ON A LONG-TERM CONTRACT PROBLEM

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1. SUMMARY

This study deals with a method to determine the optimal policy about the term of contract over a given horizon under a fluctuating market whose behaviors are known probabilistically. We define policy as that which gives the optimal term of contract corresponding to the latest information about the market. In establishing contract policy over a planning horizon, uncertainty of the market from which services or commodities are obtained is the most influential factor to be taken into consideration.

In this paper, the uncertainty will be treated as a Markov process and the dynamic programming approach will take an essential part in deriving a contract policy, which will be compared with other policies that will seem rather intuitive.

2. MARKET SITUATION

Market situation involved in this problem is characterized by the following properties.

(1) Contract is to be renewed at the end of a specific time interval immediately when the previous contract is expired but before the following state of the market is not informed, and the term of contract is limited in its length.

(2) The market is classified as strong or as weak through each time interval and the strong market over a time interval will be followed by the strong market over the following time interval with certain probabilities that are functions of the duration of the strong market up until the time interval, and this is so for the weak market.

(3) The prices of the service or the commodity obtained from the market over a certain specific time interval are the functions of the market situation (strong or weak) and of the term of contract that

comes to be valid from that specific time interval.

(4) The amount of the service or the commodity required is constant over the planning horizon and no other sudden changes in demand and supply relations will be taken into account.

We designate the market situation at the beginning of the k-th time interval by s_k which will be state variables in dynamic programming formulation presented in later sections. The above property (2) states that any s_k is represented by the two elements i_k and m_k, and is denoted as

$$s_k=\{i_k|m_k\} \tag{1}$$

where, at the beginning of the k-th time interval

$$i_k=\begin{cases}0 & \text{for weak market}\\ 1 & \text{for strong market}\end{cases}$$

and,

m_k=the duration of $i_k(=0 \text{ or } 1)$ up until the end of the k-th time interval.

The longest duration of strong (or weak) market is assumed to be finite (I_0(or I_1)) from the past analysis of the market fluctuations, and we have only finite number of s_k's at each time interval. We set $K=I_0+I_1$ to denote the total number of states at each time interval. Also, we have a set of conditional probabilities

$$Pr(i_{k+1}=0|s_k) \tag{2}$$

which are, in verbal statement, the conditional probabilities that the market will be weak over the following time interval, given that a certain market situation i_k has been lasting for m_k consecutive time intervals. Since there are only two mutually exclusive market situations,

$$Pr(i_{k+1}=1|s_k)=1-Pr(i_{k+1}=0|s_k) \tag{3}$$

will hold.

Since we have a set of probability values of (2), it is easy to get

$$Pr(s_{k+1}|s_k) \tag{4}$$

for all possible combinations of s_{k+1}'s and s_k's, both having the form like (1). Furthermore, by convolution operation, we can compute

$$Pr(s_{k+j}|s_k) \text{ for } j=2,\ 3,\ \cdots\cdots,\ J. \tag{5}$$

The set of probability values (5) for each j has clearly a relationship

$$Pr(i_{k+j}=0|s_k)=\sum_{i_{k+j}=0} Pr(s_{k+j}|s_k) \tag{6}$$

for each fixed s_k, the summation being taken over all terms having $i_{k+j}=0$ in the left hand entry of s_{k+j}'s.

The price for the service will be expressed as

a_h=the price per unit time interval in a contract that comes to be valid under weak market and has h consecutive time intervals as its term,

and,

b_h=similar to a_h but for a contract that comes to be valid under strong market.

3. FORMULATION

The objective is taken to minimize the expected total payoff to the service over a planning horizon without considering discount.

Let N be the planning horizon.

We introduce

$F_N(s_k)$=the minimum expected total payoff over N time intervals starting from the next time interval knowing that the present market state is s_k and using an optimal contract policy.

Then, for $N=1$, clearly

$$F_1(s_k)=a_1Pr(i_{k+1}=0|s_k)+b_1Pr(i_{k+1}=1|s_k) \tag{7}$$

For $N=2$, we have a choice between two one-interval contracts and one two-interval contract. Thus

$$F_2(s_k)=\min\begin{bmatrix}2\{a_2Pr(i_{k+1}=0|s_k)+b_2Pr(i_{k+1}=1|s_k)\}\\ a_1Pr(i_{k+1}=0|s_k)+b_1Pr(i_{k+1}=1|s_k)\\ \quad +a_1Pr(i_{k+2}=0|s_k)+b_1Pr(i_{k+2}=1|s_k)\end{bmatrix} \tag{8}$$

Here, we introduce a notation

$$G_h(s_{k+j}|s_k)=a_hPr(i_{k+j}=0|s_k)+b_hPr(i_{k+j}=1|s_k) \tag{9}$$

and by using this notation, we can rewrite

$$F_1(s_k)=G_1(s_{k+1}|s_k) \tag*{(7)$'$}$$

and

$$F_2(s_k)=\min\begin{bmatrix}2G_2(s_{k+1}|s_k)\\ G_1(s_{k+1}|s_k)+G_1(s_{k+2}|s_k)\end{bmatrix} \tag*{(8)$'$}$$

Also for $N=3$, provided that the longest possible term of contract (H) is greater than three, we have,

$$F_3(s_k)=\min\begin{bmatrix} 3G_3(s_{k+1}|s_k) \\ 2G_2(s_{k+1}|s_k)+G_1(s_{k+3}|s_k) \\ G_1(s_{k+1}|s_k)+\sum\limits_{s_{k+1}} F_2(s_{k+1})Pr(s_{k+1}|s_k) \end{bmatrix} \tag{10}$$

where, the summation in the bottom formula is to be taken over all possible s_{k+1} reached from s_k.

We can go on this way to define functions $F_4(s_k)$, ……, $F_H(s_k)$ and we shall here show $F_H(s_k)$ only.

$$F_H(s_k)=\min\begin{bmatrix} HG_H(s_{k+1}|s_k) \\ (H-1)G_{H-1}(s_{k+1}|s_k)+G_1(s_{k+H}|s_k) \\ (H-2)G_{R-2}(s_{k+1}|s_k)+\sum\limits_{s_{k+H-2}} F_2(s_{k+H-2})Pr(s_{k+H-2}|s_k) \\ \vdots \\ G_1(s_{k+1}|s_k)+\sum\limits_{s_{k+1}} F_{H-1}(s_{k+1})Pr(s_{k+1}|s_k) \end{bmatrix} \tag{11}$$

The top formula corresponds to the expected total payoff when H-term contract is made and the bottom one does to the situation that one-term contract is made and is followed by the optimal contract policy with the modified state variable s_{k+1} over a horizon equals to $H-1$ time intervals. It may make these recurrence relations more obvious when $F_{H+1}(s_k)$ is added and then we can present the recurrence equations that hold for N, provided that $N>H>0$ (this proviso is true when H is rather short in comparison to the horizon N and this seems to be the case for most long term planning).

In this context, for $N=H+1$,

$$F_{H+1}(s_k)=\min\begin{bmatrix} HG_H(s_{k+1}|s_k)+G_1(s_{k+H+1}|s_k) \\ (H-1)G_{H-1}(s_{k+1}|s_k)+\sum\limits_{s_{k+H-1}} F_2(s_{k+H-1})Pr(s_{k+H-1}|s_k) \\ \vdots \\ G_1(s_{k+1}|s_k)+\sum\limits_{s_{k+1}} F_H(s_{k+1})Pr(s_{k+1}|s_k) \end{bmatrix}$$

Finally, for $N>H+1$,

$$F_N(s_k)=\min_h[hG_h(s_{k+1}|s_k)+\sum_{s_{k+h}} F_{N-h}(s_{k+h})Pr(s_{k+h}|s_k)] \tag{13}$$

where h on values 1, 2, ……H.

4. COMPUTATIONAL ASPECTS

As a set of original data we must have all the values of probabilities defined by (4) which amount to $K^2\times H$ values corresponding to different combination of states for different terms of contracts. According to the numerical values in formura (4), however, it is likely that many entries in the set of probability values are zero for small j as will be seen in an example, we may have fewer than $K^2 \times H$ significant numbers.

Also as a set of functional values initially used for obtaining $F_n(s_k)$ successively for $n=H+1, \cdots\cdots, N$, we must have values of h $G_h(s_{k+1}|s_k)$ as in (9) for $h=1, 2, \cdots\cdots, H$, hence we must compute $K \times H$ values.

Another set of values is those of $G_1(s_{k+j}|s_k)$ for $j=2, 3, \cdots\cdots, H+1$, thus we need other $K \times H$ values.

The final set of necessary values is those of $F_1(s_k)$, $F_2(s_k)$, $\cdots\cdots$, $F_{N+1}(s_k)$ that appear in the functional equation (13).

We need K values of $F_n(s_k)$ for $n=1, \cdots\cdots, H+1$, hence $K \times (H+1)$ values should be obtained.

In total, the number of necessary values for solving (13) is

$$K \times H + K \times H + K \times (H+1) = K \times (3H+1)$$

for values of initial functions.

Putting all the necessy values together, we need to have

$$K^2H + K(3H+1)$$

values, and for $K=10$ and $H=10$, say, this figure amounts to $10^2 \times 10 + 10(30+1) = 1310$. We believe that this figure will give some ideas about the computation when a computor is used.

5. AN EXAMPLE

We assume the probability values of (2) as in Table 1 and the price for service depending upon i_k and h as in Table 2. From Table 1 we compute the values of (5) and show the results in Table 3.

Table 1
Values of $Pr(i_{R+1}=\delta|S_R)$

S_k	(0|1)	(0|2)	(0|3)	(0|4)	(0|5)
$\delta=0$	1.00	0.90	0.60	0.40	0.00
S_k	(1|1)	(1|2)	(1|3)	(1|4)	
$\delta=1$	0.90	0.80	0.20	0.00	

Table 2
Values of a_h, b_h

h	1	2	3	4
a_h	5	6	8	7
b_h	12	10	9	8

The resulting $F_n(s_k)$ until the planning horizon $N=30$ is shown in

Table 3. Values of $Pr(s_{k+j}|s_k)$ for j=1, 2, 3 & 4

<table>
<tr><th>$s_k=$ $(i_k m_k)$</th><th>1 (0|1)</th><th>2 (0|2)</th><th>3 (0|3)</th><th>4 (0|4)</th><th>5 (0|5)</th><th>6 (0|6)</th><th>7 (0|7)</th><th>8 (0|8)</th><th>9 (0|9)</th></tr>
<tr><td>$k=1$</td><td>(0|3) 1.000</td><td>(0|3) 0.900</td><td>(0|4) 0.600</td><td>(0|5) 0.400</td><td>(1|1) 1.000</td><td>(1|2) 0.900</td><td>(1|3) 0.800</td><td>(1|4) 0 200</td><td>(0|1) 1.000</td></tr>
<tr><td></td><td></td><td>(1|1) 0.011</td><td>(1|1) 0.400</td><td>(1|1) 0.600</td><td></td><td>(0|1) 0.100</td><td>(0|1) 0.200</td><td>(0|1) 0.800</td><td></td></tr>
<tr><td>$k=2$</td><td>(0|3) 0.900</td><td>(0|4) 0.540</td><td>(0|5) 0.240</td><td>(1|1) 0.400</td><td>(1|2) 0.900</td><td>(1|3) 0.720</td><td>(1|4) 0.160</td><td>(0|1) 0.200</td><td>(0|2) 1.000</td></tr>
<tr><td></td><td>(1|1) 0.100</td><td>(1|1) 0.360</td><td>(1|1) 0.360</td><td>(1|2) 0.540</td><td>(0|1) 0.100</td><td>(0|1) 0.180</td><td>(0|1) 0.640</td><td>(0|2) 0.800</td><td></td></tr>
<tr><td></td><td></td><td>(1|2) 0.090</td><td>(1|2) 0.360</td><td>(0|1) 0.060</td><td></td><td>(0|2) 0.100</td><td>(0|2) 0.200</td><td></td><td></td></tr>
<tr><td></td><td></td><td>(0|1) 0.010</td><td>(0|1) 0.060</td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td>$k=3$</td><td>(0|1) 0.010</td><td>(0|1) 0.054</td><td>(0|1) 0.108</td><td>(0|1) 0.148</td><td>(0|1) 0.180</td><td>(0|1) 0.576</td><td>(0|1) 0.160</td><td>(0|2) 0.200</td><td>(0|3) 0.900</td></tr>
<tr><td></td><td>(0|4) 0.540</td><td>(0|2) 0.009</td><td>(0|2) 0.040</td><td>(0|2) 0.060</td><td>(0|2) 0.100</td><td>(0|2) 0.180</td><td>(0|2) 0.640</td><td>(0|3) 0.720</td><td>(1|1) 0.100</td></tr>
<tr><td></td><td>(1|1) 0.360</td><td>(0|5) 0.216</td><td>(1|1) 0.240</td><td>(1|2) 0.360</td><td>(1|3) 0.720</td><td>(0|3) 0.090</td><td>(0|3) 0.180</td><td>(1|1) 0.080</td><td></td></tr>
<tr><td></td><td>(1|2) 0.090</td><td>(1|1) 0.325</td><td>(1|2) 0.324</td><td>(1|3) 0.432</td><td></td><td>(1|1) 0.010</td><td>(1|1) 0.020</td><td></td><td></td></tr>
<tr><td></td><td></td><td>(1|2) 0.324</td><td>(1|3) 0.288</td><td></td><td></td><td>(1|4) 0.144</td><td></td><td></td><td></td></tr>
<tr><td></td><td></td><td>(1|3) 0.072</td><td></td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td>$k=4$</td><td>(0|1) 0.054</td><td>(0|1) 0.155</td><td>(0|1) 0.319</td><td>(0|1) 0.418</td><td>(0|1) 0.576</td><td>(0|1) 0.145</td><td>(0|1) 0.002</td><td>(0|1) 0.008</td><td>(0|1) 0.010</td></tr>
<tr><td></td><td>(0|2) 0.010</td><td>(0|2) 0.054</td><td>(0|2) 0.108</td><td>(0|2) 0.148</td><td>(0|2) 0.180</td><td>(0|2) 0.576</td><td>(0|2) 0.160</td><td>(0|3) 0.180</td><td>(0|4) 0.540</td></tr>
<tr><td></td><td>(0|5) 0.216</td><td>(0|5) 0.008</td><td>(0|3) 0.036</td><td>(0|3) 0.054</td><td>(0|3) 0.090</td><td>(0|3) 0.162</td><td>(0|3) 0.576</td><td>(0|4) 0.432</td><td>(1|1) 0.360</td></tr>
<tr><td></td><td>(1|1) 0.324</td><td>(1|1) 0.217</td><td>(1|1) 0.004</td><td>(1|1) 0.006</td><td>(1|1) 0.010</td><td>(0|4) 0.054</td><td>(0|4) 0.108</td><td>(1|1) 0.308</td><td>(1|2) 0.090</td></tr>
<tr><td></td><td>(1|2) 0.324</td><td>(1|2) 0.293</td><td>(1|2) 0.216</td><td>(1|2) 0.288</td><td>(1|2) 0.144</td><td>(1|1) 0.054</td><td>(1|1) 0.136</td><td>(1|2) 0.072</td><td></td></tr>
<tr><td></td><td>(1|3) 0.072</td><td>(1|3) 0.259</td><td>(1|3) 0.259</td><td>(1|3) 0.086</td><td></td><td>(1|2) 0.009</td><td>(1|2) 0.018</td><td></td><td></td></tr>
<tr><td></td><td></td><td>(1|4) 0.014</td><td>(1|4) 0.058</td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
</table>

Table 4. Values of $F_N(s_k)$ and the optimal term.

State	1	2	3	4	5	6	7	8	9
$N=2$	10.7 (1)	12.8 (2)	15.2 (2)	16.8 (2)	20.0 (2)	19.2 (2)	16.7 (1)	11.4 (1)	10.0 (1)
3	17.8 (1)	21.3 (1)	25.2 (3)	25.8 (3)	27.0 (3)	25.3 (2)	22.0 (1)	17.0 (1)	15.0 (1)
4	26.3 (1)	28.4 (4)	29.6 (4)	30.4 (4)	32.0 (4)	30.6 (2)	27.8 (1)	23.6 (1)	22.8 (1)
5	33.4 (1)	35.4 (1)	36.4 (4)	39.1 (3)	37.4 (4)	36.8 (2)	34.7 (1)	32.0 (1)	31.3 (1)
6	40.4 (1)	42.1 (1)	42.2 (4)	41.8 (4)	43.5 (4)	43.8 (2)	42.9 (1)	39.4 (1)	38.4 (1)
7	47.1 (1)	48.0 (1)	50.4 (1)	48.7 (4)	50.8 (4)	51.8 (2)	50.2 (1)	46.4 (1)	45.4 (1)
8	53.0 (1)	54.6 (1)	55.7 (4)	56.1 (4)	58.5 (4)	59.0 (2)	57.2 (1)	53.2 (1)	52.1 (1)
9	59.6 (1)	61.7 (1)	63.1 (4)	63.7 (4)	65.8 (4)	65.9 (2)	63.8 (1)	59.2 (1)	58.0 (1)
10	66.7	69.1	70.4	70.7	72.6	72.5	69.9	65.7	64.6
11	74.1	76.3	77.3	77.4	79.2	78.8	76.5	72.7	71.7
12	81.3	83.1	83.8	83.8	85.5	85.4	83.6	80.0	78.1
13	88.1	89.7	90.3	90.3	92.1	92.5	90.9	87.3	86.3
14	94.7	9.62	97.0	97.2	99.2	99.8	98.1	94.2	93.1
15	101.2	103.0	104.1	104.4	106.5	106.9	104.9	100.8	99.7
16	108.0	110.1	111.2	111.6	113.6	113.7	111.4	107.3	106.2
17	115.1	117.2	118.3	118.5	120.4	120.3	118.1	114.0	113.0
18	122.2	124.2	125.1	125.2	127.0	126.9	124.9	121.1	120.1
19	129.2	130.9	131.7	131.8	133.6	133.8	131.9	128.2	127.2
20	135.9	137.6	138.4	138.6	140.5	140.8	139.0	135.2	134.2
21	142.6	144.4	145.3	145.5	147.5	147.8	145.9	142.0	140.9
22	149.4	151.2	152.3	152.6	154.5	154.8	152.7	148.7	147.6
23	156.3	158.2	159.3	159.5	161.5	161.5	159.4	155.4	154.4
24	163.2	165.2	166.2	196.4	168.3	168.3	166.2	162.3	161.3
25	170.2	172.1	173.0	173.1	175.0	175.1	173.1	169.2	168.2
26	177.1	178.9	179.7	179.9	181.8	181.9	180.0	176.2	175.2
27	183.9	185.7	186.6	186.7	188.7	188.9	187.0	183.1	182.1
28	190.7	192.5	193.5	193.7	195.6	195.8	193.9	189.9	188.9
29	197.5	199.4	200.4	200.6	202.6	202.7	200.7	296.7	195.7
30	204.4	206.3	207.3	207.5	209.4	209.5	207.5	203.5	202.5

The optimal ferms in the brackets are the same for $N \geqq 9$.

Table 4 with the optimal term of contracts at every state and stage which make up the policy.

From Table 4 it can be said that the optimal policy for N in terms of h's corresponding to s_k where $N>h$ will be {1, 1, 4, 4, 4, 2, 1, 1, 1} to which the policy converged.

This is the solution in policy space correspondig to the recurrence equation (13) and the resulting values of $F_N(s_k)$ are also shown in Table 4.

Since our objective was to minimize the expected total payoff without discount, there would be no objection in using this policy, yet from the viewpoint of testing the solution, we would like to know the stability of the policy.

By stability we mean the standard deviation of the distribution of the actual outcoming total payoff over sequences of market situations generated from a certain specified state, which is (0|1) in this example, and using the optimal policy. We get results like Figure 1. The sample mean $\bar{x}$ is 204.8 and this value is quite close to $F_{30}(1)=204.4$ and, as a

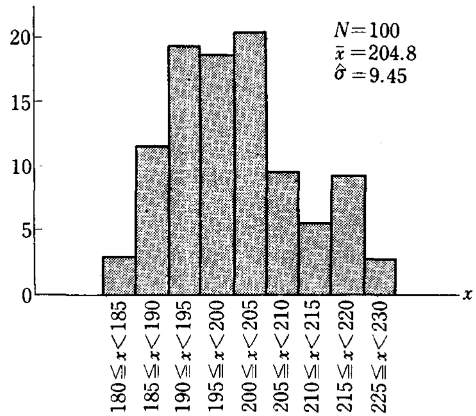

Fig. 1 The distributin of actual total payoff for $N=30$, $S_k=(0|1)$ using the optimal policy on Table 4.

measure of stability we get 9.45, which is an estimate of the standard deviation. The coefficient of variation is about 0.05 and is thought to be sufficiently small in terms of stability.

6. OTHER POLICIES

Since we know about the market fluctuations as those assumed, we may select other policies which would be based on our intuition. A couple of policies would be

(1) To renew the contract so that the next contract will come to be valid under the weak market with highest probability.

(2) To renew the contract so that it will result in the lowest expected payoff.

The policy (1) which we call the H. P. Policy (Highest Probability Policy) looks into the dynamical aspect of the problem qualitatively and reasonably, while the policy (2), or the L. C. Policy (Least Cost Policy), looks at just the present and ignores the dynamic aspect.

The derivation of the H. P. Policy and the L. C. Policy will be as follows:

(i) The H. P. Policy

As was defined by (2), $Pr(i_{k+1}=0|s_k)$ means that the probability that the $(k+1)$th market would turn out to be weak, given the k-th market situation. Similarly, $Pr(i_{k+j}=0|s_k)$ will be defined that the probability that the $(k+j)$th market will turn out to be weak, given the k-th market situation.

By the H. P. Policy the term of contract will be determined so that the following contract will come to be valid under the weak market with highest probability, thus, h for any s_k will be determined that which minimizes

$$Pr(i_{k+h+1}=0|s_k) \tag{14}$$

for $h=1, 2, \cdots\cdots, H(=4$ in this example).

We should know, from (14), the values $Pr(i_{k+2}=0|s_k)$ $Pr(i_{k+H+1}=0|s_k)$ and the last of these values have appeared on Table 3, so we need another series of computation described before. The values of $Pr(i_{k+j}=0|s_k)$, for $j+2, \cdots\cdots, H$ are obtained from Table 3, such that, for instance,

$$\begin{aligned} Pr(i_{k+2}=0|s_k) &= \sum_{i_{k+2}=0} Pr(s_{k+2}|s_k) \\ &= Pr(1|s_k)+Pr(2|s_k)+\cdots\cdots+Pr(5|s_k) \end{aligned} \tag{15}$$

and for $s_k=2=(0|2)$, this results in 0.550. The H. P. Policy is shown on

Table 5.

Table 5. The H. P. Policy

s_k	1	2	3	4	5	6	7	8	9
h	1	1	4	4	4	3	2	1	1

(ii) L. C. Policy

The derivation of this policy will be to obtain h that minimizes

$$a_h Pr(i_{k+1}=0|s_k)+b_h Pr(i_{k+1}=1|s_k) \tag{16}$$

and the results are shown on Table 6.

Table 6. The L. C. Policy

s_k	1	2	3	4	5	6	7	8	9
h	1	1	4	4	4	4	4	1	1

7. COMPARISON OF POLICIES

To ensure the optimality of the first policy in comparison to the other two policies described in the preceeding section, we need to simulate on the same sequence of market situation by which we tested the stability of the first policy.

As the result for 100 runs of simulation, we get

$\bar{x}=209.7$ for the H. P. Policy

$\bar{x}=211.3$ for the L. C. Policy

both of which are slightly higher than that for the first policy. This difference, of course, depending upon the values of a_h's, b_h's and the probability values that determine the behavior of the market, is rather small in the example above.

8. SOME REMARKS

We showed a way to derive an optimal contract policy under a certain type of market variation and compared the policy with two other policies. Although the optimal policy did not manifest an overwhelming superiority to the other policies in the numerical example, we except that the further analytical study will reveal it.

The most critical aspect of this problem may lie in the fact that the variations of the market is described as a set of conditional probability. We can define $Pr(s_{k+1}|s_k)$ is the transition probability from a state s_k to a state s_{k+1} by one transition, thus a sequence of market state $\{s_k, s_{k+1}\cdots\}$ forms a Markov chain.

This type of market variation with an associated payoff was studied in detail in Howard, where decisions are made at each stage, and a method for finding the optimal policy through the iteration process was developed.

In this problem, however, decisions are not necessarily made at every stage, in other words, if a decision is to contract for h time intervals, then the next decision will have to be made h intervals later, when the state variables will change from s_k to s_{k+h} with probability $Pr\ (s_{k+h}|s_k)$. So, the sequence of state variables relevant to the decisions skipps some time intervals (h) that are the term a decision is valid for.

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REFERENCES

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