ON SOME PREVENTIVE MAINTENANCE POLICIES

HAJIME MAKABE and HIDENORI MORIMURA

Yokohama National University and Tokyo Institute of Technology
(Received September, 6, 1963)

SUMMARY

Continuing to the last paper [5], some discussions on Policy III and its ramifications are dealt with. Usefulness of Policy III will be shown from various points of view.

§ 1. INTRODUCTION

In the last paper [5], we proposed a new preventive maintenance policy and named it as Policy III. This policy may be used to maintain a complex system consists of many equipments of same type. These objects are same to that of Policy II proposed by R. Barlow and L. Hunter [2].

For convenience sake to proceed our discussion, we shall describe the definitions and main results of these policies.

DEFINITIONS OF PREVENTIVE MAINTENANCE POLICIES

Policy I: Perform preventive maintenance after t_0 hours of continuing operation time without failure. $(0<t_0\leqq\infty)$ If the system fails before t_0 hours have elapsed, perform maintenance at the time of failure. Preventive maintenance at the time is rescheduled. For this policy, we assume that the system is as good as new after any time of maintenance (or replacement) is performed.

Policy II: Perform preventive maintenance on the system after it has been operating a total of t^* hours regardless of the number of intervening failures. $(0<t^*\leqq\infty)$ We assume that after each failure only minimal repair is made and that the system failure rate [see NOTATION in below] is no disturbed after performing minimal repair.

Policy III: Perform preventive maintenance (that is "overhaul") at k-th failure of the system, but for first $(k-1)$ times of it, perform minimal repairs only. If we overhaul the system, we can regard as the system is replaced by a new system and the failure rate is not effected by any minimal repair. This circumstance is same as Policy II, but in this case the life time distributions of replacing systems may not necessarily be identical.

NOTATIONS AND ASSUMPTION

In below, we consider some large systems consisted of a number of equipments of same type. If we overhaul the system, it can be seen that the system is renewal. Thus, we shall consider that the system is replaced by a new system with a specified life time distribution function at the time epoch when the overhaul completes. We shall assume that the mean life time are $\mu_1, \mu_2, \ldots, \mu_M$ which appear with probabilities $p_1, p_2, \ldots, p_M$, respectively. Of course, we assume that all μ's are positive and finite.

We shall denote the life time distribution of i-th system as $F_i(x)$, and its density function as $f_i(x)$. Of course, $F_i(x) \equiv f_i(x) \equiv 0$ for all $x<0$.

Put

$$q_i(x)=\frac{f_i(x)}{1-F_i(x)} \qquad \text{[failure rate]}, \qquad (1.1)$$

$$\mu_i=\int_0^\infty x dF_i(x) \qquad \text{[mean life time]}, \qquad (1.2)$$

$$\mu=\sum_{i=1}^{M} p_i \mu_i \qquad \text{[average mean life time]}, \qquad (1.3)$$

and denote as

T_m: the mean minimal repair time,

T_s: the mean maintenance time for an overhaul.

MAIN RESULTS

1. *Limiting Efficiencies in the Above Circumstances*

Policy II;

$$Eff_{\infty}^{(2)}=\frac{t^*}{t^*+T_m\sum_{i=1}^{M}p_i\int_0^{t^*}q_i(t)dt+(T_s-T_m)} \tag{1.4}$$

Policy III:

$$Eff_{\infty}^{(3)}=\frac{\overline{U}(k)}{\overline{U}(k)+(k-1)T_m+T_s} \tag{1.5}$$

where

$$\overline{U}(k)=\sum_{i=1}^{M}p_iE(U_i^{(k)}) \tag{1.6}$$

and $U_i^{(k)}$ is the total operating time (without any repair time) of the system having $F_i(x)$ as its life time distribution.

2. *Optimal Policy of Type* III *Which Maximize the Limiting Efficiency*

General form:

The optimal policy is to overhaul the system at k_0-th failure which is given by unity plus the largest integer such as

$$\varphi(k)\overset{(\text{def})}{=}\left(1-\frac{\overline{U}(k)}{\overline{U}(k+1)}\right)\left(k+\frac{T_s}{T_m}\right)>1 \tag{1.7}$$

A special case: (Weibull type distribution of life time)

If the life time distribution of systems are given by Weibull distributions with identical shape parameter, then opimal k_0 is given by

$$\left.\begin{array}{ll} k_0=\left[\frac{1}{\beta-1}\frac{T_s-T_m}{T_m}\right]+1\ , & \beta>1,\quad T_s>T_m \\ k_0=1 & ,\ \beta>1,\quad T_s\leqq T_m \\ k_0=\infty & ,\ \beta=1,\quad T_s\geqq T_m \\ k_0=1 & ,\ \beta=1,\quad T_s<T_m \\ k_0=\infty & ,\ \beta<1 \end{array}\right\} \tag{1.8}$$

3. *Merits of Policy* III

(i) The limiting efficiency for Policy III may be considerd higher than the one of Policy II.

(ii) When the life time distributions are Weibull type with identical shape patameter β, the optimal policy of type III is independent of scale parameter α. Thus, we can take an policy for many systems with different scale parameters but with common shape parameters.

(iii) The practical operation for Policy III is simpler than the one for Policy II. Because, we may only count the number of times of failure for Policy III, but it is somewhat tedious to accumulate the true running times for Policy II.

(iv) Policy III has a robustness for the variation of the mean life time.

Remark 1.1 The main reason why Policy III has the higher efficiency than Policy II is that each replacement period is long or short corresponding as the life time of the system. Further, if we adopt Policy II, we shall give up the system as soon as the total running time build up to t^* hours, but under Policy III we can continue the operation till the next failure. Thus, we can get a higher efficiency. Considering the second reason, we shall improve Policy II slightly in § 5 of the present paper.

§ 2. OPTIMAL POLICY OF TYPE III IN THE SENSE OF MINIMUM MAINTENANCE COST RATE

In the above, we measured the effectiveness of mainenance policy by limiting efficiency that is operating time rate. But, sometimes, the mean minimal repair time is rather long but the cost is low compared with the ones for overhaul. In that cases, we must consider the mean cost per unit hour during a long time. In order to make our theory available in such a cases, we shall introduce the concept of cost into our models. There are some methods of introduction, but we shall consider the maintenance cost rate which is defined as follows.

Assume that a minimal repair of the system requires cost C_m and an overhaul requires cost C_s in average. An additional assumption is that any non-operating times of the system are regarded as losses which evaluated as C_0 per unit hour.

We shall measure the effectiveness of our policy by the total average cost for maintenance per unit hour (i.e., maintenance cost rate). When it is denoted as $C_\infty^{(3)}(k)$, we can see that

$$C_\infty^{(3)}(k)=\frac{(k-1)(C_0T_m+C_m)+C_0T_s+C_s}{U(k)+(k-1)T_m+T_s} \tag{2.1}$$

for Policy III. This formula is derived in [5].

Now, we shall find the opimal k to minimize $C_\infty^{(3)}(k)$ that is the largest integer $k+1$ such as

$$\Delta C_\infty^{(3)}(k)\equiv C_\infty^{(3)}(k+1)-C_\infty^{(3)}(k)<0$$

or

$$\begin{aligned} &H(k)[k(C_0T_m+C_m)+(C_0T_s+C_s)] \\ &\quad<H(k+1)[(k-1)(C_0T_m+C_m)+(C_0T_s+C_s)] \end{aligned} \tag{2.2}$$

where

$$H(k)=\overline{U}(k)+(k-1)T_m+T_s. \tag{2.3}$$

(2.2) may be rewritten as

$$\frac{k\overline{U}(k)-(k-1)\overline{U}(k+1)+T_s}{\overline{U}(k+1)-U(k)+T_m}<\frac{C_0T_s+C_s}{C_0T_m+C_m}. \tag{2.4}$$

If we put $C_0=1$ and $C_m=C_s=0$, it is checked that (2.4) consist with (1.7). In the other words, this formula (2.4) is an extension of (1.7) to the present case. Thus, at the present stage, we shall introduce the assumption of Weibull type life time distributions of the systems to proceed our further discussions as well as in [5].

Without any loss of generality, we can put $C_0=1$, and henceforth we shall assume $C_0=1$. Now, when the life time distributions of the systems are given by

$$\begin{aligned} F_i(x)&=1-e^{-\alpha_i x^\beta} \quad , \quad x>0 \\ &=0 \quad , \quad x\leqq 0, \end{aligned} \tag{2.5}$$

$\overline{U}(k)$ is expressed by

$$\overline{U}(k)=\frac{\mu\beta}{B\left(\frac{1}{\beta},\ k\right)}. \tag{2.6}$$

Hence, the left hand side of (2.4) becomes

$$\frac{\mu\beta\left[\frac{k}{B\left(\frac{1}{\beta},\ k\right)}-\frac{k-1}{B\left(\frac{1}{\beta},\ k+1\right)}\right]+T_s}{\mu\beta\left[\frac{1}{B\left(\frac{1}{\beta},\ k+1\right)}-\frac{1}{B\left(\frac{1}{\beta},\ k\right)}\right]+T_m}$$

$$=\frac{k-\frac{(k-1)\left(\frac{1}{\beta}+k\right)}{k}+\frac{T_s}{\mu\beta}B\left(\frac{1}{\beta},\ k\right)}{\left(\frac{1}{\beta k}+1-1\right)+\frac{T_m}{\mu\beta}B\left(\frac{1}{\beta},\ k\right)}$$

$$=\frac{k\beta-k+1+\frac{k}{\mu}T_sB\left(\frac{1}{\beta},\ k\right)}{1+\frac{k}{\mu}T_mB\left(\frac{1}{\beta},\ k\right)}.$$

Thus, (2.4) implies that

$$\left[(T_m+C_m)(1-\beta)+\frac{T_s+C_s-(T_m+C_m)}{k}\right]>\frac{C_mT_s-C_sT_m}{\mu}B\left(k,\ \frac{1}{\beta}\right). \tag{2.7}$$

In particular, when C's and T's have some special relations, the corresponding formulae may be deduced. These results are summed up in the following

Theorem 2.1 The optimal policy of type III to minimize the cost rate (2.1) with $C_0\equiv 1$ is that perform minimal repairs at first (k_0-1) failures and perform an overhaul at the k_0-th failure, where k_0 is unity plus the largest integer such as

$$\frac{kU(k)-(k-1)U(k+1)+T_s}{U(k+1)-U(k)+T_m}<\frac{T_s+C_s}{T_m+C_m}, \tag{2.8}$$

Theorem 2.2 When the life time distributions of the systems are Weibull type with a common shape parameter $\beta>1$, (2.8) in the above theorem will be rewritten as

$$\left[(T_m+C_m)(1-\beta)+\frac{T_s+C_s-(T_m+C_m)}{k}\right]>\frac{C_mT_s-C_sT_m}{\mu}B\left(k,\frac{1}{\beta}\right), \quad (2.7)$$

Corollary 2.1 When the life time distributions of the systems are Weibull type, the optimal k_0 of Policy III will be calculated in some special cases, i.e.,*)

(i) if $C_m=C_s=0$, then $k_0=\left[\frac{1}{\beta-1}\left(\frac{T_s}{T_m}-1\right)\right]+1$ (2.9)

(ii) if $T_m=T_s=0$, then $k_0=\left[\frac{1}{\beta-1}\left(\frac{C_s}{C_m}-1\right)\right]+1$ (2.10)

(iii) if $\frac{T_s}{T_m}=\frac{C_s}{C_m}$, then $k_0=\left[\frac{1}{\beta-1}\left(\frac{T_s'}{T_m'}-1\right)\right]+1$ (2.11)

where

$T_j'=T_j+C_j \quad (j=m \text{ or } s).$ (2.12)

Remark 2.1 To this corollary, we shall add some remarks.

(i) (2.9) consists with (1.8). This fact was also checked in the above, in a more general form.

(ii) When $T_m=T_s=0$, any repair or overhaul has no time. In this case, we must minimize the maintenance cost (rate). R. Barlow and L. Hunter [2] discussed on this case and get the relation (2.10).

(iii) In this case, we are interesting that (2.11) can be expressed in an analogous form to (2.9) or (2.10), substituting T_j' for T_j or C_j.

Remark 2.2 As illustrated in Fig. 2.1, the both sides of (2.7) are monotone decreasing with k if $T_s+C_s>T_m+C_m$. And the left hand side

---

*) $[x]$ denotes the largest integer not beyond x as usual for positive x, but it may be interpreted as zero for all negative x.

is always negative for all k if $T_s+C_s<T_m+C_m$. Since the right side will vanish for infinite k and the left side will converge to $(T_m+C_m)\ (1-\beta)$, there exists an optimal $k_0<\infty$ for $\beta>1$. More precisely, if

$$(T_s+C_s)-\beta(T_m+C_m)<\frac{C_mT_s-T_mC_s}{\mu}\beta, \tag{2.13}$$

we have $k_0=1$. For example, if C_m is sufficiently large compared with the other C's and T's, this inequality may hold. Hence, we have $k_0=1$. Of course, when the cost of a minimal repair is expensive, we must overhaul the system at each failure.

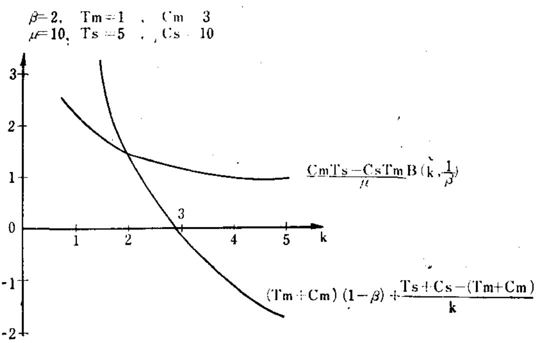

Fig. 2.1

§ 3. INTERVAL RELIABILITY FOR POLICY III

Interval reliability (or strategic reliability) is defined as the probability that the system is operating at an observed epoch and continue to operate for a preassigned mission time. This interval reliability is discussed by R. Barlow and L. Hunter [3] and A.J. Truelove [7] for the failure distribution of exponential type or the preventive maintenance policy of type I.

In this section,*) we shall firstly calculate the interval reliability

*) For the sake of simplicity we shall assume here that T_m, T_s and α not vary.

for our Policy III using the so-called "key renewal theorem," and then proceed to find the optimal policy of type III which maximize the interval reliability and add some comments on a relation with the limiting efficiency of it (discussed already in [5]).

Let $N_{00}(t)$ be the number of times of the overhaul of the system in $[0, t)$. Defining as

$$\psi_{00}(t)=E\{N_{00}(t)\} \tag{3.1}$$

we can see that in our case $\psi_{00}(y+\Delta y)-\psi_{00}(y)$ is the probability to complete the overhaul of the system at time interval $[y, y+\Delta y)$ for sufficiently small Δy. Hence under Policy III, the following relation can be easily shown:

$$R(x,t)=P\,[\text{the system is operating at time } t \text{ and continue to operate for the mission time } x]$$

$$=1-F(t+x)+\int_0^t d\psi_{00}(t)\cdot\sum_{\nu=0}^{k-1}P(E_\nu) \tag{3.2}$$

where events E_ν are defined as

$E_\nu=$ {from y till t, minimal repairs of just ν-times are performed and the system is operating at time t and continue to operate for a preassigned time x under the condition given by the above event} (3.3)

If we assume the Weibull type as the failure distribution, then we have

$$P[E_\nu]=\frac{\alpha^\nu(t-y-\nu T_m)^{\nu\beta}}{\nu!}e^{-(t-y-\nu T_m+x)^\beta} \tag{3.4}$$

Hence, inserting (3.4) into (3.2) we have

$$R(x,t)=\sum_{\nu=0}^{k-1}\int_0^{t-\nu T_m}\frac{\{\alpha(t-y-\nu T_m)^\beta\}^\nu}{\nu!}e^{-\alpha(t-y-\nu T_m+x)^\beta}d\psi_{00}(y)$$

$$=\sum_{\nu=0}^{k-1}\int_0^{t-\nu T_m}f_\nu(y)e^{-\alpha(t-y-\nu T_m+x)^\beta}d\psi_{00}(y), \tag{3.5}$$

where

$$f_\nu(y)=\frac{\{\alpha(t-y-\nu T_m)^\beta\}^\nu}{\nu\,!}, \quad y\leqq t-\nu T_m$$
$$=0 \qquad\qquad\qquad , \quad y>t-\nu T_m. \tag{3.6}$$

Next, we may rewrite the integrand of (3.5) using the kernel function

$$\Psi(T-y)=\frac{\{\alpha(T-y)^\beta\}^\nu}{\nu\,!}e^{-\alpha(T+x-y)^\beta} \tag{3.7}$$

where $T=t-\nu T_m$. This kernel function $\Psi(t)$ has the following relation:

1. it is bounded for $t\geqq 0$,
2. $\Psi(t)\epsilon L_1$ that is $\Psi(t)$ is absolutely integrable,
3. $\lim_{t\to\infty}\psi(t)=0$,
4. $F(t)\epsilon\mathfrak{S}$ that is the k-th convolution has absolutely continuous part,
5. $\mu<\infty$.

Hence we can apply key renewal theorem due to W. L. Smith [6] for the calculation of

$$\lim_{t\to\infty}R(x,t)=\frac{1}{\mu_k}\int_0^\infty\left\{\sum_{\nu=0}^{k-1}\frac{(\alpha t^\beta)^\nu}{\nu\,!}\right\}e^{-\alpha(t+x)^\beta}dt, \tag{3.8}$$

where μ_k is the expected length of a renewal cycle and can be expressed by

$$\mu_k=(k-1)T_m+T_s+E(U^{(k)})$$
$$=(k-1)T_m+T_s+\mu\beta\Big/B\left(k,\frac{1}{\beta}\right). \tag{3.9}$$

As a consequence of the above results we have the following

Theorem 3.1 For Policy III, the interval reliability $R_k^{(3)}(x)$ of the system with a Weibull type failure distribution is given by

$$R_k^{(3)}(x)=\frac{1}{(k-1)T_m+T_s+\dfrac{\mu\beta}{B\left(k,\dfrac{1}{\beta}\right)}}\sum_{\nu=0}^{k-1}\int_0^\infty\frac{(\alpha t^\beta)^\nu}{\nu!}e^{-\alpha(t+x)^\beta}dt\ . \tag{3.10}$$

Remark 3.1 Especially, we have the identy

$$R_k^{(3)}(0)=Eff_\infty^{(3)}(k) \tag{3.11}$$

which is seen to be hold intuitively. A rigorous proof of this assertion is as follows: We have

$$\begin{aligned}R_k^{(3)}(0)&=\frac{1}{\mu_k}\sum_{\nu=0}^{k-1}\int_0^\infty\frac{(\alpha t^\beta)^\nu}{\nu!}e^{-\alpha t^\beta}dt\\&=\frac{1}{\mu_k}\frac{1}{\alpha^{1/\beta}\cdot\beta}\sum_{\nu=0}^{k-1}\frac{\Gamma\left(\nu+\dfrac{1}{\beta}\right)}{\Gamma(\nu+1)}\ .\end{aligned} \tag{3.12}$$

On the other hand, since

$$Eff_\infty^{(3)}(k)=\frac{E(U^{(k)})}{\mu_k}=\frac{1}{\mu_k}\cdot\frac{\Gamma\left(k+\dfrac{1}{\beta}\right)}{\Gamma(k)\cdot\alpha^{1/\beta}} \tag{3.13}$$

we can conclude that (3.11) is true, showing the following relation

$$\frac{\Gamma\left(k+\dfrac{1}{\beta}\right)}{\Gamma(k)}=\frac{1}{\beta}\sum_{\nu=0}^{k-1}\frac{\Gamma\left(\nu+\dfrac{1}{\beta}\right)}{\Gamma(\nu+1)}\ , \tag{3.14}$$

which may be readly verified by mathematical induction.

Now, we shall consider the maximization problem of $R_k^{(3)}(x)$ obtained in the above. To solve this problem, we shall at first find the greatest k such as

$$R_{k+1}^{(3)}(x)-R_k^{(3)}(x)>0 \tag{3.15}$$

and add unity. (This is denoted by k_0.) This k_0 is our solution which gives the optimal policy of type III for a preassigned x.

Putting

$$f(k)=\sum_{\nu=0}^{k-1}\int_0^\infty \frac{(\alpha t^\beta)^\nu}{\nu!}e^{-\alpha(t+x)^\beta}\,dt \tag{3.16}$$

we have

$$\frac{f(k+1)}{\mu_{k+1}} \geqq \frac{f(k)}{\mu_k} \tag{3.17}$$

from (3.15), where

$$\mu_k=(k-1)T_m+T_s+\frac{\mu\beta}{B\left(k,\frac{1}{\beta}\right)}. \tag{3.18}$$

Thus, if we calculate numerically (i.e., tabulate) the function $f(k)$ for a preassigned x, then we can obtain k_0. It must be remarked here that k_0 is dependent on α, β, $X=x/\mu$, $\tau_m=T_m/\mu$ and $\tau_s=T_s/\mu$.

For the sake of its complicacy, we shall get out here of the tabulation of $f(k)$, and proceed to consider an approximated relation. If X is small compared with 1, each term of $f(k)$ may be calculated by the following way

$$\int_0^\infty \frac{(\alpha t^\beta)^\nu}{\nu!}e^{-\alpha(t+x)^\beta}\,dx$$

$$=\int_0^\infty \frac{u^\nu}{\nu!}e^{-\left(u^{1/\beta}+\Gamma\left(1+\frac{1}{\beta}\right)X\right)^\beta}\cdot\frac{u^{\frac{1}{\beta}-1}}{\beta\alpha^{1/\beta}}\,du$$

$$\doteqdot\int_0^\infty \frac{u^\nu}{\nu!}e^{-u}\frac{u^{\frac{1}{\beta}-1}}{\beta\alpha^{1/\beta}}\,du+\Gamma\left(1+\frac{1}{\beta}\right)\cdot$$

$$X\int_0^\infty \frac{u^\nu}{\nu!}\left(-\beta u^{\frac{\beta-1}{\beta}}\right)e^{-u}\cdot\frac{u^{\frac{1}{\beta}-1}}{\beta\alpha^{1/\beta}}\,du$$

$$=\frac{1}{\beta\alpha^{1/\beta}}\;\frac{\Gamma\left(\nu+\frac{1}{\beta}\right)}{\Gamma(\nu+1)}+\Gamma\left(1+\frac{1}{\beta}\right)\cdot\frac{1}{\alpha^{1/\beta}}\cdot X$$

$$=\mu\left(\frac{1}{B\left(\nu,\frac{1}{\beta}\right)}+X\right). \tag{3.19}$$

Hence, we have

$$f(k)=\frac{1}{\alpha^{1/\beta}}\frac{\Gamma\left(k+\frac{1}{\beta}\right)}{\Gamma(k)}+k\mu X$$

$$=\mu\left(\frac{\beta}{B\left(k,\frac{1}{\beta}\right)}+kX\right). \tag{3.20}$$

Inserting this relation into (3.17) we have

$$\frac{\dfrac{\beta}{B\left(k+1,\frac{1}{\beta}\right)}+(k+1)X}{\dfrac{\beta}{B\left(k+1,\frac{1}{\beta}\right)}+k\tau_m+\tau_s}>\frac{\dfrac{\beta}{B\left(k,\frac{1}{\beta}\right)}+kX}{\dfrac{\beta}{B\left(k,\frac{1}{\beta}\right)}+(k-1)\tau_m+\tau_s}, \tag{3.21}$$

which is rewritten as

$$\frac{k(\tau_m-X)+(\tau_s-X)}{\dfrac{\beta}{B\left(k+1,\frac{1}{\beta}\right)}+k\tau_m+\tau_s}<\frac{(k-1)(\tau_m-X)+(\tau_s-X)}{\dfrac{\beta}{B\left(k,\frac{1}{\beta}\right)}+(k-1)\tau_m+\tau_s}, \tag{3.22}$$

or

$$\frac{k\tau_m'+\tau_s'}{\dfrac{\beta}{B\left(k+1,\frac{1}{\beta}\right)}+k\tau_m+\tau_s}<\frac{(k-1)\tau_m'+\tau_s'}{\dfrac{\beta}{B\left(k,\frac{1}{\beta}\right)}+(k-1)\tau_m+\tau_s}. \tag{3.23}$$

It is very interesting to see that the above inequality (3.23) is the same with (2.2), referring (2.1) and (2.6) with $C_0=1$. Hence, we may say that the optimal policy to minimize the cost rate be regarded as the one to maximize the interval reliability by the following interpretation:

$$\left.\begin{array}{l}\tau_m \Rightarrow T_m,\ \tau_s \Rightarrow T_s \\ X \Rightarrow C_m \text{ or } C_s \\ \tau_m' = \tau_m - X \Rightarrow T_m + C_m,\ \tau_s' = \tau_s - X \Rightarrow T_s + C_s\end{array}\right\} \quad (3.24)$$

Example 3.1 A table and a graph of $R_k^{(3)}(x)$ will be shown for the case: $\beta=2$, $\tau_m=\sqrt{\frac{2}{\pi}}\cdot\frac{2}{5}$ and $\tau_s=\sqrt{\frac{2}{\pi}}\cdot 2$. From the graph, for example, we can say that if the optimal value k_0 of Policy III is settled, this policy does not lose its optimality for any X on $[0, \xi_1)$, and even in the case X is on $[\xi_1, \xi_2)$, the interval reliability is not far from that of optimal one.

Table 3.1

X \ k	1	2	3	4	5
0	0.39	0.44	0.46	0.47	0.47
$1/\sqrt{2\pi}$	0.24	0.24	0.24	0.22	0.21
$2/\sqrt{2\pi}$	0.12	0.12	0.11	0.10	0.09
$4/\sqrt{2\pi}$	0.02	0.02	0.01	0.01	0.01

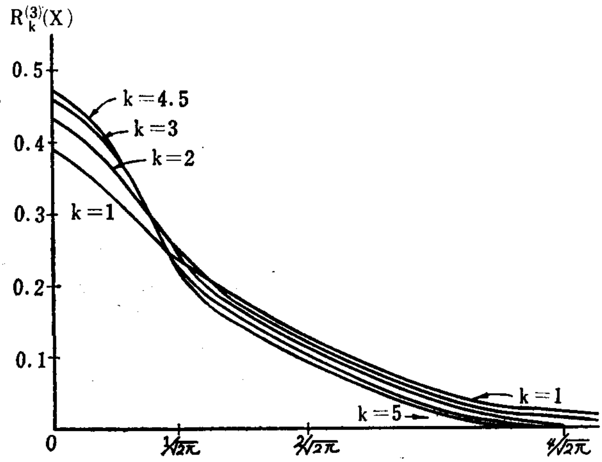

Fig. 3.1

§ 4. INTERVAL RELIABILITY FOR POLICY II AND IV

We define Policy IV by the following

Definition 4.1 When the total running time amounts to t^* or when the number of times of failure counted from the last overhaul is k_0, perform an overhaul. Otherwise, perform a minimal repair. This preventive maintenance policy IV which is a combined policy of Policy II and III. The system which have been overhauled is considered to enter into a new generation.

Interval reliability for this policy can be calculated along an analogous direction to § 3. Thus, we have

$$R^{(4)}(k, t^*; x)$$

$$=\lim_{t\to\infty}\sum_{\nu=0}^{k-1}\int_{t-\nu T_m+x-t^*}^{t-\nu T_m}\frac{\{\alpha(t-\nu T_m-y)^\beta\}^\nu}{\nu\,!}e^{-\alpha(t-\nu T_m+x-y)^\beta}\,d\psi_{00}(y) \tag{4.1}$$

Putting $T=t-\nu T_m$ and noting that

($\nu+1$)-th term of te right hand side of (4.1)

$$=\lim_{T\to\infty}\int_{T-(t^*-x)}^{T}\frac{\{\alpha(T-y)^\beta\}^\nu}{\nu\,!}e^{-\alpha(T-y+x)^\beta}\,d\psi_{00}(y), \tag{4.2}$$

the kernel function in (4.2) may be seen to be as

$$\begin{aligned}\Psi(z)&=\frac{(\alpha z^\beta)^\nu}{\nu\,!}e^{-\alpha(z+x)^\beta}, \quad 0\leqq z<t^*-x \\ &=0 \qquad\qquad\qquad\quad, \text{ otherwise.}\end{aligned} \tag{4.3}$$

Hence by the key renewal theorem we have as the interval reliability for Policy IV

$$R^{(4)}(k, t^*; x)=\frac{1}{\mu'_{k,t^*}}\sum_{\nu=0}^{k-1}\int_0^{t^*-x}\frac{(\alpha z^\beta)^\nu}{\nu\,!}e^{-\alpha(z+x)^\beta}\,dz \tag{4.4}$$

where μ'_{k,t^*} is an expected time length of a renewal cycle and is calculated as

$$\mu'_{k,t^*}=\left[(k-1)T_m+\frac{\mu\beta}{B\left(k,\frac{1}{\beta}\right)}\right](1-S_k)$$
$$+\sum_{\nu=0}^{k-1}(t^*+\nu T_m)p_\nu+T_s, \tag{4.5}$$

where

$$S_k=\sum_{\nu=0}^{k-1}\frac{(\alpha t^{*\beta})^\nu}{\nu\,!}e^{-\alpha t^{*\beta}},$$
$$p_k=\frac{(\alpha t^{*\beta})^k}{k\,!}e^{-\alpha t^{*\beta}}. \tag{4.6}$$

In the above equality (4.5), letting $k\to\infty$, we can see that Policy IV approaches to Policy II and we have as the interval reliability for Policy II

$$R_{t^*}^{(2)}(x)=\lim_{k\to\infty}R^{(4)}(k,t^*;x)$$
$$=\frac{\int_0^{t^*-x}e^{-\alpha(z+x)^\beta+\alpha z^\beta}dz}{t^*+\alpha t^{*\beta}+T_m} \tag{4.7}$$

As a special case, we have $Eff_\infty^{(2)}$ derived by R. Barlow and L. Hunter [2] putting $x=0$.

Now, solving $\frac{\partial R_{t^*}^{(2)}(x)}{\partial t^*}=0$ for t^*, we have the optimal policy of type II to maximize the interval reliability for a preassigned x. Inserting (4.7) into this relation we have

$$e^{-\alpha t^{*\beta}+\alpha(t^*-x)^\beta}\cdot(t^*+\alpha t^{*\beta}+T_s)$$
$$=(1+\alpha\beta t^{*\beta-1})\cdot\int_0^{t^*-x}e^{-\alpha(z+x)^\beta+\alpha z^\beta}dz. \tag{4.8}$$

Similarly to the treatment in §3, we shall consider the approximate relation for a sufficiently small x. It is as follows,

$$e^{-\alpha\beta t^{*(\beta-1)}x}\cdot(t^*+\alpha t^{*\beta}+T_s)=(1+\alpha\beta t^{*\beta-1}) \tag{4.9}$$

from (4.8). Solving the equation in x we have

$$x=\frac{1}{\alpha\beta t^{*\beta-1}}\log\frac{t^*+\alpha t^{*\beta}+T_s}{1+\alpha\beta t^{*\beta-1}}. \tag{4.10}$$

A graph of the right hand side of (4.10) shows the relation between t^* and x which gives the optimal t^*. We are going to construct it.

§ 5. POLICY V AND OPTIMAL TYPE OF PREVENTIVE MAINTENANCE POLICY

We have introduced the Policy III based on the idea that we wish to use for a long time the system with a long life and the one with a short life for a short time. And, as was shown already, Policy III is more efficient and simpler than Policy II. In this section, we shall further show that Policy III becomes the optimal policy in all preventive maintenance policies from a practical view point. If we speak mathematically, our discussion is still in vagueness to assert the optimality of Policy III. However, it seems that the further detailed rigorous discussions are rather tedious and difficult but have little values for practical purposes.

Now, we know the sample values of $(U_1, U_2, \ldots, U_k)$ by the k-th failure as our information, where U_i is the total running time excluding any repair time till the i-th failure after an overhaul (or a replacement). In Policy II, we use only $U(t)$ which is the total running time till the time t, and in Policy III we use only the number of times of failures. Since Policy II is less efficient than Policy III, then we shall consider to use the information of $(U_1, U_2, \ldots, U_k)$.

By the way, under the assumption of Weibull type with a known shape parameter as the life time distribution of the system, the unbiased and sufficient estimator of α is given by

$$\hat{\alpha}=\frac{k-1}{U_k^{\beta}}. \tag{5.1}$$

This deduction will be given in Appendix. Thus, we shall decide at each failure whether a minimal repair or an overhaul must be performed based on only U_k. Of course, when Weibull type does not assumed as the life time distributions of the systems, the above restriction on using our information has no neccesity. But, for many practical applications the assumption of Weibull type life time distribution may be allowable.

Hence, it seems to be reasonable to decide whether minimal repair or overhaul according to which inequality

$$U_k \lesseqqgtr a_k \tag{5.2}$$

holds. Along this idea, our policy may be defined by the sequence (a_1, a_2, ...). We shall now consider the maximization problem of the limiting efficiency in terms of the sequence $\{a_i\}$. This problem can also be regarded as that of random walk with an absorbing barrier $\{a_i\}$. [see Fig. 5.1].

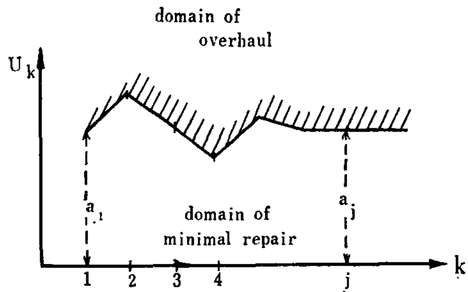

Fig. 5.1

When the sequence is arbitrary, our discussion is rather complex, hence we shall restricted our consideration into the following three cases.

a) $a_1 \leqq a_2 \leqq a_3 \leqq \cdots\cdots$

If $u_k \leqq a_k$, perform an overhaul at k-th failure, and if $U_k >$ a_k perform a minimal repair. [see Fig. 5.2]

b) $a_1 \geqq a_2 \geqq \cdots\cdots$

If $U_k > a_k$, perform an overhaul at k-th failure, and if $U_k \leqq a_k$ perform a minimal repair. [see Fig. 5.3]

c) $a_1 \leqq a_2 \leqq a_3 \leqq \cdots\cdots$

If $U_k > a_k$, perform an overhaul at k-th failure, and if $U_k \leqq a_k$ perform a minimal repair. [see Fig. 5.4]

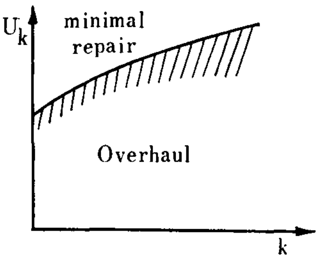

Fig. 5.2

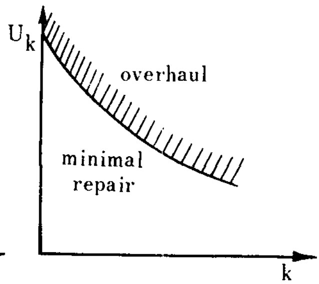

Fig. 5.3

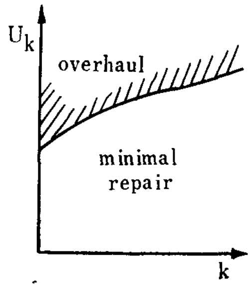

Fig. 5.4

Before we proceed further discussion, in order to make it concrete we shall introduce the following definitions.

Definition 5.1 Let $\{a_i\}$ is a preassigned sequence of nonnegative numbers (allowable the infinity), and at i-th failure ($i=1, 2, \dots, k, \dots$), perform an overhaul if $U_i>a_i$ and perform an minimal repair if $U_i\leqq a_i$. This preventive maintenance policy will be called as Policy V.

In particular, we shall call the policy as Policy V_D when $\{a_i\}$ is a monotone non-increasing sequence and Policy V_A when $\{a_i\}$ is a monotone non-decreasing sequence.

Definition 5.2 In Policy V, converse the signs of inequality. This preventive maintenance policy will be called as Policy V′. Policy V'_D and Policy V'_A are also defined analogously.

Definition 5.3 Putting $a_i=\tilde{t}$ in the above definition of Policy V, we get a similar policy to Policy II. But, in this case, the system may be operated somewhat after $U(t)$ build up to $\tilde{t}$. So, we shall name this preventive maintenance policy as Policy II′.

Definition 5.4 Similarly to the above, putting $a_i=\tilde{t}$ for $i<k$, and $a_i=0$ for $i\geqq k$, we get a similar policy to Policy IV. Thus, we shall call this policy as Policy IV′.

Definition 5.5 Furthermore, putting $a_1=a_2=\dots=a_{k_1}=\infty$, $a_{k_1+1}=\dots=a_{k_2}=\tilde{t}$, $a_{k_2+1}=\dots=0$ in Policy V, we get a extended policy of Policy

IV′. Thus, we shall name this preventive maintenance policy as Policy IV″.

Now, under these preparations, we shall consider the optimal policy of type V. First of all, we shall prove the following

Theorem 5.1 Under the assumption that the life time distributions of the systems are Weibull type with common shape parameter $\beta>1$, an optimal policy of type V_D to maximize the limiting efficiency $Eff_\infty^{(5)}$ has the same type to the one of the following policies: II′, IV′, IV″, III.

Furthermore, we can say the same assertion concerning with Policy V_A and Policy V'_A.

Proof. We shall prove only the first assertion. The second part of the theorem may be shown similarly.

Let the probability that an overhaul is performed at i-th failure be $P(i)$, then we have

$$P(1)=\int_{a_1}^{\infty}\alpha\beta u_1^{\beta-1}e^{-\alpha u_1^\beta}du_1,$$

$$P(i)=\left[\int_{a_i}^{a_{i-1}}\int_{u_{i-1}}^{\infty}+\int_0^{a_i}\int_{a_i}^{\infty}\right]\frac{\alpha^{i-1}\beta u_{i-1}^{(i-1)\beta-1}}{(i-2)!}\alpha\beta u_i^{\beta-1}e^{-\alpha u_i^\beta}du_{i-1}du_i,\ \text{for } i\geqq 2 \tag{5.3}$$

Because, the event that an overhaul is performed at i-th failure will occurs in the two ways:

$$\left.\begin{array}{ll}\text{(A)} & a_i<U_i\leqq a_{i-1}\\ \text{(B)} & U_{i-1}\leqq a_i \text{ and } a_i<U_i\end{array}\right\} \tag{5.4}$$

[see Fig. 5.5], and further, the density function $h(u_{i-1}, u_i)$ of the joint distribution of (U_{i-1}, U_i) is given by

$$\begin{aligned}
&h(u_{i-1}, u_i)\, du_{i-1}\, du_i\\
&=P(u_{i-1}\leqq U_{i-1}<u_{i-1}+du_{i-1})\ P(u_i\leqq U_i<u_i+du_i\,|\,U_i>u_{i-1})\\
&=\left[\frac{(\alpha u_{i-1}^{\beta})^{(i-2)}}{(i-2)!}e^{-\alpha u_{i-1}^\beta}\cdot\alpha\beta u_{i-1}^{\beta-1}du_{i-1}\right]\cdot\left[\frac{\alpha\beta u_i^{\beta-1}e^{-\alpha u_i^\beta}}{e^{-\alpha u_{i-1}^\beta}}du_i\right]\\
&=\frac{\alpha^i\beta^2u_{i-1}^{(i-1)\beta-1}\ u_i^{\beta-1}}{(i-2)!}e^{-\alpha u_i^\beta}\,du_{i-1}\ du_i.
\end{aligned} \tag{5.5}$$

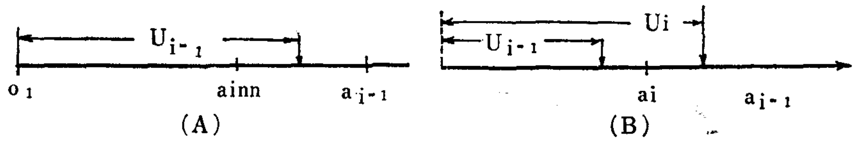

Fig. 5.5

Hence we have

$$E_V(U)=\int_{a_1}^{\infty} u_1\alpha\beta u_1^{\beta-1}e^{-\alpha u_1^{\beta}}du_1$$

$$+\sum_{i=2}^{\infty}\left(\int_{a_i}^{a_{i-1}}\int_{u_{i-1}}^{\infty}+\int_0^{a_i}\int_{a_i}^{\infty}\right)\frac{\alpha^i\beta^2u_{i-1}^{(i-1)\beta-1}u_i^{\beta}}{(i-2)!}\, e^{-\alpha u_i^{\beta}}\, du_{i-1}\, du_i \qquad (5.6)$$

and

$$E_V(K)=\sum_{i=1}^{\infty} i\ P(i) \qquad (5.7)$$

where $E_V(U)$ is the expected running time of the system in a renewal cycle under Policy V and $E_V(K)$ is the expected times of failure in the same cycle.

By the way, using these notations, the limiting efficiency of Policy V_D will be given by

$$Eff_{\infty}^{(5)}=\frac{E_V(U)}{E_V(U)+\{E_V(K)-1\}T_m+T_s}. \qquad (5.8)$$

Putting the partial derivative of $Eff_{\infty}^{(5)}$ $(a_1, a_2, \ldots, a_i, \ldots)$ with respect to a_i equal to zero, we have

$$E_V(U)\cdot\frac{\partial}{\partial a_i}E_V(K)=\frac{\partial}{\partial a_i}E_V(U)\left\{\frac{T_s-T_m}{T_m}+E_V(K)\right\}. \qquad (5.9)$$

Since we have, from (5.3), (5.6) and (5.7), for all $i\geqq 1$

$$\frac{\partial}{\partial a_i}E_V(U)=\frac{\alpha^i\beta a_i^{\beta i-1}}{(i-1)!}\left\{\int_{a_i}^{\infty}\alpha\beta u_{i+1}^{\beta}e^{-u_{i+1}^{\beta}}du_{i+1}-a_ie^{-\alpha^{\beta_i}}\right\}, \qquad (5.10)$$

$$\frac{\partial}{\partial a_i} E_V(K) = \frac{\alpha^i \beta a_i^{i\beta-1}}{(i-1)!} e^{-\alpha a_i^\beta}, \tag{5.11}$$

or

$$\frac{\partial}{\partial a_i} E_V(U) = \frac{(\alpha a_i^\beta)^{i-1} e^{-\alpha a_i^\beta}}{(i-1)!} \cdot \alpha\beta a_i^{\beta-1}\left\{E(Z \mid Z \geqq a_i) - a_i\right\}, \tag{5.12}$$

$$\frac{\partial}{\partial a_i} E_V(K) = \frac{(\alpha a_i^\beta)^{i-1} e^{-\alpha a_i^\beta}}{(i-1)!} \cdot \alpha\beta a_i^{\beta-1} \tag{5.13}$$

where Z is a random variable obeying to Weibull distribution with the scale parameter α and the shape parameter β.

From these formulae we can easily see that

$$\frac{\partial}{\partial a_i} E_V(U)\Big]_{a_i=0} = \frac{\partial}{\partial a_i} E_V(K)\Big]_{a_i=0} = 0, \tag{5.14}$$

hence we have

$$\frac{\partial}{\partial a_i} Eff_\infty^{(5)}(a_1, \ldots\ldots, a_{i-1}, 0, a_{i+1}, \ldots\ldots) = 0. \tag{5.15}$$

And, for $a_i \neq 0$, (5.9), (5.12) and (5.13) imply that

$$\frac{E_V(U)}{\{(T_s - T_m)/T_m + E_V(K)\}} = E(Z \mid Z \geqq a_i) - a_i. \tag{5.16}$$

Since the existence of an optimal policy of type V_D is ovbious we shall denote it as $\{a_i^0\}$, i.e.,

$$\sup_{\{a_i\}} Eff_\infty^{(5)}(a_1, a_2, \ldots\ldots) = Eff_\infty^{(5)}(a_1^0, a_2^0, \ldots\ldots). \tag{5.17}$$

We shall consider here the policy $(a_1^0, a_2^0, \ldots\ldots, a_{j-1}^0, a_j, a_{j+1}^0, \ldots\ldots)$. From the monotoness of the sequence $\{a_i\}$ and the continuity of $aff_\infty^{(5)}(a_1^0, a_2^0, \ldots\ldots, a_{j-1}^0, a_j, a_{j+1}^0, \ldots\ldots)$, it may be seen that the limiting efficiency has the maximum at $a_j = a_j^0 \leqq a_{j-1}^0$ and $a_j^0 = 0$ or $= \tilde{a}_j$ or $= a_{j-1}^0$, where $\tilde{a}_j$ is a root of (5.16) which is different from zero. Of course, $a_j^0 = 0$ implies that $a_k^0 = 0$ for all $k \geqq j$.

Next, we shall say that if $a_j^0=\tilde{a}_j<a_{j-1}^0$ then $a_{j-1}^0=a_{j-2}^0=\cdots\cdots=a_1^0=\infty$. To show the fact, suppose $a_{j-1}^0<\infty$. The above argument gives $a_{j-1}^0=\tilde{a}_{j-1}$ or $=a_{i-2}^0<\infty$. Repeating this, we can conclude that there exists a_j^0 such as

$$a_i^0=\tilde{a}_i<\infty \text{ and } \tilde{a}_j<\tilde{a}_i$$

Because, when there exist no a_i^0 such as (5.18) for $i>1$, the relation

$$a_{j-1}^0=a_{j-2}^0=\cdots\cdots=a_1^0<\infty \tag{5.19}$$

must be true, and since a_1^0 may be chosen in $(0, \infty)$ to maximize the efficiency, (5.19) implies that $a_1^0=\tilde{a}_1<\infty$.

If a_j^0 and a_i^0 equal to $\tilde{a}_j$ and $\tilde{a}_i$, respectively, (5.16) must be hold for both a_j^0 and a_i^0. This is contrary. Then, we can conclude that the optimal policy $\{a_i^0\}$ satisfy the one of the following relations.

$$a_1^0=\cdots\cdots=a_{k_1}^0=\infty,\ a_{k_1+1}^0=\cdots\cdots=a_{k_2}^0=\tilde{t},\ a_{k_2+1}^0=\cdots\cdots=0, \tag{5.20}$$

$$a_1^0=\cdots\cdots=a_k^0=\tilde{t},\ a_{k+1}^0=\cdots\cdots=0, \tag{5.21}$$

$$a_1^0=\cdots\cdots=\tilde{t}, \tag{5.22}$$

$$a_1^0=\cdots\cdots=a_k^0=\infty,\ a_{k+1}^0=\cdots\cdots=0. \tag{5.23}$$

These policies can be illustrated in Fig. 5.6~5.9, respectively. This completes the proof of the first part of the theorem. The second part of the theorem may be shown analogously and the proof of it is omitted.

In the present theorem, we did not discuss on which policy has the highest efficiency in these. It is tedious and difficult to do this exactly,

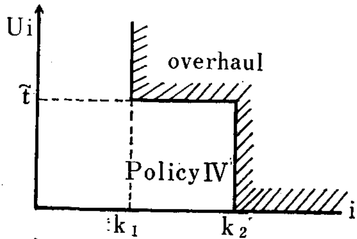

Fig. 5.6 (5.20)

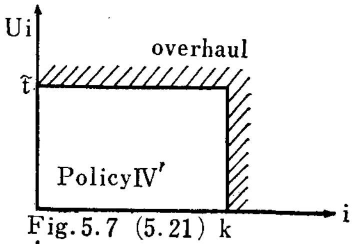

Fig. 5.7 (5.21)

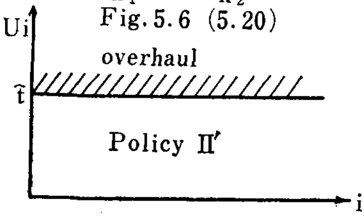

Fig. 5.8 (5.22)

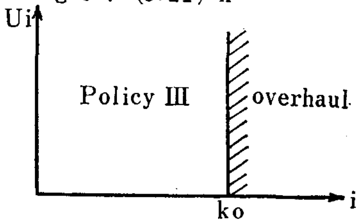

Fig. 5.9 (5.23)

but we knew that the theorem has a sufficient sense from a practical view point. More precisely, Policy III is more efficient and simpler than Policy II and has a robustness for the variety of the scale parameter in the case of Weibull type life time. [see [5]] And, we can see physically that Policy II' is has a slightly higher efficiency than Policy II, hence, that the efficiency of Policy III, perhaps at least, is comparable to the one of Policy II'. Thus, for the sake of the simple procedure and the robustness, Policy III may be taken up in practical uses. Since these Policy IV' and Policy IV'' are both some slight modifications of Policy II and III, our intuitive consideration will deduce that these policies may be refused in practical uses.

Finally, we shall sumarize above considerations.

1. A preventive maintenance policy is a decision rule to perform whether a minimal repair or an overhaul.
2. Our informations to make the decision at time t are
   (i) number of times of failure,
   (ii) total running time $U(t)$ till t excluding any repair time,
   (iii) history of $U(t)$.

3. When the life time distribution of each system is Weibull type with common shape parameter, the unknown parameter is scale parameter (α) only. Thus, if we estimate α at first and use the knowledge to our decision, we may get a preventive maintenance procedure with a higher efficiency.
4. An sufficient estimator of α in that case is depend only U_k which is the total running time till the k-th failure from the last overhaul.
5. We shall consider the maintenance policy to decide whether an overhaul ought to be performed or not based on U_k at each failure. (Policy V) The decision rule in Policy V is given by the boundary $\{a_i\}$. [see Fig. 5.1]
6. There is an optimal policy of type V in the category of policies of types of II′, IV′, IV″, and III. [see Theorem 5.1]
7. The first three types of these policies will be less suitable than Policy III in our practical uses.

Thus, we can recognize intuitively that Policy III is the most available type in all preventive maintenance policies.

Remark 5.1 The optimal value t^* of Policy II′ is the root of the following transcendental equation.

$$e^{\alpha t^{*\beta}}\int_{t^*}^{\infty} ue^{-\alpha u^{\beta}}\alpha(\alpha u^{\beta}) = \frac{\alpha t^{*\beta}T_m+T_s}{(\alpha t^{*\beta}-1)T_m+T_s}\cdot t^* \tag{5.24}$$

and the efficiency is shown to be

$$Eff_{\infty}^{(2')}=\frac{t^*}{t^*+(\alpha t^{*\beta}-1)T_m+T_s}\,. \tag{5.25}$$

In the following, we shall prove this assertion. Putting as $a_1=a_2=\cdots\cdots=a_i=\cdots\cdots=t^*$ in (5.3), (5.6) and (5.7) we have

$$P(i)=\int_0^{t^*} du_{i-1}\int_{t^*}^{\infty}\frac{\alpha^{i-1}\beta u_{i-1}^{(i-1)\beta-1}}{(i-2)!}\alpha\beta u_i^{\beta-1}e^{-\alpha u_i^{\beta}}du_i$$

$$=\frac{(\alpha t^{*\beta})^{i-1}}{(i-1)!}e^{-\alpha t^{*\beta}}, \tag{5.26}$$

$$E_V(U)=\sum_{i=1}^{\infty}\int_0^{t^*}du_{i-1}\int_{t^*}^{\infty}\frac{\alpha^{i-1}\beta u_{i-1}^{(i-1)\beta-1}}{(i-2)!}\alpha\beta u_i^{\beta}e^{-\alpha u_i^{\beta}}du_i$$

$$=\sum_{i=1}^{\infty}\frac{(\alpha t^{*\beta})^{i-1}}{(i-1)!}\int_{t^*}^{\infty}u_ie^{-\alpha u_i^{\beta}}d(\alpha u_i^{\beta})$$

$$=e^{\alpha t^{*\beta}}\int_{t^*}^{\infty}ue^{-\alpha u^{\beta}}d(\alpha u^{\beta}) \tag{5.27}$$

and

$$E_V(K)=\sum_{i=1}^{\infty}\frac{(\alpha t^{*\beta})^{i-1}}{(i-1)!}e^{-\alpha t^{*\beta}}=1+\alpha t^{*\beta} \tag{5.28}$$

Inserting (5.26)~(5.28) into (5.16) we get

$$e^{\alpha t^{*\beta}}\int_{t^*}^{\infty}ue^{-\alpha u^{\beta}}d(\alpha u^{\beta})$$

$$=\left[e^{\alpha t^{*\beta}}\int_{t^*}^{\infty}ue^{-\alpha u^{\beta}}d(\alpha u^{\beta})-t^*\right]\cdot\left(\frac{T_s}{T_m}+\alpha t^{*\beta}\right), \tag{5.29}$$

from which we have (5.24). If we insert these relations into $Eff_{\infty}^{(5)}$ (for example, (5.8)), then we have (5.25).

§ 6. OTHER REMARKS

Remark 6.1 In the above section, our discussion be proceeded under the assumption that the life time distribution of the systems are of the Weibull type. However, we shall suppose here that the failure rate is expressed by $q(t)$ in general form. In the Weibull case, the sufficiency of U_k as a estimator of the scale parameter is shown. [see § 5 and Appendix]. Here, in the present case, we shall try a similar discussion.

Let the joint probability density function of $(U_1, U_2, \ldots\ldots, U_k)$ be $p(u_1, u_2, \ldots\ldots, u_k)$, then we have

$$p(u_1, u_2, \ldots\ldots, u_k)=q(u_1)q(u_2)\ldots\ldots q(u_k)e^{-Q(u_k)}, \tag{6.1}$$

where

$$Q(u)=\int_0^u q(t)dt.$$

Suppose that the scale parameter α is contained in $q(u)$ in the form

$$q(u)=\lambda\left(\frac{u}{\alpha}\ ;\ \beta\right), \tag{6.2}$$

then we can rewrite (6.1) as

$$p(u_1,\ u_2,\ \ldots\ldots,\ u_k)=\lambda\left(\frac{u_1}{\alpha}\ ;\ \beta\right)\cdots\lambda\left(\frac{u_k}{\alpha}\ ;\ \beta\right)e^{-\int_0^{u_k}\left(\frac{u}{\alpha}\ ;\ \beta\right)du} \tag{6.3}$$

Further, we shall add the following assumption

$$q(u)=\lambda\left(\frac{u}{\alpha}\ ;\ \beta\right)=\alpha^a\lambda'(u\ ;\ \beta), \tag{6.4}$$

in order to guarantee the sufficiency of U_1 as a estimator of α. (6.4) is also satisfied by Γ type life time distributions as well as Weibull type distributions. In such a case, we can always decide our Policy III independently of α.

Next, we shall consider again on the optimality of Policy III. In order to be able to continue our analogous discussions to the ones in § 5, what condition must be imposed on $q(u)$? This can be elucidated considering that the formula of $P(i)$ is expressed by

$$P(i)=\left[\int_{a_i}^{a_{i-1}}du_{i-1}\int_{u_{i-1}}^{\infty}du_i+\int_0^{a_i}du_{i-1}\int_{a_i}^{\infty}du_i\right]h(u_{i-1},\ u_i)$$

$$=\left[\int_{a_i}^{a_{i-1}}du_{i-1}\int_{u_{i-1}}^{\infty}du_i+\int_0^{a_i}du_{i-1}\int_{a_i}^{\infty}du_i\right]\frac{\{Q(u_{i-1})\}^{i-2}}{(i-2)!}q(u_{i-1})e^{-Q(u_i)}q(u_i) \tag{6.5}$$

in this case. That is, since

$$E_V(U)=\int_{a_1}^{\infty}u_1e^{-Q(u_1)}q(u_1)du_1$$

$$+\sum_{i=2}^{\infty}\left[\int_{a_i}^{a_{i-1}}\int_{u_{i-1}}^{\infty}+\int_0^{a_i}\int_{a_i}^{\infty}\right]u_i\frac{\{Q(u_{i-1})\}^{i-2}}{(i-2)!}q(u_{i-1})q(u_i)e^{-Q(u_i)} \tag{6.6}$$

and

$$E_V(K)=\sum_{i=1}^{\infty}iP(i), \tag{6.7}$$

we can differenciate $E_V(U)$ and $E_V(K)$ such as

$$\frac{\partial}{\partial a_j}E_V(U)=\frac{\{Q(a_j)\}^{j-1}q(a_j)}{(j-1)!}\left[\int_{a_1}^{\infty}u_{j+1}e^{-Q(u_{j+1})}q(u_{j+1})du_{j+1}-a_je^{-Q(a_j)}\right]$$

$$=\frac{\{Q(a_j)\}^{j-1}q(a_j)}{(j-1)!}e^{-Q(a_j)}[E(U|U\geqq a_j)-a_j] \tag{6.8}$$

and

$$\frac{\partial}{\partial a_j}E_V(K)=\frac{\{Q(a_j)\}^{j-1}q(a_j)}{(j-1)!}e^{-Q(a_j)} \tag{6.9}$$

Hence the condition $\frac{\partial}{\partial a_j}Eff_{\infty}^{(5)}=0$ implies that

$$\frac{\{Q(a_j)\}^{j-1}q(a_j)}{(j-1)!}e^{-Q(a_j)}[E_V(U)-\{E(U|U\geqq a_j)-a_j\}$$

$$\times\left\{\frac{T_s-T_m}{T_m}-E_V(K)\right\} \tag{6.10}$$

Thus, under the assumptions (6.1) and (6.4), Theorem 5.1 will be also true.

Remark 6.2 If the mean life time of the systems vary, for example, take the values μ_1 and μ_2 with probability p_1 and p_2, respectively, then we have the following limiting efficiency for Polity V_D

$$Eff_{\infty}^{(5)}=\frac{pE_V(U_1)+qE_V(U_2)}{p[E_V(U_1)+\{E_V(K_1)-1\}T_m+T_s]+q[E_V(U_2)+\{E_V(K_1)-1\}T_m+T_s]}$$

$$=\frac{pE_V(U_1)+qE_V(U_2)}{p\frac{E_V(U_1)}{Eff_{\infty}^{(5)}(1)}+q\frac{E_V(U_2)}{Eff_{\infty}^{(5)}(2)}}, \tag{6.11}$$

where $Eff_{\infty}^{(5)}(i)$ is the efficiency of the system with mean life time μ_i, and U_i, K_i is the total running time and the number of times of minimal repair of it, respectively. Thus, it is easy to see that in this case the particular sequence

$$a_1 = a_2 = \cdots = a_i = \cdots \ (\neq 0) \tag{6.12}$$

does not satisfy the relation

$$\frac{\partial}{\partial a_i} Eff_{\infty}^{(5)} = 0. \tag{6.13}$$

In the other words, Policy II may be not an optimal type in such a case.

Remark 6.3 For many electronic and mechanical systems, we may experience that the mixed or composite type of Weibull distribution is more adequate than the simple type of it for the life time distribution. But, in practice, for the case of mixed type (whose graph of failure rate is given in Fig. 6.1), we can count the number of times of failure after γ and apply Policy III, where γ is the location parameter of the Weibull distribution corresponding to wear out failure. Usually, we can suppose that the elements in the system have recieved "aging" already, hence that Policy III may be not effected by the distribution corresponding to catastrophic failure. Or, in the case that we can examine failed elements to distinguish whether their failure types belong to catastrophic or wear out, it is also that Policy III does not depend on γ. Thus, we can say that Policy III has the robustness for varying γ also in usual practical cases.

Remark 6.4 Sometimes, we must consider on some preventive maintenance policies for the case where we can not recognize its failure at a glance. In such a case, we need to determine the time intervals to check the system which characterize the "checking procedure". Discussions on optimal checking procedure is scarecely treated in literatures. In [1], the optimal time interval is obtained for the case of exponential life time distribution.

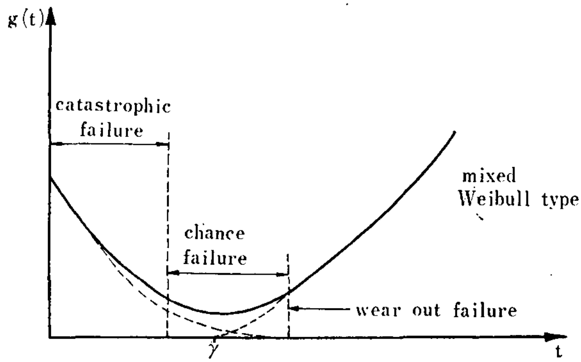

Fig. 6.1

By the way, the repairs are performed when and only when the failures detected, and the accurate number of times of failure can be counted for any checking procedure. Hence, we can apply the optimal policy of type III independently of the checking procedre. In the other words, we can assign an optimal checking procedure in a cycle from an overhaul to the next, and apply the optimal policy of type III to decide whether a minimal repair or an overhaul to be performed.

APPENDIX

Deduction of (5.1)

The probability density function of $(U_1, U_2, \dots, U_k)$ is given by

$$p(u_{k1}, u_2, \dots, u_k)=\alpha\beta u_1^{\beta-1}\cdot\alpha\beta u_2^{\beta-1}\dots\alpha\beta u_2^{\beta-1}\cdot e^{-\alpha u_k^\beta}, \qquad \text{(A.1)}$$

and using the method of maximum likelihood estimate we have

$$\frac{\partial p}{\partial\alpha}=\frac{\partial}{\partial\alpha}\left[\alpha^k\beta^k(u_1u_2\dots u_k)^{\beta-1}\cdot e^{-\alpha u_k^\beta}\right]=0$$

or

$$k\alpha^{k-1}\beta^k(u_1\dots u_k)^{\beta-1}e^{-\alpha u_k^\beta}=\alpha^k\beta^k(u_1\dots u_k)^{\beta-1}u_k^\beta e^{-\alpha u_k^\beta}. \tag{A.2}$$

Hence, we can say that the maximum likelihood estimator of α is

$$\hat{\alpha}=\frac{k}{U_k^\beta}. \quad \text{(A.3}$$

But in order to make this estimator unbiased, (A.3) must be replaced by

$$\hat{\alpha}=\frac{k-1}{U_k^\beta},$$

which is (5.1).

Next, from (A.1) we have

$$\begin{aligned} p(u_1, u_2, \dots, u_k) &= \alpha^k u_k^{\beta-1}e^{-\alpha u_k^\beta}\cdot\beta u_1^{\beta-1}\dots u_{k-1}^{\beta-1} \\ &= h(\alpha_i u_k)\,q\,(u_1, \dots, u_{k-1}) \end{aligned} \tag{A.4}$$

which shows that the joint probabilty of $(U_1, U_2, \dots, U_{k-1})$ conditioned by $U_k=a$ does not depend on α. Thus, we may say that (5.1) is an unbiased and sufficient estimator of α.

REFERENCES

[1] R. Barlow and L. Hunter; Mathematical models for system reliability, The Sylvania Technologist, vol. 13, No. 1 and 2 (1960)

[2] ——————; Optimal preventive maintenance policies, Opns. Res., vol. 8, pp. 90–100 (1960)

[3] ——————; Reliability analysis of a one-unit system, Opns. Res., vol. 9, pp. 200–208 (1961)

[4] R. Barlow, A. Marshall and F. Proschan; Properties of probability distributions with monotone hazard rate, Ann. Math. Stat., vol. 34, pp. 375–389 (1963)

[5] H. Makabe and H. Morimura; A new policy for preventive maintenance, Journ. Op. Res. Soc. Japan, vol. 5, pp. 110–124 (1963)

[6] W.L. Smith; Asymptotic renewal theorems, Proc. Royal Soc. Edinburgh, vol. 64, pp. 9–48 (1954)

[7] A. J. Truelove; Strategic reliability and preventive maintenance, Opns. Res., vol. 9, pp. 22–29 (1961)