T'}\Big\{y(t+\alpha)-\sum_{i=1}^{n}\int_0^{T_i}x_i(t-\tau)dK_i\Big\}^2 dt : \min \tag{4-2}$$

The necessary and sufficient condition for (4–2) turns out

$$\rho_i(\tau+\alpha)=\sum_{j=1}^{n}\int_0^{T_i}\varphi_{ij}(\tau-\sigma)dk_j(\sigma)\quad \tau\geqq 0\ (i=1,\ 2,\ \dots,\ m)$$

where ρ_i and φ_{ij} is the auto-correlation and cross-correlation function.

If we assume that

$$\left.\begin{aligned}\rho_i(\tau)&=\sum_{k=1}^{N_i}\phi_{ik}e^{-\gamma_{ik}\tau}\\ \varphi_{ij}(\tau)&=\sum_{k=1}^{N_{ij}}\phi_{ijk}e^{-\gamma_{ijk}\tau}\end{aligned}\right\}\tag{4-4}$$

$$(i,\ j=1,\ 2,\ \dots,\ n)$$

then we have

$$K_i(\tau)=\sum_{q=1}^{l_i}(-1)^{q-1}[a_{iq}^{(1)}\delta^{(q-1)}(\tau)+b_{iq}^{(1)}\delta^{(q-1)}(T-\tau)]+\sum_{r=1}^{m_i}[a_{ir}^{(2)}e^{-\alpha_{ir}\tau}+b_{ir}^{(2)}e^{-\alpha_{ir}(T-\tau)}]+K_{iq}(\tau)\quad \begin{pmatrix}0\leqq\tau\leqq T\\ i=1,2,\dots,n\end{pmatrix}\tag{4-5}$$

where, T is the maximum value of T_i,

$a_{iq}^{(1)}$, $b_{iq}^{(1)}$, $a_{ir}^{(2)}$, and $b_{ir}^{(2)}$ are solution of simultaneous equations

$$\left.\begin{aligned} &\sum_{q=1}^{l_j} a_{jq}^{(1)} \gamma_{ijk}^{q-1} + \sum_{r=1}^{m_j} \left(\frac{a_{jr}^{(2)}}{a_{jr}+\gamma_{ijk}} - \frac{b_{jr}^{(2)} e^{-\alpha_{jr}T}}{\alpha_{jr}+\gamma_{ijk}} \right) = c_{ijk}, \\ &\sum_{q=1}^{l_j} b_{iq}^{(1)} \gamma_{ijk}^{q-1} + \sum_{r=1}^{m_j} \left(\frac{b_{jr}^{(2)}}{\alpha_{jr}+\gamma_{ijk}} - \frac{a_{jr}^{(2)} e^{-\alpha_{jr}T}}{\alpha_{jr}+\gamma_{ijk}} \right) = d_{ijk} \end{aligned}\right\} \quad \begin{pmatrix} i,\ j=1, 2, \dots, n \\ k=1, 2, \dots, N_{ij} \end{pmatrix} \tag{4-6}$$

c_{ijk} and d_{ijk} are given by

$$\left.\begin{aligned} c_{ijk} &= \frac{\sum_{j'=1}^{n} P_{Mij'} [K_{j'p}(\sigma),\ e^{\gamma_{ijk}\sigma}] - P_{Li}[\rho_i(\sigma+\alpha), e^{\gamma_{ijk}\sigma}]}{2\gamma_{ijk}\phi_{ijk} \prod_{u=1}^{n(j)} \prod_{v=1}^{N_{ij(k)}} (\gamma^2_{ijk}-\gamma^2_{iuv})} \Bigg|_{\sigma=0} \\ d_{ijk} &= \frac{P_{Li}[\rho_i(\sigma+\alpha),\ e^{-\gamma_{ijk}(\sigma-T)}] - \sum_{j'=1}^{n} P_{Mij'}[K_{j'p}(\sigma),\ e^{-\gamma_{ijk}(\sigma-T)}]}{2\gamma_{ijk}\phi_{ijk} \prod_{u=1}^{n(j)} \prod_{v=1}^{N_{ij(k)}} (\gamma^2_{ijk}-\gamma^2_{juv})} \Bigg|_{\sigma=T} \end{aligned}\right. \tag{4-7}$$

where P_{Li} and $P_{Mij'}$ are the bilinear concomitants, and $\prod_{u=1}^{n(j)}$ indicates that u takes on all integer values from 1 to n with the exception of the integer j. [5] [See Appendix (I)]

5. The Extrapolation of Discrete Time Series.

We shall predict the future value $Y_{i+\alpha}(\alpha>0)$ for a discrete time series Y_i from informations $X_{jr}(j=1, 2, \dots, n;\ r=i-s_j, \dots, i)$.

(a) The length T_j of the influence period are not necessarily assumed to be equal for each time series X_j.

We shall introduce coefficients of prediction K_{jr} so that the mean square

$$\sigma^2 \equiv \frac{1}{m} \sum_{i=1}^{m} (Y_{i+\alpha} - \sum_{j=1}^{n} \sum_{r=0}^{s_j} X_{ji-r}K_{jr})^2, \quad i>r \tag{5-1}$$

takes the minimum value.

By means of the conditions,

$$\frac{\partial \sigma^2}{\partial K_{jr}}=0,$$

we have

$$\sum_{j=1}^{n}\sum_{r=0}^{s_j}\left(\sum_{r=0}^{m} X_{ji-r}X_{j'i-r'}\right)K_{jr}=\sum_{i=1}^{m} X_{j'i-r'}Y_{i+\alpha} \tag{5-2}$$

$$\begin{pmatrix} i>r & r'=0,\ 1,\ 2,\ \dots,\ s'_j \\ i'>r' & j'=1,\ 2,\ \dots,\ n \end{pmatrix}$$

which turn out to a simultaneous linear equation for K_{jr}. We can determine the coefficients of prediction.

As we have

$$\sigma^2[K+\varepsilon M]-\sigma^2[K]=\frac{1}{m}\sum_{i=1}^{m}\left(\sum_j\sum_r X_{ji-r}M_{jr}\varepsilon_{jr}\right)^2 \tag{5-3}$$

We know that

$$\begin{aligned}&\sigma^2[K_{10}+\varepsilon_{10}M_{10},\ K_{11}+\varepsilon_{11}M_{11},\ \dots,\ k_{nSn}+\varepsilon_{nSn}M_{nSn}]\\ &-\sigma^2[K_{10},\ K_{11},\ \dots,\ K_{nSn}]\geqq 0\end{aligned} \tag{5-4}$$

indicating that (5-2) is the sufficient condition for (5-1) should take the minimum value.

(b) We define

$$\rho_{ji}=\frac{1}{m}\sum_{h=1}^{m} Y_{i+h}Y_{jh}$$

$$\varphi_{jki}=\frac{1}{m}\sum_{h=1}^{m} X_{ji+h}X_{kh} \tag{5-5}$$

$$\text{assuming that}\quad \overline{Y}=\overline{X}=0 \tag{5-6}$$

then we define

$$\rho_{ji+\alpha}=\sum_{h=1}^{n}\sum_{\sigma=1}^{s}\varphi_{jhi-\sigma}K_{h\sigma} \tag{5-7}$$

Again simultaneous equation for $K_{k\sigma}$ is obtained.

When the assumptions (5–6) of zero mean hold, then the (5–7) coincide with (5–2). For many practical cases where (5–6) does not hold, we should make use of the first method which is more general than the second one.

As already mentioned in section 3, an application of the first method to predict the balance sheet for certain company has been carried out. In this case the effective influence period is fixed as 3 years prior and reasonable values are obtained.

6. Prediction of a Time Series by Means of Predicted Values of Other Series.

Now, we shall study the method of an economic system. The status of a system is expressed by variables $y_1(t)$, $y_2(t)$, $y_3(t)$.... By the investigation of the system, which is sometimes called as a cross-section analysis, we suppose that the relationship among variables has been established.

The situation or environment of the system are expressed by the other variable $x(t)$ and the dependence of the inner variables $y_i(t)$ on $x(t)$ are assumed to be known and $x(t)$ is sometimes called as input or output.

If the future value of $x(t+\alpha)$ is prescribed, then $y(t+\alpha)$ can be determined, by use of the relation between x and y. But if an exraordinary value of $x(t+\alpha)$ were assigned, then it might have some effects on the economic system. Therefore the properties of growth of $y_i(t)$ should be taken into account.

Here again, we come to the case of multiple prediction explained in section 2.

We have

$$y_i(t+\alpha)=f(x)+\sum\int_0^{T_i} y_j(t-\sigma)K_{ij}(\sigma)d\sigma \tag{6-1}$$

where some of K vanish in many cases.

Suppose, there is a relation, like

$$y(t)=cx(t)+\varepsilon(t)$$

then we have

$$y(t+\alpha)=c\int_0^\infty x(t-\sigma)dK_1(\alpha)+\int_0^\infty x(t-\sigma)dK_2(\sigma)$$

then the equations for K_1 and K_2 are

$$\left.\begin{aligned}&\varphi(\tau-\alpha)=c\int_0^\infty \rho(\tau-\sigma)dK_1(\sigma)\,,\\ \text{and}\quad&\\ &\rho(\tau+\alpha)=c\int_0^\infty \rho(\tau-\sigma)dK_2(\sigma)\end{aligned}\right\}\qquad(6\text{-}2)$$

where ρ is an autocorrelation function of $x(t)$ and φ is the cross correlation function of $x(t)$ and $y(t)$.

7. An Application to Macroscopic Economic Systems.

As an example of the preceeding section, we shall examine the prediction of the national economic system.

By the analysis of the mechanism, we have

$$\left.\begin{aligned}&V=yV\\ &C=cyV\\ &M=myV\\ &S=syV\\ &I=syV+A\\ &X=(y-1)V+A\\ &B=(1-cy-sy)V-A\end{aligned}\right\}\qquad(7\text{-}1)$$

Hence Y is not influenced by X and depends on V therefore

$$Y(t+\alpha)=y\,[\int_0^T V(t-\alpha)K_{YV}(\sigma)d\sigma]+\int_0^T Y(t-\sigma)K_{YY}(\sigma)d\sigma \tag{7-2}$$

We can introduce the equation for K_{YY} and K.
On the other hand

$$X(t+\alpha)=(y-1)[\int_0^T V(t-\sigma)K_{XV}(\sigma)d\sigma] + \int_0^T X(t-\sigma)K_{YXX}(\sigma)d\sigma+A(t+\alpha) \tag{7-3}$$

where K_{SS} and K_{VV} can be determined.
For $S(t+\alpha)$ we have

$$S(t+\alpha)=sy\,[\int_0^T V(t-\sigma)K_{SV}(\sigma)d\sigma]+\int_0^T Y(t-\sigma)K_{SS}(\sigma)d\sigma \tag{7-4}$$

In the similar way, by means of the equation (7-1), we can simplify the equation of the multiple prediction.

Detailed investigations are still going on and the results will be reported on the other chance.

Conclusions and Aknowledgements

Several methods of prediction of a multiple time series are investigated. This is the efforts of combining the statistical analysis with the econometrics theory to get the proper prediction, based upon a better understanding of economic systems.

(1) Wiener's theory on stationary time series is extended to include the prediction of multiple time series. Simulteneous integral equations for predictor matrix are introduced which can be solved by means of Wiener-Hopf technique.

(2) Prediction problems for finite influence time are solved analytically.

(3) The method for determining the influence time for the series are investigated.

(4) The methods of multiple prediction for discrete time series

are obtained and examined. A practical application has been investigated.
(5) A method for prediction of the future status of an economic system is introduced and an application for a macroscopic analysis is studied.

Still some problems are remaining unsolved and the research should be continued to overcome these difficulties.

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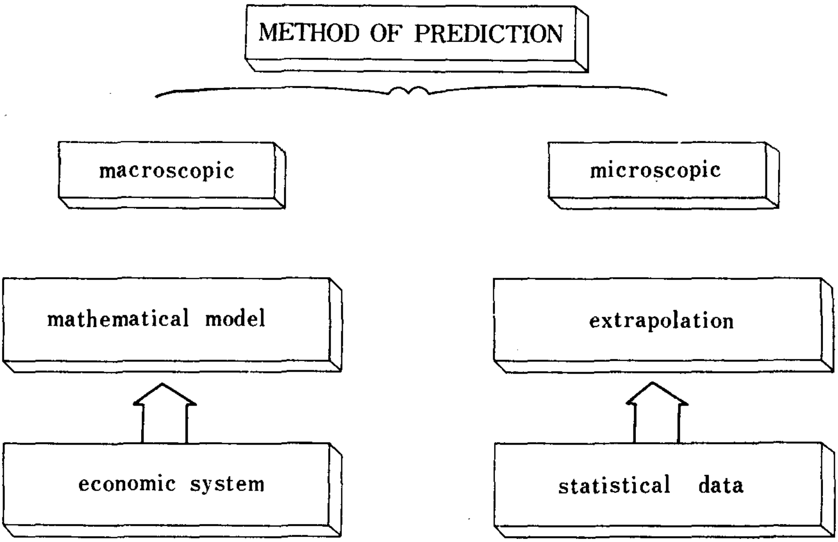
METHOD OF PREDICTION
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mathematical model
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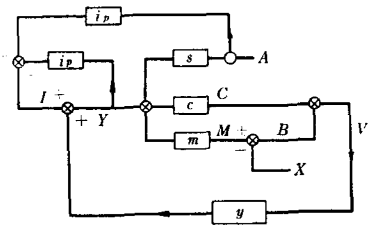
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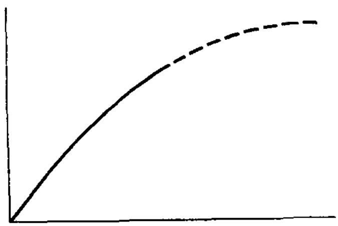
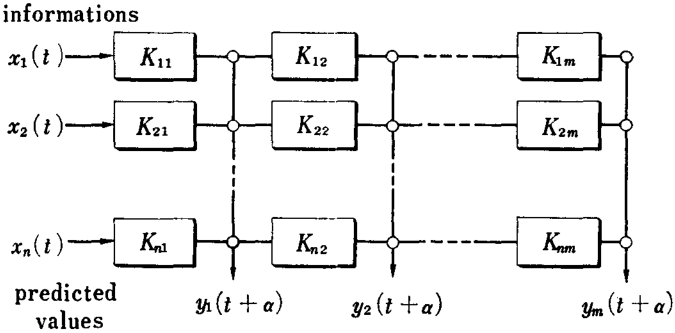
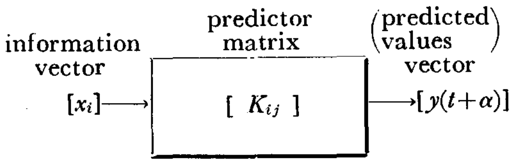
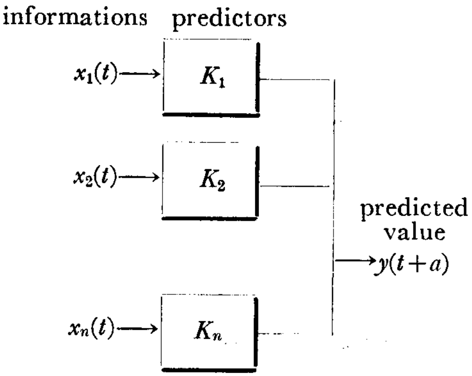






$$\rho_{ij}(\alpha+\tau)-\sum_{h=1}^{n}\int_0^\infty \varphi_{ih}(\tau\sigma)dK_{hj}(\sigma)=0, \qquad \begin{pmatrix} i=1,\ 2,\ \dots,\ n\\ j=1,\ 2,\ \dots,\ m\end{pmatrix}$$

$$\left.\begin{aligned}
&\varphi_{ij}(t)=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}x_i(t+\tau)\overline{x_j(t)}dt, && \left.\begin{matrix} i\\ j\end{matrix}\right\}=1,\ 2,\ \dots,\ n\ ;\\
&\rho_{ij}(t)=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}y_i(t+\tau)\overline{x_j(t)}dt, && \begin{matrix} i=1,\ 2,\ \dots,\ m\ ;\\ j=1,\ 2,\ \dots,\ n\ ;\end{matrix}\\
&\rho_{ii}(t)=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}y_i(t+\tau)\overline{y_i(t)}dt, && i=1,\ 2,\ \dots,\ m
\end{aligned}\right\}$$



$$y(t+\alpha)=\int_0^\infty x_1(t-\tau)dK_1(\tau)+\int_0^\infty x_2(t-\tau)dK_2(\tau)+\dots$$
$$+\int_0^\infty x_n(t-\tau)dK_n(\tau)$$

$$p_i(\tau+\alpha)=\sum_j\int_0^\infty \varphi_{ij}(\tau-\sigma)dK_j(\sigma), \qquad \tau\geqq 0$$
$$(i,\ j=1,\ 2,\ \dots,\ n)$$

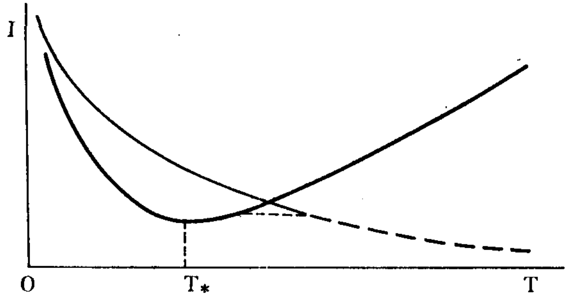
$$\left.\begin{aligned}\varphi_{ij}(t)&=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}x_i(t+\tau)x_j(t)dt,\\ \rho_i(t)&=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}y(t+\tau)x_i(t)dt,\\ \rho_0(t)&=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}y(t+\tau)y(t)dt\end{aligned}\right\}$$

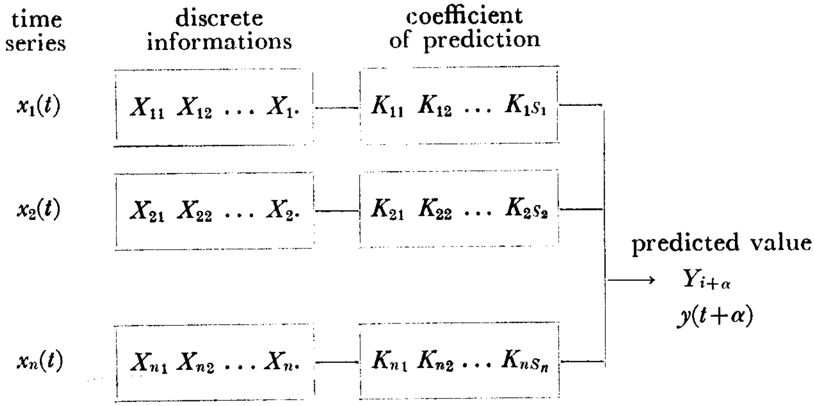
Determination of Influence Time

$$I[K(\tau),T]=\lim_{T'\to\infty}\frac{1}{2T'}\int_{T'}^{T'}\left\{f(t+\alpha)-\int_0^T f(t-\tau)\,dK(\tau)\right\}^2dt$$

$$\varphi(\tau+\alpha)=\int_0^T\varphi(\tau-\sigma)dK_T(\sigma) \qquad \tau\geqq 0$$

$$\frac{\partial I}{\partial T}=0 \qquad \longrightarrow \qquad T^*$$





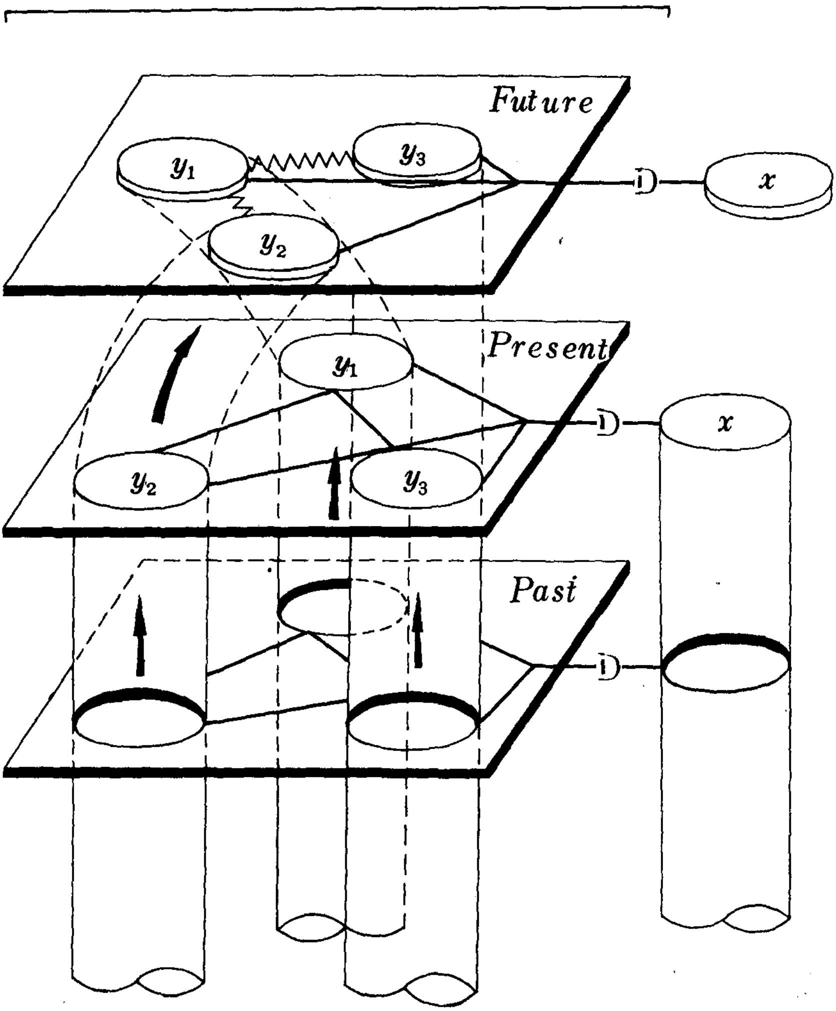
$$\sigma^2 \equiv \frac{1}{m}\sum_{i=1}^{m}\left(Y_{i+\alpha}-\sum_{j=1}^{n}\sum_{r=0}^{Sj} X_{j\cdot i-r}K_{jr}\right)^2$$

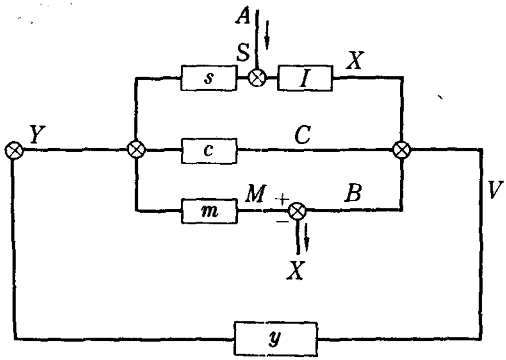
$$\sum_{j=1}^{n}\sum_{r=0}^{Sj}\left(\sum_{i=1}^{m} X_{j\cdot i-r}\ X_{j'\cdot i-r'}\right)K_{jr}=\sum_{i=1}^{m} X_{j'\cdot i-r'}\ Y_{i+\alpha}$$

$$\begin{cases} i>r,\ r' \\ r'=0,\ 1,\ 2,\ \dots,\ S_{j'} \\ j'=1,\ 2,\ \dots,\ n \end{cases}$$

$$\rho_{j\cdot i+\alpha}-\sum_{h=1}^{n}\sum_{r=0}^{Sk}\varphi_{jh\cdot i-r}\ K_{nr}=0$$

inside of an economic systems





A Outside Investment

B Balance

C Consumption

I Investment

M Manufacturing for Exportation

S Saving

X Importation

Y Gross Income

V Gross National Production