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BATCH-ARRIVAL QUEUEING PROBLEM

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I. INTRORUCTION

In the recent years, the study of the queueing system with batch arrival was done by R. G. Miller [2] and F. G. Foster [3]. R. G. Miller formulated two general models in which batches arrive at a single server system and are served in batches, the size of both the "arrival" batches and the "service" batches being random variables with known probability distributions, and obtained explicit results only for the type M/G/1 in the case "Model I". F. G. Foster has made an investigation of the queueing system of the type GI/M/1, the size of arrival batch being a known constant and the units of the batch are served individually. If in [3] we think of the size of the batch as a random variable then we have a generalization of the case of unit arrivals as treated in [1] and of batch arrivals in [3]. In this paper we deal with such a general system. This generalization will be useful in many practical applications. Since our system is analyzed with reference to the imbedded Markov chain, we first consider an irreducible Markov chain associated with such a queueing system. By the properties of the chain we obtain limiting distributions of the queue size and of the waiting time for an arbitrary unit.

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II. THE ASSOCIATED MARKOV CHAIN

We consider a Markov chain $\{\xi_n, n=0, 1, 2, \ldots\}$ defined on the space

$\{0, 1, 2,\ldots\}$. Let r be an integer, $r\geqq 1$ and let $\{a_n, n=0, 1, 2,\ldots\}$ be a sequence satisfying the conditions

$$a_n>0 \text{ for all } n \text{ and } \sum_{n=0}^{\infty} a_n=1.$$

We assume $P\{\xi_0=k\}=1$ and we define the transition probability matrix $P=(p_{ij})$ as follows:

$$p_{ij}=\begin{cases}\alpha_{r+i-1} & (j=0)\\ a_{r+i-j} & (j\geqq 1)\end{cases} \tag{1}$$

where we put $a_n\equiv 0$ $(n<0)$ and $\alpha_{r+i-1}=1-\sum_{n=0}^{r+i-1} a_n$.

Let $A(z)=\sum_{n=1}^{\infty} a_n z^n$ for $|z|\leqq 1$.

Lemma 1. If (a) $|w|<1$ or (b) $|w|\leqq 1$, $\sum_{n=0}^{\infty} na_n>r$, then the equation

$$z^r=wA(z) \tag{2}$$

has exactly r roots $z=\gamma_n(w)$ $(n=1, 2,\ldots\ldots,)$ in the unit circle $|z|<1$.

Proof: In case (a), we have

$$|wA(z)|<1 \text{ if } |z|=1.$$

Therefore, by Rouche's theorem, the equation (2) has exactly r roots in the circle $|z|<1-\varepsilon$ where ε is a sufficiently small positive number. In case (b), $A(1-\varepsilon)$ and $(1-\varepsilon)^r$ are monotone decreasing functions of ε $(0\leqq\varepsilon\leqq 1)$, they agree at $\varepsilon=0$ and their derivatives at $\varepsilon=0$ are $-\sum_{n=1}^{\infty} na_n$ and $-r$ respectively. Hence if $\sum_{n=1}^{\infty} na_n>r$, we have $A(1-\varepsilon)<(1-\varepsilon)^r$ if ε is a sufficiently small positive number. Therefore, we have

$$|wA(z)|\leqq|A(z)|\leqq\sum_{n=0}^{\infty} a_n|z|^n<(1-\varepsilon)^r=|z|^r$$

if $|z|=1-\varepsilon$ and ε is small enough. Also, by Rouche's theorem, the equation (2) has exactly r roots in the circle $|z|<1-\varepsilon$. Let us denote these roots by $\gamma_n(w)$ $(n=1, 2,\ldots\ldots, r)$.

Lemma 2. If $|w|<1$, then

$$\sum_{n=0}^{\infty} p_{00}^{(n)} w^n=\frac{1}{1-w}\prod_{n=1}^{r}(1-\gamma_n(w)). \tag{3}$$

Proof: We have

$$p_{ik}^{(n+1)}=\sum_{j=0}^{\infty}p_{ij}p_{jk}^{(n)}$$
$$=\sum_{j=0}^{r+i-1}a_j p_{r+i-j,k}^{(n)}+\alpha_{r+i-1}p_{0,k}^{(n)},\ (n\geqq 0). \tag{4}$$

Now let k be fixed and for $|z|\leqq 1$ introduce the generating function

$$U_n(z)=\sum_{i=0}^{\infty}p_{ik}^{(n)}z^i,\ (n\geqq 0) \tag{5}$$

and let

$$A_n(z)=\sum_{i=0}^{n}a_i z^i,\ (n=0,\ 1,\ 2,\dots\dots,\ r-2)$$

then multiplying (4) by z^i and summing it from $i=0$ to $i=\infty$, we get, after simplification,

$$z^r U_{n+1}(z)-A(z)U_n(z)=p_{0,k}^{(n)}\{\alpha_{r-2}z^r+zA_{r-2}(z)-A(z)\}/(1-z)$$
$$-p_{1,k}^{(n)}zA_{r-2}(z)-p_{2,k}^{(n)}z^2A_{r-3}(z)-\dots-p_{r-1,k}^{(n)}z^{r-1}A_0(z),\ (n\geqq 0) \tag{6}$$

where $U_0(z)=z^k$, $A_{-1}=0$ and $\alpha_{-1}=1$.

Further if we introduce the generating function

$$\Omega(z,\ w)=\sum_{n=0}^{\infty}U_n(z)w^n=\sum_{n=0}^{\infty}\sum_{i=0}^{\infty}p_{ik}^{(n)}z^i w^n \tag{7}$$

which is convergent if $|z|\leqq 1$ and $|w|<1$, by (6) we obtain that

$$\Omega(z,\ w)=\frac{1}{(z^r-wA(z))}\Bigg[z^{r+k}+w\Bigg\{\frac{\alpha_{r-2}z^r+zA_{r-2}(z)-A(z)}{1-z}B_0^{(k)}(w)$$
$$-zA_{r-2}(z)B_1^{(k)}(w)-z^2A_{r-3}(z)B_2^{(k)}(w)-\dots-z^{r-1}A_0(z)B_{r-1}^{(k)}(w)\Bigg\}\Bigg] \tag{8}$$

where $B_i^{(k)}(w)=\sum_{n=0}^{\infty}p_{ik}^{(n)}w^n$ $(i=0,\ 1,\ 2,\dots\dots,\ r-1)$.

Here $\Omega(z,\ w)$ is a regular function of z if $|z|\leqq 1$ and $|w|<1$. By Lemma 1. (a), the denominator of (8) has exactly r roots $z=\gamma_i(w)$ $(i=1,\ 2,\dots\dots,\ r)$ in the unit circle $|z|<1$. These must also be the roots of the numerator. Therefore we have the relations

$$\{(w\alpha_{r-2}-1)\gamma_i^r(w)+wA_{r-2}(\gamma_i(w))\gamma_i(w)\}B_0^{(k)}(w)$$
$$-wA_{r-2}(\gamma(w))\gamma_i(w)(1-\gamma_i(w))B_1^{(k)}(w)$$
$$-wA_{r-3}(\gamma_i(w))\gamma_i^2(w)(1-\gamma_i(w))B_2^{(k)}(w)$$
$$\vdots$$
$$-wA_0\gamma_i^{r-1}(w)(1-\gamma_i(w))B_{r-1}^{(k)}(w)$$
$$=-\gamma_i^{r+1}(w)(1-\gamma_i(w)),\ (i=1,\ 2,\dots\dots,\ r) \tag{9}$$

Since

$$\Delta(w)=\begin{vmatrix}(w\alpha_{r-2}-1)\gamma_1{}^r+wA_{r-2}(\gamma_1)\gamma_1 & -wA_{r-2}(\gamma_1)\gamma_1(1-\gamma_1)\cdots & -wA_0\gamma_1{}^{r-1}(1-\gamma_1)\\ \vdots & \vdots & \vdots\\ (w\alpha_{r-2}-1)\gamma_r{}^r+wA_{r-2}(\gamma_r)\gamma_r & -wA_{r-2}(\gamma_r)\gamma_r(1-\gamma_r)\cdots & -wA_0\gamma_r{}^{r-1}(1-\gamma_r)\end{vmatrix}$$

$$=A_0{}^{r-1}(w-1)\Delta^*(w)\prod_{n=1}^{r}\gamma_n(w) \qquad (10)$$

where

$$\Delta^*(w)=\begin{vmatrix}1 & \gamma_1(w)\cdots\cdots\cdots & \gamma_1{}^{r-1}(w)\\ \vdots & \vdots & \vdots\\ 1 & \gamma_r(w)\cdots\cdots\cdots & \gamma_r{}^{r-1}(w)\end{vmatrix} \qquad (11)$$

$B_i{}^{(k)}(w)$ ($i=0, 1, 2,\ldots\ldots, r-1$) will be determined from the relations (9). Now to obtain $B_0{}^{(0)}(w)=\sum_{n=0}^{\infty}p_{00}^{(n)}w^n$, we consider the special case with $i=k=0$ in the relations (9). Then we have $B_0{}^{(0)}(w)=\Delta_0(w)/\Delta(w)$, where

$$\Delta_0(w)=\begin{vmatrix}-\gamma^1{}_r(1-\gamma_1) & -wA_{r-2}(\gamma_1)\gamma_1(1-\gamma_1)\cdots\cdots\cdots & -wA_0\gamma_1{}^{r-1}(1-\gamma_1)\\ \vdots & \vdots & \vdots\\ -\gamma_r{}^r(1-\gamma_r) & -wA_{r-2}(\gamma_r)\gamma_r(1-\gamma_r)\cdots\cdots\cdots & -wA_0\gamma_r{}^{r-1}(1-\gamma_r)\end{vmatrix}$$

$$=-A_0{}^{r-1}w^{r-1}\Delta^*(w)\prod_{n=1}^{r}\gamma_n(w)(1-\gamma_n(w)), \qquad (12)$$

that is,

$$B_0{}^{(0)}(w)=\sum_{n=0}^{\infty}p_{00}^{(n)}w_n=\frac{1}{1-w}\sum_{n=1}^{r}(1-\gamma_n(w)).$$

Let $f_0{}^{(n)}$ be the probabilty that starting from state 0 the system returns to the state 0 for the first time at the nth step.

Theorem 1. For $|w|<1$

$$\sum_{n=1}^{\infty}f_0{}^{(n)}w^n=\frac{\prod_{n=1}^{r}(1-\gamma_n(w))+w-1}{\prod_{n=1}^{r}(1-\gamma_n(w))} \qquad (13)$$

Proof: We have that

$$p_{00}^{(n)}=f_0{}^{(n)}+f_0{}^{(n-1)}p_{00}^{(1)}+\ldots+f_0{}^{(1)}p_{00}^{(n-1)},\ (n\geqq 1). \qquad (14)$$

Forming the generating function we get by (14) that

$$\sum_{n=1}^{\infty} f_0^{(n)} w^n = \sum_{n=1}^{\infty} p_{00}^{(n)} w^n / \sum_{n=0}^{\infty} p_{00}^{(n)} w^n.$$

By Lemma 2. We have the theorem.

Our Markov chain is clearly irreducible and aperiodic, then all states in the chain are recurrent nonnull if $f_0 = \sum_{n=1}^{\infty} f_0^{(n)} = 1$ and $\mu_0 = \sum_{n=1}^{\infty} n \cdot f_0^{(n)} < \infty$. Now, from Lemma 1 (6) and Theorem 1, it follows that

$$f_0 = \lim_{w \to 1} \sum_{n=1}^{\infty} f_0^{(n)} w^n = 1 \quad \text{if} \quad \sum_{n=1}^{\infty} n a_n > r$$

and

$$\mu_0 = \lim_{w \to 1} \left(\frac{d}{dw} \sum_{n=1}^{\infty} f_0^{(n)} w^n \right) = 1 / \sum_{n=1}^{r} (1 - \gamma_n(1)) \quad \text{if} \quad \sum_{n=1}^{\infty} n a_n > r.$$

Hence the states of the Markov chain $\{\xi_n\}$ are recurrent nonull states if $\sum_{n=1}^{\infty} n a_n > r$.

III. QUEUE WITH BATCH ARRIVAL

We consider the following single server queueing system:

(i) Batches arrive at the sequence of instants $\tau_1, \tau_2, \ldots, \tau_n, \ldots$ where the process $\{\tau_n\}$ is a renewal prosess, that is the inter-arrival periods $\tau_{n+1} - \tau_n > 0$ ($n = 1, 2, \ldots$) are independently and identically distributed random variables with common distribution $F(x)$. The size N of an arrival batch is a random variable with distribution $c_i = P\{N = i\}$ ($i = 1, 2, \ldots\ldots, r$). We can assume $c_r > 0$.

Let $\varphi(s)$ denote the Laplace transform of $F(x)$, that is

$$\varphi(s) = \int_0^{\infty} e^{-sx} dF(x), \quad (R(s) \geqq 0)$$

and let us assume that the expectation

$$\frac{1}{\lambda} = \int_0^{\infty} x dF(x)$$

exists.

(ii) Units are ssrved individually by a single server. Since the units of a batch arrive simultaneously, we shall suppose that they are ordered for purpose of service. Batches are served in order of arrival. Let χ_n be the service time of the nth unit to be served. We assume the $\{\chi_n\}$ is a sequence of identically distributed independent non-negative random variables, independent also of the sequence $\{\tau_n\}$, and that their common distribution function, $H(x)$, is the exponential distribution:

$$H(x)=P\{\chi_n \leqq x\}=1-e^{-\mu x}, \ (x\geqq 0)$$

Then $\frac{1}{\mu}=\int_0^\infty x dH(x)$. We define $\rho=\lambda/\mu$.

If a batch arrives to find the server idle, then the unit of the batch receive service at once, if the batch arrives to find the server busy, then the batch joins the queue.

Let $\xi(t)$ be the number of unite in the system, including the one being served, at the instant t and let $\xi_n=\xi(\tau_n-0)$, $(n=1, 2, \ldots)$. We wish to determine the limiting distributions,

$$p_j=\lim_{n\to\infty} P\{\xi_n=j\}$$

and

$$\pi_j=\lim_{t\to\infty} P\{\xi(t)=j\}.$$

Let $\{\nu_n, n=1, 2, \ldots\}$ be a sequence of identically distributed independent random variables with distribution

$$k_j=P\{\nu_n=j\}=\int_0^\infty e^{-\mu x}\frac{(\mu x)^j}{j!}dF(x), \ (j=0, 1, 2, \ldots).$$

It is easy to see that

$$\xi_{n+1}=[\xi_n+(\text{the size of a batch arrived at instant } \tau_n)-\nu_n]^+, \quad (15)$$

where $[a]^+$ means the positive part of a, i. e., $[a]^+=\max(a, 0)$.

Put $K(z)=\sum_{j=0}^\infty k_j z^j$ for $|z|\leqq 1$, then $K(z)=\varphi[\mu(1-z)]$. By virtue of the characteristic property of the expontial distribution, the sequence of random variables $\{\xi_n\}$ forms a Markov chain. We obtain that the matrix of transition probabilities (p_{ij}) has the following form:

$$p_{ij}=\begin{cases}\alpha_{r+i-1} & (j=0)\\ a_{r+i-j} & (j\geqq 1)\end{cases}$$

where $$a_{r+i-j}=\begin{cases}c_r k_{r+i-j}+c_{r-1}k_{r+i-j-1}+\cdots+c_1 k_{i-j+1} & (j\leqq i+1)\\ c_r k_{r+i-j}+c_{r-1}k_{r+i-j-1}+\cdots+c_{j-i}k_0 & (j>i+1)\end{cases}$$

and $$\alpha_{r+i-1}=1-\sum_{n=0}^{r+i-1} a_n.$$

Since $k_j>0$ for all j and $c_r>0$, $\{a_n\}$ satisfies the condition

$$a_n>0 \text{ for all } n \text{ and } \sum_{n=0}^\infty a_n=1.$$

Using the notations of the preceding section, we have

$$A(z)=\varphi[\mu(1-z)](c_1z^{r-1}+c_2z^{r-2}+\cdots+c_{r-1}z+c_r).$$

Then if $\sum_{n=1}^{\infty} na_n>r$, that is $\rho\sum_{n=1}^{r} nc_n<1$, the equation

$$K(z)\sum_{n=1}^{r} c_n z^{r-n}=z^r$$

has exactly r roots (distinct or coincident) inside the circle $|z|=1$. Denote the roots by $z=\gamma_n^*(n=1, 2,\ldots r)$.

From the results of the preceding section, it is clear that the distribution $\{p_j\}$ exists and is independent of the initial state of the system if $\rho\sum_{n=1}^{r} nc_n<1$. For the relation between the two distributions $\{p_j\}$ and $\{\pi_j\}$ we shall eximine at the end of this section. Define $P(z)=\sum_{n=1}^{\infty} p_n z^n$ for $|z|\leqq 1$.

Theorem 2.

$$P(z)=\prod_{n=1}^{r}\frac{1-\gamma_n^*}{1-\gamma_n^* z} \qquad (16)$$

Proof: The random variable, the right-hand side of (15), has, in the limit as $n\rightarrow$, the generating function,

$$P(z)K(z^{-1})C(z)+\sum_{n=0}^{\infty} b_n(1-z^{-n})$$

where $C(z)=\sum_{n=1}^{r} c_n z^n$ and $\{b_n, n=0, 1, 2,\ldots\}$ is a sequence of real non-negative constants for which $\sum_{n=0}^{\infty} b_n=p_0$. More precisely, $b_n=\sum_{i=0}^{\infty}\sum_{j=1}^{r} p_i c_j k_{i+j+n}$, $(n=0, 1, 2,\ldots)$.
Therefore

$$P(z)=\sum_{n=0}^{\infty} b_n(1-z^{-n})/\{1-C(z)K(z^{-1})\}.$$

We now have to determine the constants b_n, and for this purpose we consider the denominator. Clearly, these are exactly r roots outside the unit circle, and these are $1/\gamma_n^*(n=1, 2,\ldots, r)$.

Since $P(z)$ is a regular function of z for $|z|\leqq 1$, the function

$$D(z)=P(z)\prod_{n=1}^{r}(1-\gamma_n^* z)$$

must be regular for $|z|\leqq 1$. Now let $D(z)$ be defined for $|z|>1$. by

$$D(z)=\sum_{n=0}^{\infty} b_n(1-z^{-n})\prod_{n=1}^{r}(1-\gamma_n{}^*z)/\{1-C(z)K(z^{-1})\}.$$

Since in this expression all the roots of the denominator outside the unit circle $|z|=1$ are also roots of the numerator, it follows that $D(z)$ is regular for $|z|>1$. Therefore, by analytic continuity, $D(z)$ is defined and regular for all z. Since, moreover, $D(z)=O(|z|)$ as $z\rightarrow\infty$, it follows that $D(z)=D$, a constant independent of z. Therefore

$$P(z)=D/\sum_{n=1}^{r}(1-\gamma_n{}^*z)$$

and from $P(1)=1$ we obtain (16).

Example 1. We suppose

$$F(x)=1-e^{-\lambda x},\ (x\geqq 0).$$

Then $\varphi(s)=\lambda/(\lambda+s)$, Therefore $K(z)=\varphi[\mu(1-z)]=\rho/(\rho+1-z)$. Hence the r roots $\gamma_n{}^*$ are the roots of

$$(c_1z^{r-1}+\cdots+c_r)\frac{\rho}{\rho+1-z}=z^r,$$

inside the unit circle $|z|=1$. This equation can be written,

$$(1-z)z^r+\rho(z^r-c_1z^{r-1}-\cdots-c_r)=0 \tag{17}$$

which, being a polynominal equation of degree $r+1$, has exactly $r+1$ roots, and one of them is $z=1$. Now if we consider the expression

$$(1-z)\prod_{n=1}^{r}(1-\gamma^*z)$$

the roots of this expression are 1, $\gamma_1{}^{*-1}$, $\gamma_2{}^{*-1},\ldots,\gamma_r{}^{*-1}$. But since these are the reciprocals of the roots of equatioh (17), it follows that they are the roots of the equation

$$(1-z^{-1})z^r+\rho(z^{-r}-c_1z^{-(r-1)}-\cdots-c_r)=0,$$

that is,

$$1-z\{1+\rho(1-C(z))\}=0. \tag{18}$$

Thus we have the identity

$$(1-z)\prod_{n=1}^{r}(1-\gamma_n{}^*z)=1-z\{1+\rho(1-C(z))\}.$$

From this we have

$$P(z)=(1-z)\prod_{n=1}^{r}(1-\gamma_n{}^*)/[1-z\{1+\rho(1-C(z))\}].$$

Letting $z\to 1-0$ we obtain $\prod_{n=1}^{r}(1-\gamma_n^*)=1-\rho\sum_{n=1}^{r}nc_n$. Thus, in this case we have

$$P(z)=(1-z)\left(1-\rho\sum_{n=1}^{r}nc_n\right)/[1-z\{1+\rho(1-C(z))\}], \qquad (19)$$

so that the generating function $P(z)$ can be expressed in a form not explicitly involving the roots γ_n^*.

Example 2. We suppose

$$F(x)=1-\sum_{j=0}^{k-1}\frac{(\lambda kx)^j}{j!}e^{-\lambda kx},\ (x\geqq 0).$$

Then $\varphi(s)=\{\lambda k/(\lambda k+s)\}^k$ and $K(z)=\{\rho/(\rho+(1-z)k^{-1})\}^k$. The roots γ_n^* are the roots of the equation

$$\rho^k(c_1z^{r-1}+\cdots+cr)=z^r\{\rho+(1-z)k^{-1}\}^k \qquad (20)$$

inside the unit circle $|z|=1$. This equation, being a polynomial equation of degree $r+k$, has $r+k$ roots. It is clear that $r+1$ of them are 1, γ_1^*, γ_2^*, $\gamma_3^*,\cdots\gamma_x$. Let us denote the other $k-1$ roots lying outside the circle $|z|=1$ by

$$\beta_1,\ \beta_2,\ldots,\ \beta_{k-1}.$$

Now the equation whose roots are the reciprocals of the roots of (20) is

$$\rho^k(c_1z^{-(r-1)}+\cdots+c_r)=z^{-r}\{\rho+(1-z^{-1})k^{-1}\}^k,$$

that is,

$$(\rho k)^k(c_1z^{k+1}+\cdots+c_rz^{k+r})-\{\rho k+1)z-1\}^k=0.$$

Therefore we have

$$(-1)^{k+1}(1-z)\prod_{n=1}^{r}(1-\gamma_n^*z)\prod_{n=1}^{k-1}(1-\beta_nz)$$
$$=(\rho k)^kz^kC(z)-1\{(\rho k+1)z-1\}^k.$$

From this we have

$$P(z)=\frac{(-1)^{k+1}(1-z)\prod_{n=1}^{k-1}(1-\beta_nz)\prod_{n=1}^{r}(1-\gamma_n^*)}{(\rho kz)^kC(z)-\{(\rho k+1)z-1\}^k}.$$

Letting $z\to 1-0$ we obtain

$$\prod_{n=1}^{r}(1-\gamma_n^*)=(\rho k)^{k-1}k\left(1-\rho\sum_{n=1}^{r}nc_n\right)/(-1)^{k+1}\sum_{n=1}^{k-1}(1-\beta_n),$$

so that finally

$$P(z)=\frac{(\rho k)^{k-1}k(1-z)\left(1-\rho\sum_{n=1}^{r}nc_n\right)}{(\rho kz)^kC(z)-\{(\rho k+1)z-1\}^k}\prod_{n=1}^{k-1}\frac{1-\beta_n z}{1-\beta_n}. \tag{21}$$

Thus in the case of Erlang inter-arrival times, the generating function $P(z)$ can be expressed in a form which involves explicitly only the roots β_n of (20), lying outside the unit circle.

We define a process $Z(t)$ on the state space $\{0, 1, 2,\ldots\}$, as having transitions only at the sequence of instants $\tau_1, \tau_2,\ldots$. Between transitions, the value of $Z(t)$ is the value of $\xi(t)$ at the last Z transitions and $Z(\tau_n+0)=\xi(\tau_n-0)$. Let $\delta(t)$ be the time since the last transition of the Z-process and let

$$\begin{aligned}G(\delta, n)&=\lim_{t\to\infty}P\{\delta(t)\leqq\delta,\ Z(t)=n\}\\&=\lambda\int_0^\delta\{1-F(x)\}dx\lim_{t\to\infty}P\{Z(t)=n\}\\&=\lambda p_n\int_0^\delta\{1-F(x)\}dx.\end{aligned}$$

Therefore we have

$$\begin{aligned}\pi_n&=\lim_{t\to\infty}P\{\xi(t)=n\}\\&=\sum_{i=0}^{\infty}\int_0^\infty P\{\text{a batch of an arbitrary size } k \text{ arrives and } k+i-n\\&\qquad\text{units depart in time } \delta\}d_\delta G(\delta, i).\end{aligned}$$

IV. THE WAITING TIME DISTRIBIBUTION

In this section we assume $\rho\sum_{n=1}^{r}nc_n<1$. We consider first the waiting time of the first unit in an arbitrary arriving batch. The waiting time is defined as the time which elapses between the instant at which the unit arrives and the instant at which its service begins. Let $\eta(t)$ be the waiting time which the unit would have if it arrived at the instant t, and define $\eta_n=\eta(\tau_n-0)$. Thus η_n is the waiting time of the first unit in the nth arriving batch. We considr the limiting distriribution,

$$W(x)=\lim_{n\to\infty}P\{\eta_n\leqq x\}.$$

We define $\Omega(s)=\int_0^\infty e^{-sx}dW(x)$ for $R(s)\geqq 0$.

Theorem 3.

$$\Omega(s)=\prod_{n=1}^{r}\frac{1-\gamma_n^*}{1-\frac{\mu\gamma_n^*}{\mu+s}} \tag{22}$$

Proof: Each unit has a service time whose distribution is exponetial with Laplace transform $\mu/(\mu+s)$. From the characteristic property of the exponetial distribution, we can suppose that the service time of the unit at the head of the queue recommences at the instant of the arrival of a batch. If an arriving batch finds n units in the system, the waiting time of the first unit in the batch will have Laplace transform $\{\mu/(\mu+s)\}^n$.

The stationary probability of n units in the system is p_n. Using Theorem 2. we have

$$\Omega(s)=\sum_{n=0}^{\infty}p_n\left(\frac{\mu}{\mu+s}\right)^n=P\left(\frac{\mu}{\mu+s}\right)$$

which gives the required result.

Corollary. The Laplace transform of the waiting time distribution of a random unit in a batch is

$$\sum_{n=1}^{r}\sum_{m=0}^{n-1}\frac{c_n}{n}\left(\frac{\mu}{\mu+s}\right)^m\prod_{n=1}^{r}\frac{1-\gamma_n^*}{1-\frac{\mu\gamma_n^*}{\mu+s}}. \tag{23}$$

Example 3. We consider the case in which the inter-arrival times are exponentially distributed. Then

$$\Omega(s)=P\left(\frac{\mu}{\mu+s}\right)=\left(1-\rho\sum_{n=1}^{r}nc_n\right)\Big/\left\{1-\frac{\lambda}{s}\left(1-\sum_{n=1}^{r}c_n\left(\frac{\mu}{\mu+s}\right)^n\right)\right\}. \tag{24}$$

$f_0^{(n)}$ in the section I is the probability that a busy period consists of services of n arrival batches. Then the mean number of units being served in the mean busy period is

$$\mu_0\sum_{n=1}^{r}nc_n=\frac{\sum_{n=1}^{r}nc_n}{\prod_{n=1}^{r}(1-\gamma_n^*)}. \tag{25}$$

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