THE PROVISION OF SPARE CABLE

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1. INTRODUCTION

The telecommunication cable plays an important role in telecommunication service. Once a cable trouble happens, hundreds and thousands of subscribers have to face with service interruption.

If there is no spare cable in stock, many customers are put to inconvenience during a long lead time until a new cable is received. Therefore there is a strong need for spare cables to be in stock ready for such cable troubles.

A cable can be classified into two kinds; one is an aerial cable and the other is a cable in an underground conduit. In case of an aerial cable, the required length of the cable to be replaced can be obtained by cutting the spare cable in stock and a jointed cable can be often used. On the contrary, in case of a cable in a conduit, it is impossible to put the jointed cable into the conduit. For the above reasons, the spare cable problems are quite different between the above two kinds of cables.

For the spare aerial cable in stock, an expansion of the methods used in a usual inventory control system can be easily taken. However, for the spare cable in a conduit to be stocked, a new method is necessary to be developed.

Many subscribers are provided the telecommunication service through cables in conduits. Therefore, shortage of spare cables in stock creates inconvenience to the subscribers to a great extent. The cable in a conduit is high in price and so the inventory cost also becomes high.

The length of a cable conduit is influenced by the geographical condition of the area to be served. When a spare conduit cable of maximum length is available in stock, it can be cut to meet the required length.

This article applies the Markov chain theory to determine what will happen to the remaining part of the cable, and prove that the ergodic condition is determined by how the scrapping criterion for the remaining part of the cable is set. Furthermore, it discusses how to determine the necessary quantity of spare cables of maximum length under the given shortage rate.

2. THE TRANSITION OF THE NUMBER OF SPARE CABLE PIECES

The distribution of cable length to be replaced is mainly based on conduit length. According to data, the demand distribution is nearly uniform from about 3m to the maximum conduit length. At least one spare cable piece of maximum conduit length is to be in stock and when it is used, a reorder is placed immediately. Actually the maximum conduit length cable is not necessarily always used. If the cable length to be replaced is shorter than the maximum conduit length, a short piece of the remaining cut part is produced.

It is impossible to put the jointed short piece cable into the conduit.

A short piece less than the minimum conduit length is usually scrapped. Under this scrapping criterion, it is necessary to explain what will happen to the spare cables in stock.

As explained later, if short pieces are held in stock for the reason that there will be demand for them in the future, the amout of these short pieces will increase day by day. Therefore, we have to set a certain criterion to scrap them regardless of their usableness in the future. These scrapped short pieces are sent to the cable manufacturers where copper and lead are recovered, thereby avoiding a large loss.

Now we have to decide the proper scrapping criterion by taking account of both the scrap loss and holding cost. Let us discuss first about the transition of the number of cable pieces under the scrapping criterion of minimum conduit length and then about the proper scrapping criterion.

Let's assume that the minimum conduit length is Lm. We expect that there are demands such as Lm, $2Lm$, $3Lm$......, Then, the demand would be subject to uniform distribution such as 1, 2,......$N+1$, in Lm units.

When an N length piece is cut, it is replenished immediately. If a short piece is available to meet the demand, it is used.

Now, we shall consider a point of N dimensions $(X_1, X_2, \ldots\ldots, X_n)$, where each X_i is a non negative integer.

This point of N dimensions indicates the cable length and the amount of cable pieces of each. In other words, (0, 0,……, 0) indicates the state that there is no piece from 1m to Nm. Since there is at least 1 cable of $N+1m$ length, it is not shown specifically.

1m length pieces are exluded from the consideration because all the pieces of minumum conduit or less length are scrapped. That is, (1, 0,…, 0) indicates the state that there is one piece of 2m length and no other pieces.

Now, let the state (0, 0,…, 0) be expressed by 0, the state (0, 0, …, 1) by 1 and the state (0, 0,…, 1, 0) by 2, and so forth, then they are denumerable and form a Markov chain.

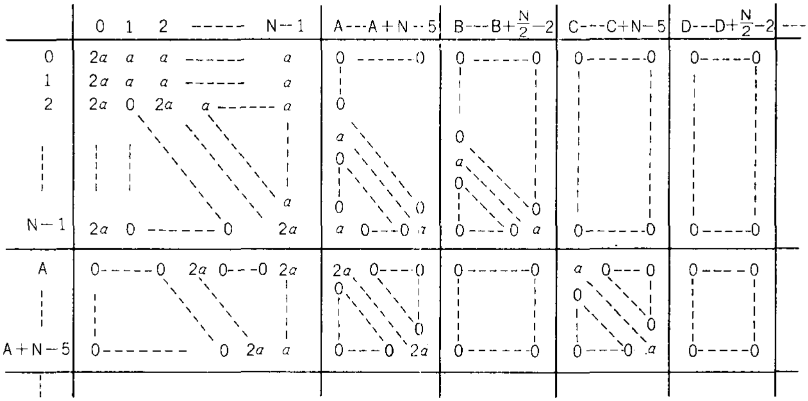
Now, the states are defined as follows:

$(0,0,0,\ldots,0)\rightarrow 0$	short piece 0
$(0,0,0,\ldots,1)\rightarrow 1$	short piece 1
$(0,0,0,\ldots,1,0)\rightarrow 2$	
$(1,0,0,\ldots,0)\rightarrow N-1$	
$(1,0,\ldots,1^{*1},0,\ldots,0)\rightarrow A$	short piece 2
$\vdots$	
$(1,1,0,\ldots,0)\rightarrow A+N-2-3^{*3}$	
$(0,0,\ldots,2^{*2},0,\ldots,0)\rightarrow B$	
$(2,0,0,\ldots,0)\rightarrow B+\frac{N}{2}-2^{*4}$	
$\vdots$	
$(2,0,\ldots,1^{*1},0,\ldots,0)\rightarrow C$	short piece 2
$(2,1,0,\ldots,0)\rightarrow C+N-2-3$	
$(0,0,\ldots,3,0,\ldots,0)\rightarrow D$	
$\vdots$	
$(3,0,\ldots,0)\rightarrow D+\frac{N}{2}-2$	

*1: Does not exceed $N-2$m.
*2: Does not exceed $N/2$m.
*3: As the second piece does not exceed 3m.
*4: As the minimum cable length is 2m.

Let a be equal to $\frac{1}{N+1}$, then we have the following transition probability matrix.

We have to prove whether this Markov chain is transient or persistent, and if it is persistent, whether it is null state or ergodic. This proof is very difficult for a general transition matrix. Therefore, we will study the smallest transition matrix which includes all the combinations of transitions that may occur in a general case. This condition is satisfied in the case of $N=5$ (proof omitted).



The transition matrix for $N=5$ is given by

	0	1	2	3	4	5	6	7	8	9	10	11	12 ----
0	$2a$	a	a	a	a								
1	$2a$	a	a	a	a								
2	$2a$		$2a$	a	a								
3	$2a$			$2a$	a	a							
4	$2a$				$2a$	a	a						
5				$2a$	a	$2a$		a					
6					$2a$		$2a$	a	a				
7						$2a$	a	$2a$		a			
8							$2a$		$2a$	a	a		
9								$2a$	a	$2a$		a	
10									$2a$		$2a$	a	a

where we define the states as follows:

$$\begin{aligned}
(0,0,0,0)&\rightarrow 0\\
(0,0,0,1)&\rightarrow 1\\
(0,0,1,0)&\rightarrow 2\\
(0,1,0,0)&\rightarrow 3\\
(1,0,0,0)&\rightarrow 4\\
(1,1,0,0)&\rightarrow 5\\
(2,0,0,0)&\rightarrow 6\\
(2,1,0,0)&\rightarrow 7\\
(3,0,0,0)&\rightarrow 8\\
(3,1,0,0)&\rightarrow 9\\
(4,0,0,0)&\rightarrow 10\\
(4,1,0,0)&\rightarrow 11\\
(5,0,0,0)&\rightarrow 12
\end{aligned}$$

Let y_i be the probability that the system is permanently in a transient state. From Feller's following theorem:

Theorem 2.1 Let an irreducible chain have states E_0, $E_1,\ldots,$ In order that the states be transient, it is necessary and sufficient that the system of equations

$$y_1=\sum_{j=1}^{\infty}p_{ij}y_j \quad i=1,2,\ldots\ldots \tag{3}$$

admits of a non-zero bounded solution, where P_{ij} is the probability of the transition from E_i to E_j.

In case of $N=5$, the above equations are

$$\begin{aligned}
y_i&=ay_i+ay_{i+1}+ay_{i+2}+ay_{i+3} && i=1\\
y_i&=2ay_i+ay_{i+1}+ay_{i+2} && i=2,3,4\\
y_i&=2ay_{i-2}+ay_{i-1}+2ay_i+ay_{i+2} && i=5,7,9\\
y_i&=2ay_{i-2}+2ay_i+ay_{i+1}+ay_{i+2} && i=6,8,10
\end{aligned} \tag{4}$$

(0, 0, 0): No short pieces
(0, 0, 1): One piece 4m long
(0, 1, 0): One piece 3m long
(1, 0, 0): One piece 2m long
(2, 0, 0): Two pieces 2m long and so forth.
⑤ : Transition by replacement of a 5m length cable.
$\underset{\textcircled{1}}{\longrightarrow}$: Transition by replacement of a 1m length cable.

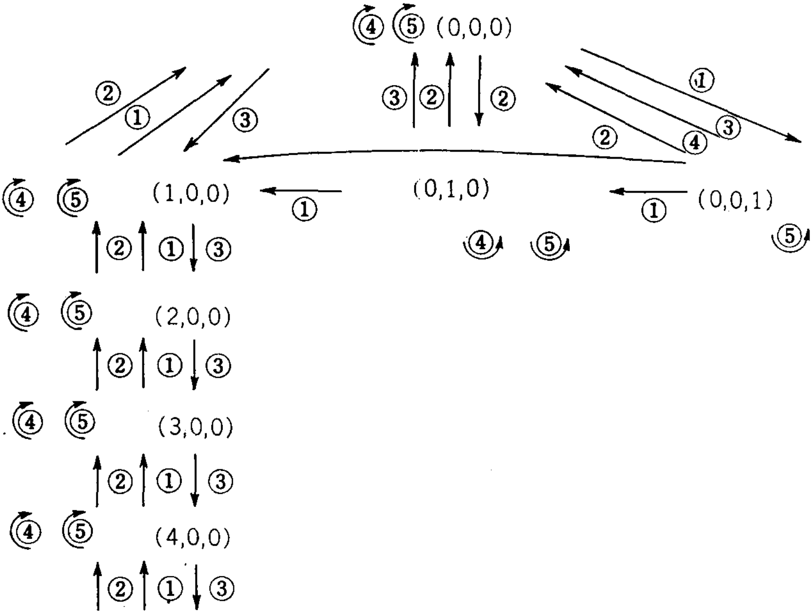


Fig. 1. Transition of States in the Case of $N=4$

The solution y_t that satisfies the above equations is only the case where it is equal to zero. Therefore it is clear that the chain is persistent. Then we have to decide whether the chain is null state or ergodic.

If the states are neither transient nor recurrent null (i. e. they are ergodic), then for every pair of states i and j

$$\lim_{n\to\infty} P_{ij}^{(n)}=\Pi_j \qquad (5)$$

and the Π_j satisfy the equations

$$\Pi_j=\sum_{i=0}^{\infty}\Pi_i P_{ij} \qquad (6)$$

where, Π_i, Π_j are the stationary probabilities of state i and j.

In this case, the state probabilities that satisfy eq. (6) are only those eqeal to zero. Therefore they are not ergodic but null states. Fig. 1 shows how the amount of short pieces continues to increase in the case of $N=4$.

In the case of $N=3$ or less the chain is ergodic. This is because the states are limited to the following three states,

$(0, 0) \quad (0, 1) \quad (1, 0)$

Therefore, the transition probability matrix is a finite as follows:

$$\begin{array}{c|ccc} (0,0) & (0,0) & (0,1) & (1,0) \\ \hline (0,0) & 2a & a & a \\ (0,1) & 2a & a & a \\ (1,0) & 2a & 0 & 2a \end{array} \tag{7}$$

The stationaly probabilities are obtained from eq. (6) and

$$\sum_j \Pi_j = 1 \tag{8}$$

Hence, $\Pi_0=\frac{1}{2},\ \Pi_1=\frac{1}{6},\ \Pi_2=\frac{1}{3}$

Thus, R, the average number of reels in stock is:

$$\mathrm{R}=1.5 \text{ reels} \tag{9}$$

where each short piece cable is wound on a separate reel.

E, the average total length of cables in stock is

$$\mathrm{E}=5.2 \text{ m} \tag{10}$$

The above (9) and (10) include one piece of $N+1=4$ m length. we will study the cases of $N=4$ or more, which are not ergodic, by the following simulation technique. (Fig. 2)

As it is clear from the example in Fig. 3 (a) (b), the results of simulation show that the number of reels and the amount of inventory in stock increase year by year in the case of $N+1=10$, 20, and furthermore, this trend increases as $N+1$ becomes larger.

3. ECONOMIC STUDY

A. The case of $N+1=4$ (the ergodic case):

In this case, we have to consider both the scrap loss and inventory cost.

(1) Scrap Loss:

Let's define the following notations.

C_1: Scrap loss (per meter)

This is the loss after recovery of copper and lead from scrapped cable pieces.

a: Scrap probability per trouble* $=\frac{1}{N+1}$

λ: Annual frequency rate of trouble.

r: Piece length to be scrapped.

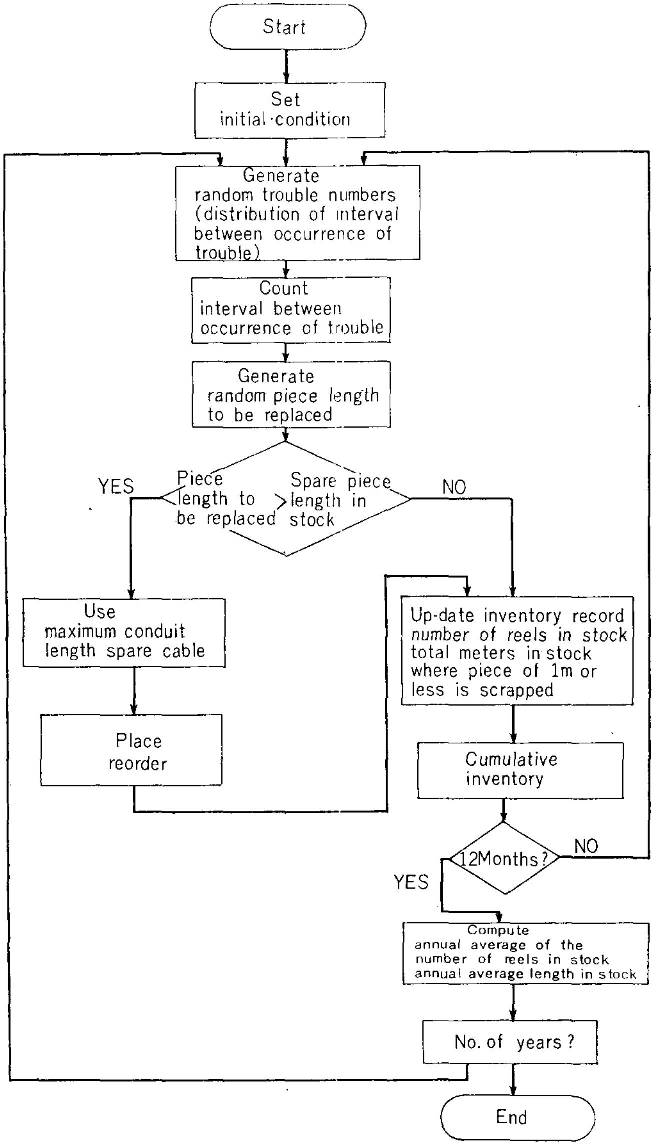


Fig. 2. Simulalion Flowchart

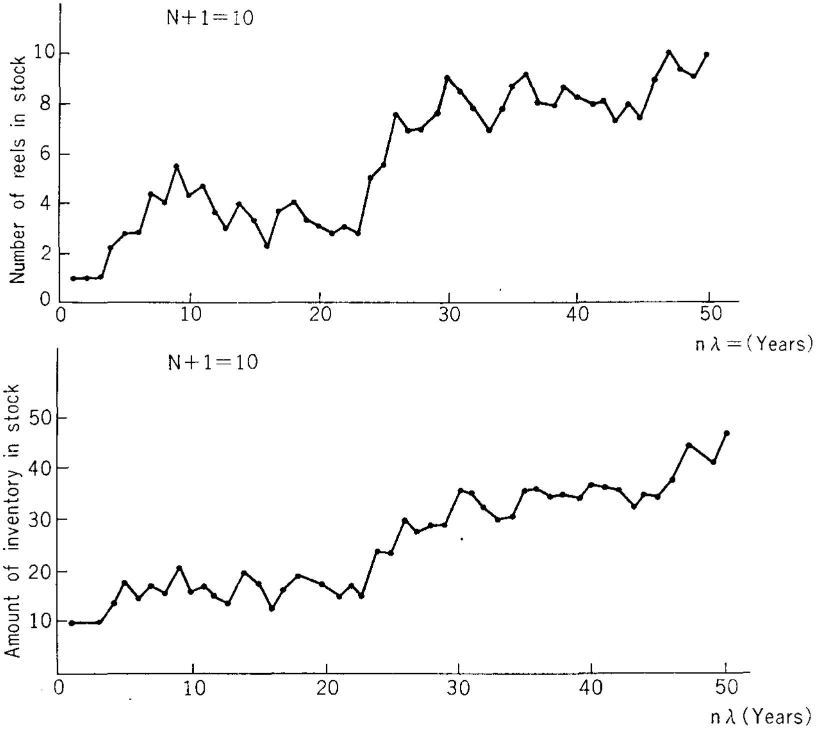


Fig. 3 (a)

Then, the scrap cost C_s is

$$C_s = C_1 r a \lambda \tag{11}$$

*According to the transition probability matrix (19), the scrap probability is a for any state.

(2) Inventory Cost:

Let's define the following notations

C_2: Inventory cost proportional to the number of reels (per reel)

C_3: Inventory cost proportional to the total length (per meter)

Then, the inventory cost C_I is

$$C_I = 1.5\, C_2 + 5.2\, C_3 \tag{12}$$

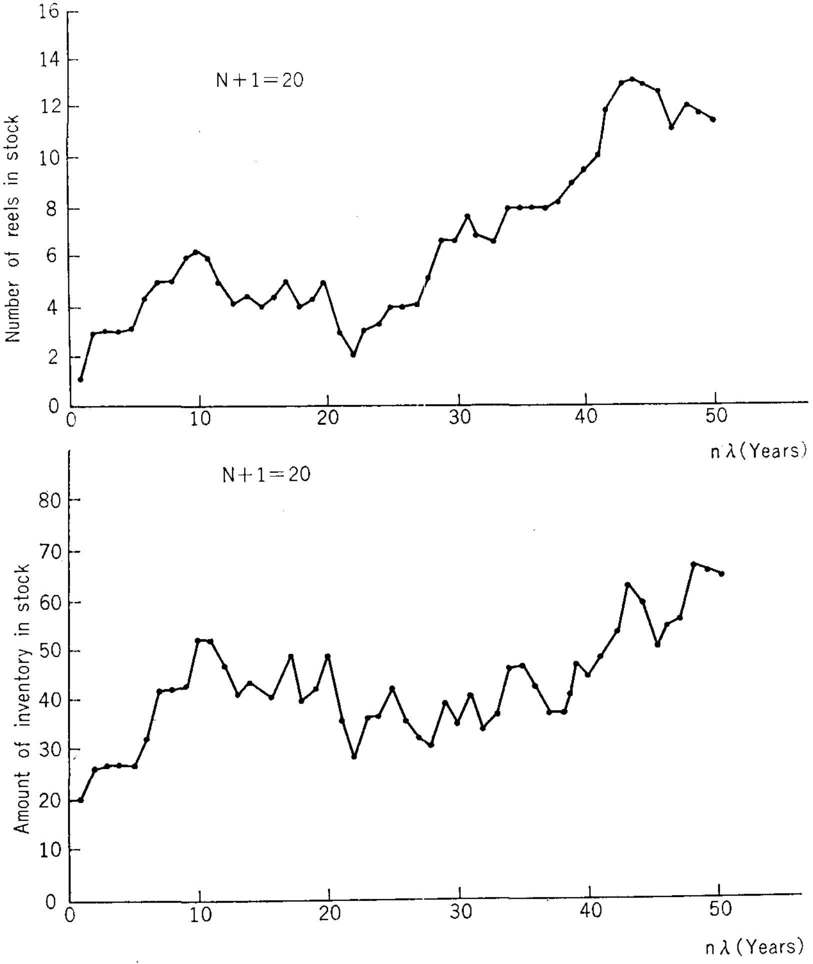


Fig. 3 (b)

Hence, the total cost in this case, $C(N+1=4)$ is

$$C(N+1=4)=C_s+C_I=C_1ra\lambda+1.5\,C_2+5.2\,C_3 \qquad (13)$$

B. The $N+1 \geqq 5$ case (the null state case):

In this case, as the number of reels and the amount of inventory in stock increase year by year, we scrap all short pieces in stock leaving only those of $N+1$ m after a given time.

(1) Scrap Loss

Annual scrap loss, Cas is

$$Cas = C_1 r 2a\lambda \tag{14}$$

where the scrap probability is $2a$ for each trouble *1.

The scrap loss Cns where all except those of $N+1$ m are scrapped every nλ th year, is

$$Cns = C_1 hn\lambda \tag{15}$$

where, h=coefficient of increase in inventory (meters per year).

(2) Inventory Cost:

The inventory cost C_{I} is

$$C_{\text{I}} = C_2(kn\lambda+1) + C_3(hn\lambda+N+1) \tag{16}$$

where, k=coefficient of increase in number of reels per year.

The total cost (annual cost), $C(N+1 \geqq 5)$ is given by *2

$C(N+1 \geqq 5) = Cas +$ annual cost of $(Cns + C_{\text{I}})$

$$= C_1 r 2a\lambda + \left\{ \int_0^{n\lambda} C_2(kt\lambda+1) + C_3(ht\lambda+N+1)] e^{-it} dt \right.$$

$$\left. + C_1 hn\lambda e^{-in\lambda} \right\} \frac{1}{1-e^{-in\lambda}} \cdot \frac{i}{1-e^{-in\lambda}}$$

$$= C_1 r 2a\lambda + \left[\frac{\kappa_1}{i}(\xi - n\lambda\chi) + k_2 \xi + c_1 hn\lambda\chi \right] \frac{1}{i\xi^2} \tag{17}$$

where,

$$\xi = \frac{1-e^{-i\lambda n}}{i}$$

$$\chi = e^{-i\lambda n}$$

$$\kappa_1 = C_2 k + C_3 h$$

$$\kappa_2 = C_2 + C_3(N+1)$$

*1 Transition probability of scrapping is $2a$, excluding the scrap at the early state of $(0, 0, \cdots\cdots, 0, 0)$ and $(0, 0, \ldots\ldots, 0, 1)$.

*2 The calculation of annual cost $(Cns + C_{\text{I}})$ is made as follows. Let's assume that all short pieces are scrapped at each $n\lambda$ th year, then, the present worth of $(Cns + C_{\text{I}})$ is obtained for the infinite future. Now let's put $\theta = (Cns + C_{\text{I}})$. Then,

The present worth of $(Cns + C_{\text{I}}) = \theta + \theta e^{-in\lambda} + \theta e^{-i2n\lambda} + \cdots = \dfrac{\theta}{1-e^{-in\lambda}}$. Hence,

The annual cost of $(Cns + C_{\text{I}}) = \dfrac{\theta}{1-e^{-in\lambda}} \cdot \dfrac{i}{1-e^{-in\lambda}}$

From eq. (17) the optimum scrapping period is obtained so as to satisfy the following condition.

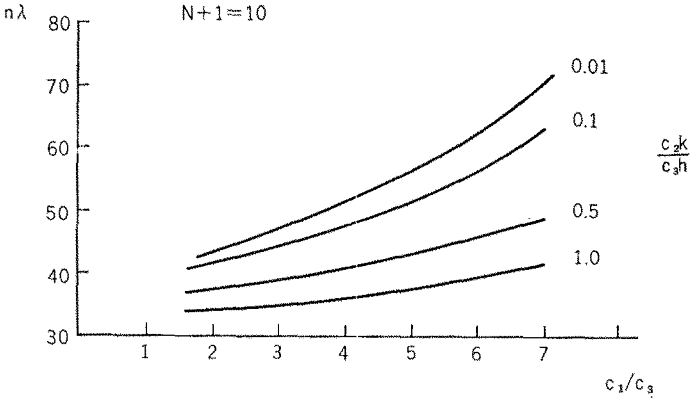


Fig. 4 (a)

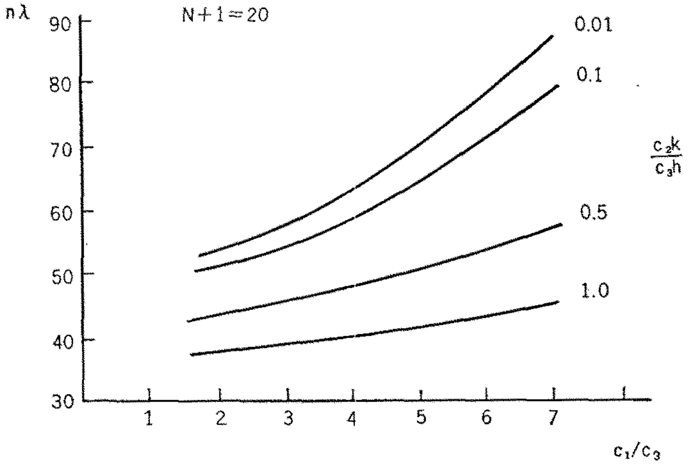


Fig. 4 (b)

$$n\lambda\left(\frac{1+\chi}{1-\chi}\right)=\frac{1}{i}\left(1-\frac{\eta_1}{\eta_2}\right) \qquad (18)$$

where,
$$\eta_1=\frac{\kappa_1}{i}+\kappa_2$$

$$\eta_2=c_1h-\frac{\kappa_1}{i}$$

The optimum scrapping period is shown in Fig. 4 (a), (b) using

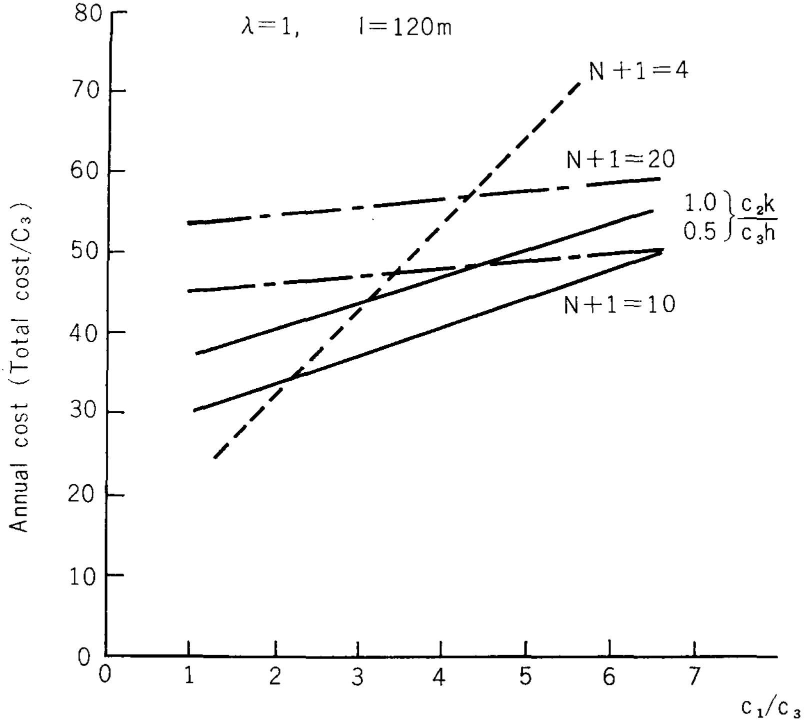


Fig. 5 (a)

eq. (18).

Let's compare the total cost for the case of scrapping after the optimum scrapping period with that for the case which meets the condition of $N+1=4$.

Fig. 5 (a), (b) shows the comparison of the costs under the condition of the maximum conduit length 1=120m, trouble frequency rates $\lambda=1,5$.

Actually, it can be assumed that c_2k/c_3h is 0.5~1.0 and c_1/c_3 is about 3. Therefore not much difference exists between both cases with respect to the total cost comparison.

In the case of $N+1=4$, when λ becomes large, the cost becomes slightly higher. In this case, however, we have a large intangible merit such as the convenience of handling in the storeroom and the saving of

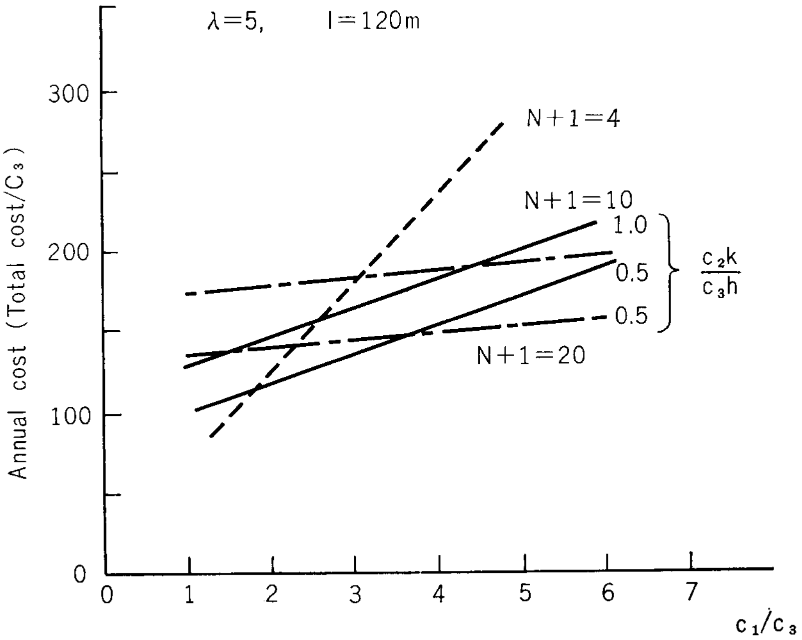


Fig. 5 (b)

storeroom space. Therefore, we come to the conclusion that it is better to meet the condition for $N+1=4$ by taking a large scrapping length.

In the above discussion, we have been considering the quantized length, so to satisfy the condition of $N+1=4$. Actually we have to set the scrapping criterion as one half of the maximum conduit length. with this criterion, the average scrapped length becomes 3/8 of the maximum conduit length.

The above economic comparison has been done under this actual condition.

4. AVERAGE NUMBER OF STEPS TILL SCRAPPING

In the ergodic case, we have to estimate the holding time of a maximum conduit length spare cable from the time of stocking till scrapping This holding time is the same as the period represented by the period represented by the average number of steps, that is the fre-

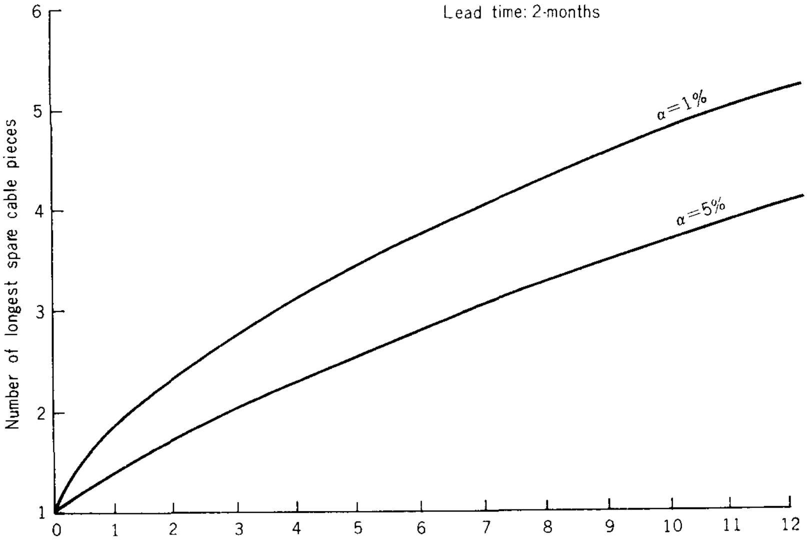


Fig. 6

quency of trouble and replacement from the time of being stocked to the time of scrapping.

In case of $N+1=4$, including the absorbing state, four states may be considered for the transition matrix (7) which are (0, 0, 0) (0, 0, 1) (0, 1, 0) (1, 0, 0)

The transition matrix in this case is given by

$$\begin{array}{c|cccc} & (0,0,0) & (0,0,1) & (0,1,0) & (1,0,0) \\ \hline (0,0,0) & a & a & a & a \\ (0,0,1) & a & a & a & a \\ (0,1,0) & a & 0 & 2a & a \\ (1,0,0) & 0 & 0 & 0 & 1 \end{array} \tag{19}$$

For such an absorbing Markovch ain, Kemeny and Snell have given the following theorems.

Theorom 4. 1 Let $M(n)$ be the average number of steps that the process move from state E_i to state E_j, then

$$M(n)=N \tag{20}$$

where $$E_i,\ E_j \in T$$

T: set of transient state

N: $N=(I-Q)^{-1}$ is the fundamental matrix for an absorbing Markov chain, where Q is a submatrix of the trsnsient state.

Theorem 4.2 Let $M(t)$ be the average number of steps that the process is in a transient state, then

$$M(t)=N\zeta \tag{21}$$

where ζ: column vector with all entries 1

From the above theorems we obtain that $M(t)$ is 4 steps, that is, srapping is done after 4 troubles on the average.

5. THE NUMBER OF THE LONGEST SPARE CABLE PIECES REQUIRED FOR STOCK

The number of the longest spare cable pieces should be decided by the frequency of troubles that will happen during the lead time. If a shortage of spare cables occur, it will bring a large economical loss. However, since it is actually difficult to evaluate this monetary loss, we decide the number of longest spare pieces so as to make the shortage rate small by using the knowledge that troubles are subject to the poisson distribution as follows:

$$\sum_{\nu}^{s}\frac{(T\lambda P_c)^{\nu}e^{-(T\lambda P_c)}}{\nu}=1-\alpha \tag{22}$$

where S: number of pieces in stock.

T: lead time.

P_C: probability of using the longest spare cable.

α: shortage rate.

Fig. 6 shows the number of longest spare cable pieces obtained from eq. (22) for each trouble frequency rate under a 2-months lead time and 1 and 5 percent shortage rate.

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The Diagram of Mutual Relations