

A NEW POLICY FOR PREVENTIVE MAINTENANCE

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SUMMARY

R. Barlow and L. Hunter [1] studied two preventive maintenance policies—one of which is most useful in maintaining simple equipment and another which is useful in maintaining large, complex systems. They called those policies as Policy I and Policy II, respectively. In the present paper, we shall propose a new policy which will be called as Policy III and which is most useful in maintaining large systems consisted with a number of equipments of same type. We can see that Policy III is more efficient and simpler than Policy II for their practical uses. Furthermore, by introducing maintenance costs, these policies will be generalized.

Some detailed discussion with numerical examples when the life time distributions of the system are Weibull type will be added. Especially, an optimum type III policy can be decided by only the shape parameter, which is usually known as characteristics of the system. And, it is remarkable that even though the scale parameter change our Policy III cease to be optimal but in such a case Policy II already cannot be optimal.

§ 1. INTRODUCTION

First of all, in order to describe the character of our Policy III explicitly, we shall quote the definitions of the two policies in [1].

Policy I For less complex equipment, such as vacuum tubes, repairs at the time of failures (or replacement) may actually correspond to general overhauls. Thus, it is sufficient that we consider only one pattern of maintenance. So, Policy I may be defined as follows: Perform preventive maintenance after t_0 hours of counting operation without failure ($0 < t_0 \leq \infty$). If the system fails before t_0 hours have elapsed, perform maintenance at the time of failure. Preventive maintenance at the

time is rescheduled. For this policy, we assume that the system is as good as new after any time of maintenance (or replacement) is performed.

Policy II For more complex systems than the above, such as computers, preventive maintenance is commonly scheduled after a certain number of operating hours have accumulated. Between maintenance periods, failures are repaired as quickly as possible. The definition of Policy II is as follows: Perform preventive maintenance on the system after it has been operating a total of t^* hours regardless of the number of intervening failures ($0 < t^* \leq \infty$). We assume that after each failure only minimal repair is made and that the system failure rate*) is not disturbed after performing minimal repair.

R. Barlow and L. Hunter [1] found the optimum maintenance periods t_0 and t^* in the sense of limiting efficiencies (availabilities) which is defined as follows to be maximized.

$$\begin{aligned}
 (1.1) \quad Eff_{\infty} &= \lim_{T \rightarrow \infty} Eff_T, && \text{with probability 1.} \\
 &= \lim_{T \rightarrow \infty} (\text{expected fractional amount of time system is} \\
 &\quad \text{on during } [0, T]).
 \end{aligned}$$

We shall here consider the similar problems to the case of Policy II along the analogous direction to [1].

In the next section, we shall propose a new policy which is more efficient than Policy II.

§ 2. POLICY III

We consider some large systems consisted with a number of equipments of same type.**) If we overhaul the system, it can be seen that the system is renewal. Thus we shall consider that the system is replaced by a new system with a specified life distribution function at the time epoch when the overhaul completes. In the case of Policy II which was introduced in the above, life distribution functions of replacing systems at every overhaul epochs are identical. But, in our case (Policy III), they are not necessarily identical. We shall assume that the mean life times are $\mu_1, \mu_2, \dots, \mu_M$ which appear with probabilities p_1, p_2, \dots, p_M , respectively. Of course, we assume that all μ 's are positive and finite.

*) This definition is given by (2.1)

**) See also Remark 6.2

We shall denote the life distribution of the i -th system as $F_i(x)$, and its density function as $f_i(x)$. Of course, $F_i(x) \equiv f_i(x) \equiv 0$ for all $x < 0$. And, put

$$(2.1) \quad q_i(x) \equiv \frac{f_i(x)}{1 - F(x)}$$

which is named as failure rate. Using these notations, we can write

$$(2.2) \quad \mu_i = \int_0^\infty x dF_i(x), \quad \mu = \sum_{i=1}^M p_i \mu_i.$$

In the present situation, if we take Policy II, then it will be too fast to overhaul in some cases and too slow in the other cases. For, when $\mu < \mu_i$, the optimum period t^* for i -th system must be longer and when $\mu > \mu_i$ conversely. Furthermore, the system may be operated for some hours after the time point of the overhaul scheduled by Policy II.

In order to avoid these losses, we shall introduce the following policy.

Definition For first $k-1$ times of failure of the system, we perform a minimal repair and at the time of k -th failure we overhaul it. This maintenance policy will be called as Policy III.

Further, we shall introduce the following notations.

T_m : the mean minimal repair time,

T_s : the mean maintenance time for an overhaul.

Under the above preparations, we can find the limiting efficiency in the case of Policy III.

Theorem 2.1 The limiting efficiency for Policy III is given by

$$(2.3) \quad Eff_\infty^{(3)} = \frac{\bar{U}(k)}{\bar{U}(k) + (k-1)T_m + T_s}$$

where

$$(2.4) \quad \bar{U}(k) = \sum_{i=1}^M p_i E(U_i^{(k)})$$

and $U_i^{(k)}$ is the total operating time (without repair time) of the system having $F_i(x)$ as its life time distribution.

Proof. Evidently, we have

$$(2.5) \quad \eta_m^{(3)} = \frac{\sum_{k=1}^m U_{j_h}^{(k)}}{\sum_{h=1}^m \left(U_{j_h}^{(k)} + \sum_{\nu=1}^k W_{j_h}^{(\nu)} \right)}$$

where $\eta_m^{(3)}$ denotes the fractional amount of operating time for Policy III during m cycles (i. e., m times of overhaul), and $U_{j_1}^{(k)}, U_{j_2}^{(k)}, \dots, U_{j_m}^{(k)}$ are total operating time of first system, second system, \dots , and m -th system, respectively. In the other words, $j_h (h=1, 2, \dots, m)$ mean some numbers in the integer set $\{1, 2, \dots, M\}$. Similarly, $W_{j_h}^{(\nu)}$ ($\nu=1, 2, \dots, k-1$) and $W_{j_h}^{(k)}$ mean the minimal repair times and maintenance time, respectively. Obviously $U_{j_h}^{(k)}, W_{j_h}^{(\nu)}$ ($\nu=1, 2, \dots, k; h=1, 2, \dots, m$) are mutually independent random variables.

We assume that the first and the second moments of $U_{j_h}^{(k)}$ and $W_{j_h}^{(\nu)}$ are finite for all i . Using here the renewal theorem due to H. Hatori [2] which is quoted below as Lemma 2.1, an analogous argument to [1] implies that $\eta_m^{(3)}$ tends as $m \rightarrow \infty$ to the right hand of (2.3) with probability 1.

Lemma 2.1 Let X_1, X_2, \dots be non-negative and mutually independent random variables. Let $N(t)$ be the number of sums X_1, X_1+X_2, \dots which are less than t . If (i) X_2, X_3, \dots have finite means a_2, a_3, \dots , respectively and there exists a positive constant L such that $a_\nu \geq L$ for $\nu=2, 3, \dots$, (ii) X_2, X_3, \dots have finite variances $\sigma_2^2, \sigma_3^2, \dots$, respectively and there exists a positive constant K such that $\sigma_\nu^2 \leq K < +\infty$ for $\nu=2, 3, \dots$, (iii) there exists

$$\lim_{n \rightarrow \infty} \frac{1}{n-1} (a_2 + \dots + a_n) = a$$

where a may be $+\infty$, then we have

$$P\left\{\lim_{t \rightarrow \infty} \frac{N(t)}{t} = \frac{1}{a}\right\} = 1.$$

Theorem 2.2 Under the same situation for Policy III, the limiting efficiency of Policy II is given by

$$(2.6) \quad \text{Eff}_\infty^{(2)} = \frac{t^*}{t^* + T_m \sum_{i=1}^M p_i \int_0^{t^*} q_i(t) dt + (T_s - T_m)}.$$

Proof. Analogous notations to the above are used here. If we write the number of times of failure in h -th cycle as $N_{j_h}(t^*)$, we have

$$(2.7) \quad \eta_m^{(2)} = \frac{mt^*}{mt^* + \sum_{h=1}^m \sum_{\nu=1}^{N_{j_h}(t^*)} W_{j_h}^{(\nu)}}.$$

Noting that

$$(2.8) \quad E\{N_{j_h}(t^*)\} = \int_0^{t^*} q_{j_h}(t) dt$$

and

$$E\{W_{j_h}^{(j)}\} = T_m, \quad \nu=1, 2, \dots, k-1 \\ = T_s, \quad \nu=k$$

for all j_h , we have (2.6) by the similar argument to the above.

§ 3. INTRODUCTION OF COST CONCEPT

In the above, we measured the effectiveness of maintenance policy by limiting efficiency that is operating time rate. But, sometimes, mean minimal repair time is rather long but the cost is small compared with the ones for overhaul. In that cases, we must consider the mean cost per unit hour during a long time. In order to make our theory available in such cases, we shall introduce the concept of cost into our model.

Assume that a minimal repair of the system requires cost C_m and an overhaul requires cost C_s in average. An adding assumption is that any non-operating times of the system are regarded as losses which evaluated as C_0 per unit hour.

We shall measure the effectiveness of our policy as the total average cost for maintenance per unit hour (maintenance cost rate). When it is denoted as $C_\infty^{(3)}$, we can see by a similar argument to the above that it is given by

$$(3.1) \quad C_\infty^{(3)} = C_0[1 - E f f_\infty^{(3)}] + \frac{(k-1)C_m + C_s}{\bar{U}(k) + (k-1)T_m + T_s}.$$

Thus we have

Theorem 3.1 The maintenance cost rate for Policy III is given by

$$(3.2) \quad C_\infty^{(3)} = \frac{(k-1)(C_0 T_m + C_m) + C_0 T_s + C_s}{\bar{U}(k) + (k-1)T_m + T_s}.$$

Proof. Evident due to Theorem 2. 1.

Remark 3.1 In the expression (3.2), if we put $C_m=C_s=0$ and $C_0=1$, (3.2) will be reduced to 1-(2.3). Thus, to minimize (3.2) is equivalent to maximize (2.3) in this case.

In this sense, we can say that the introduction of cost concept gave a generalization of our model. However, we shall discuss on this case in

the forthcoming paper.

§ 4. DESIGN OF OPTIMAL POLICY OF TYPE III

In this section, we shall restrict ourselves to deal with the limiting efficiency for Policy III and find the optimal policy. The optimal policy will be found explicitly when the life time distributions of all systems belong to Weibull distribution family with an identical shape parameter (β).

First of all, we state a general expression. When the failure rate of a system at time t is $q_i(t)$, the probability element that the first k times of failure occur and the k -th failure does at just time t is given by

$$(4.1) \quad r_i(t)dt = e^{-Q_i(t)} \frac{[Q_i(t)]^{k-1}}{(k-1)!} q_i(t)dt,$$

where

$$(4.2) \quad Q_i(t) = \int_0^t q_i(u)du.$$

Hence, we have

$$(4.3) \quad Eff_{\infty}^{(3)}(k) = \frac{\sum_{i=1}^M p_i \int_0^{\infty} tr_i(t)dt}{\sum_{i=1}^M p_i \int_0^{\infty} tr_i(t)dt + (k-1)T_m + T_s},$$

because the mean time till the k -th failure of a system having $F_i(x)$ as its life time distribution will be given by

$$(4.4) \quad E(U_i^{(k)}) = \int_0^{\infty} tr_i(t)dt.$$

We shall maximize (4.3) concerning with k , or equivalently, minimize

$$(4.5) \quad f(k) \equiv 1 - Eff_{\infty}^{(3)}(k) = \frac{(k-1)T_m + T_s}{H(k)},$$

where $H(k)$ is the denominator of (4.3) which is obviously positive. For our aim, it is sufficient to consider whether the numerator of the difference of (4.5) is positive or negative. Since we have

$$(4.6) \quad \Delta f(k) \equiv f(k+1) - f(k) = \frac{(T_s + kT_m)[H(k) - H(k+1)] + T_m H(k+1)}{H(k)H(k+1)}$$

we shall find the largest integer k (hereafter, denote it as $k_0 - 1$) such as

$$(4.7) \quad (T_s + kT_m)[H(k) - H(k+1)] + T_m H(k+1) < 0$$

or

$$(4.8) \quad \varphi(k) \equiv \left(1 - \frac{\bar{U}(k)}{\bar{U}(k+1)}\right) \left(k + \frac{T_s}{T_m}\right) > 1,$$

where

$$(4.9) \quad \bar{U}(k) = \sum_{i=1}^M p_i E(U_i^{(k)}) = \sum_{i=1}^M p_i \int_0^\infty t r_i(t) dt.$$

If we can examine the convexity of $f(k)$, the optimal policy of type III is to overhaul the system at each k_0 -th failure.

However, in order to avoid the difficulty of further discussion in general forms, we proceed to more detailed consideration adding the assumption on the failure rate. We shall assume that the life time distributions of our systems are all Weibull type with the identical shape parameter. More precisely, we deal with the systems having the following distributions of their life time :

$$(4.10) \quad \begin{aligned} F_i(x) &= 1 - e^{-\alpha_i x^\beta}, & x > 0 \\ &= 0, & x \leq 0 \end{aligned}$$

where $\alpha_i (i=1, 2, \dots)$ are all positive and $\beta > 1$. The reason what we adopt the above assumption is that the failure rate

$$(4.11) \quad q_i(x) = \alpha_i \beta x^{\beta-1}$$

will be essentially determined only by the mechanical character of our systems and the function form of (4.11) will be governed by the shape parameter β .

Now, (4.1), (4.4) and (4.11) imply that

$$(4.12) \quad E(U_i^{(k)}) = \int_0^\infty t e^{-\alpha_i t^\beta} \frac{(\alpha_i t^\beta)^{k-1}}{(k-1)!} \alpha_i \beta t^{\beta-1} dt,$$

since

$$(4.13) \quad Q_i(t) = \int_0^t q_i(x) dx = \alpha_i t^\beta.$$

From (4.12), we have putting $y = t^\beta$,

$$\begin{aligned} E(U_i^{(k)}) &= \int_0^\infty \alpha_i t^k y^{k-1+1/\beta} \frac{1}{(k-1)!} e^{-\alpha_i y} dy \\ &= \frac{\Gamma\left(k + \frac{1}{\beta}\right)}{\alpha_i^{1/\beta} \cdot (k-1)!} \end{aligned}$$

$$\begin{aligned}
 &= \frac{\mu_i \Gamma\left(k + \frac{1}{\beta}\right)}{\Gamma\left(\frac{1}{\beta} + 1\right) \Gamma(k)} \\
 (4.14) \quad &= \frac{\mu_i \beta}{B\left(\frac{1}{\beta}, k\right)},
 \end{aligned}$$

where μ_i is the mean life time of the system and it is given by

$$(4.15) \quad \mu_i = \alpha_i^{-1/\beta} \Gamma\left(\frac{1}{\beta} + 1\right)$$

from (4.10). Hence, we have

$$(4.16) \quad \bar{U}(k) = \sum_{i=1}^M p_i \frac{\mu_i \beta}{B\left(\frac{1}{\beta}, k\right)} = \frac{\mu \beta}{B\left(\frac{1}{\beta}, k\right)}$$

by (4.8) and (2.2). (4.16) implies that

$$\begin{aligned}
 \varphi(k) &= \left(1 - \frac{B\left(\frac{1}{\beta}, k+1\right)}{B\left(\frac{1}{\beta}, k\right)}\right) \left(k + \frac{T_s}{T_m}\right) \\
 &= \left(1 - \frac{k}{\left(k + \frac{1}{\beta}\right)}\right) \left(k + \frac{T_s}{T_m}\right) \\
 (4.17) \quad &= \frac{k + \frac{T_s}{T_m}}{k\beta + 1}.
 \end{aligned}$$

Thus, $k_0 - 1$ is the largest integer k such as

$$k + \frac{T_s}{T_m} > k\beta + 1$$

or

$$k < \frac{1}{\beta - 1} \frac{T_s - T_m}{T_m}, \quad (\beta > 1).$$

By the way, under the present assumptions, we have

$$\begin{aligned}
 \Delta\varphi(k) &\equiv \varphi(k+1) - \varphi(k) \\
 &= \frac{(k+1) + \frac{T_s}{T_m}}{(k+1)\beta + 1} - \frac{k + \frac{T_s}{T_m}}{k\beta + 1}
 \end{aligned}$$

$$(4.19) \quad = \frac{1 - \frac{T_s}{T_m}}{((k+1)\beta + 1)(k\beta + 1)}$$

which is always negative when $T_s > T_m$. Thus we can state the following

Theorem 4.1 If the life time distribution of systems are given by Weibull distribution (4.10) with identical shape parameter $\beta > 1$ and if $T_s > T_m$, then the optimal policy of type III is to overhaul the systems at each k_0 -th failure such as

$$(4.20) \quad k_0 = \left[\frac{1}{\beta - 1} \frac{T_s - T_m}{T_m} \right] + 1.$$

If $T_s \leq T_m$, that is, the mean minimal repair time is less than the time required by a overhaul in average, any minimal repair must be not taken.

Remark 4.1 In many practical cases, the shape parameter of Weibull distribution is larger than unity as in the above. But, if $\beta = 1$,

$$(4.21) \quad \varphi(k) = \left(k + \frac{T_s}{T_m} \right) / (k + 1).$$

Hence, $k_0 = \infty$ when $T_s \geq T_m$ and $k_0 = 1$ when $T_s < T_m$. This fact consist with the noted one by R. Barlow and L. Hunter [1].

Furthermore, if $0 < \beta < 1$, (4.17) have to be written as

$$(4.22) \quad k > \frac{1}{1 - \beta} \frac{T_m - T_s}{T_m}.$$

Then, $k_0 = \infty$ independetly of the values of T_m and T_s . This means that if the failure rate of a system decrease time by time, the system must be permanently used performing minimal repair at each failure. In practical cases, perhaps, such system will never exist.

§ 5. NUMERICAL EXAMPLES

As were stated above, when the mean life time of the system variates we can improve the efficiency using Policy III instead of Policy II. In the following, we shall show in some very simple cases, the predominance of our policy numerically and the proceed to the detailed discussions.

Example 5.1 Consider the case in which $T_s = 4$, $T_m = 1/2$, $\alpha = 1/9$ and $\beta = 2$ (hence, $\mu = 2.64$) then we have $k_0 = 7$ from Theorem 4.1.

Supposing that the scale parameters are α_1 and α_2 with probabilities $1/2$ respectively, where $\frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_2 = \alpha = 1/9$. In this examples we shall

consider two cases: (i) $\alpha_1=1/18, \alpha_2=1/6$ and (ii) $\alpha_1=1/27, \alpha_2=5/27$. Then the corresponding mean life times are (i) $\mu_1=2.18, \mu_2=3.76$ and (ii) $\mu_1=2.04, \mu_2=4.70$.

In the case that some systems having different mean life times are mixed as in this example, if we use Policy II, we must interpret μ properly. We can take at least two cases into our consideration. One of which is that Case A and the other is Case B whose definitions will be given as follows.

Case A: Suppose that we estimate firstly the value of μ which is $\mu = \frac{\mu_1 + \mu_2}{2}$ without recognizing the variation of μ , and then decide t^* as

$$(5.1) \quad t^* = \left\{ \frac{T_s}{T_m(\beta-1)} \right\}^{1/\beta} \frac{\beta\mu}{\Gamma\left(\frac{1}{\beta}\right)}$$

based on the formula (10) of [1]. This value of t^* cannot be optimal in the present case, because the formula (10) of [1] was deduced under the assumption that the mean life times of the systems are invariant. The limiting efficiency in our case may be found as the following expression based upon Theorem 2.2 and (4.13):

$$(5.2) \quad \begin{aligned} Eff_{\infty}^{(2)} &= \frac{t^*}{t^* + T_m t^{*\beta} \alpha + T_s} \\ &= \frac{1}{1 + T_m t^{*\beta-1} \alpha + T_s/t^*} \end{aligned}$$

Case B: In this case we suppose that we can estimate α_1 and α_2 individually and then regarding our systems as having identical life time distribution with the scale parameter $\bar{\alpha} = (\alpha_1 + \alpha_2)/2$, we maximize the limiting efficiency for Policy II. An analogous deduction of (5.2) implies that

$$(5.3) \quad Eff_{\infty}^{(2)} = \frac{1}{1 + \left(\frac{\beta T_s}{\beta-1}\right) \cdot \left\{ \frac{(\beta-1)\bar{\alpha} T_m}{T_s} \right\}^{1/\beta}}$$

and

$$(5.4) \quad t^* = \left\{ \frac{T_s}{(\beta-1)\bar{\alpha} T_m} \right\}^{1/\beta}$$

Numerical results obtained by inserting the given values into the formulas (5.2), (5.3), (5.4), (4.3) and (4.20) are summed up in the following

Table 5.1, in which r means the multiplier for μ , μ_1 and μ_2 . (for example, when $r=2$ in Case (i), $\mu_1=4.36$, $\mu_2=7.52$)

Table 5.1

r	Case	Policy II		Policy III
		Case A	Case B	
1/5	(i)	0.17	0.17	0.20
	(ii)	0.17	0.17	0.22
1	(i)	0.52	0.52	0.56
	(ii)	0.51	0.52	0.59
2	(i)	0.68	0.68	0.71
	(ii)	0.67	0.68	0.74

Though we can see intuitively that an optimal policy of type III is more efficient than the corresponding optimal policy for type II, the above table also shows this fact numerically.

Example 5.2 In this example, we consider the case where k_0 is larger than the one in the above example. Under the same values of T_s , T_m and α in Case (ii), if we put $\beta=4/3$, we have $k_0=21$. The limiting efficiency for Policy III and the one for Policy II in Case B are

$$Eff_{\infty}^{(3)}=0.87 \text{ and } Eff_{\infty}^{(2)}=0.78.$$

The limiting efficiency for Policy III is rather larger than the one for Policy II. But in this case, since we have $\mu_1=10.94$ and $\mu_2=3.32$ as mean life times, the difference between these limiting efficiencies not so large compared with the difference between μ 's.

In the following we shall consider theoretically the case in which k_0 is sufficiently large. Taking $T_s/T_m=\gamma \gg 1$, we have

$$(5.5) \quad k_0 \sim \frac{\gamma}{\beta-1},$$

$$(5.6) \quad B\left(k_0, \frac{1}{\beta}\right) \sim k_0^{-1/\beta} \Gamma\left(\frac{1}{\beta}\right),$$

and

$$(5.7) \quad Eff_{\infty}^{(3)} = \frac{1}{1 + \frac{T_m}{\bar{\mu}} \gamma^{1-1/\beta} \Gamma\left(\frac{1}{\beta}\right) \cdot (\beta-1)^{1/\beta-1}}$$

$$(5.8) \quad Eff_{\infty}^{(2)} = \frac{1}{1 + \frac{T_m}{\bar{\mu}'} \Gamma^{1-1/\beta} \Gamma\left(\frac{1}{\beta}\right) \cdot (\beta-1)^{1/\beta-1}},$$

where

$$(5.9) \quad \bar{\mu} = p_1 \mu_1 + p_2 \mu_2 \equiv \alpha^{-1/\beta} \cdot \Gamma\left(1 + \frac{1}{\beta}\right)$$

and

$$(5.10) \quad \bar{\mu}' = (p_1 \alpha_1 + p_2 \alpha_2)^{-1/\beta} \Gamma\left(\frac{1}{\beta}\right) \equiv \bar{\alpha}^{-1/\beta} \Gamma\left(1 + \frac{1}{\beta}\right).$$

In general, we have

$$(5.11) \quad \bar{\alpha}^{-1/\beta} \leq \bar{\alpha}^{-1/\beta}.$$

Hence we can always conclude that

$$(5.12) \quad Eff_{\infty}^{(3)} \geq Eff_{\infty}^{(2)}$$

where the equality holds only when $\mu = \bar{\mu}$. (See also Remark 6.1)

Examples 5.3 Now we proceed to the case in which k_0 is small. Putting $T_s=2$, $T_m=1/2$ and $\beta=3$, we have $k_0=2$. We calculate the limiting efficiencies in two cases: (i) $\alpha_1=5/27$, $\alpha_2=1/27$; (ii) $\alpha_1=8/45$, $\alpha_2=1/45$, and summarize these results in the following table.

Table 5.2

case		Case(i)	Case(ii)
mean life time	μ_1	1.57	1.59
	μ_2	2.68	3.18
Policy II		0.47	0.47
Policy III		0.53	0.56

As was shown numerically in the above table, the difference between the limiting efficiencies for two policies is rather large compared with the difference between μ 's. In the other words, Policy III is most useful in the case in which the period for overhaul is rather small.

§ 6. ROBUSTNESS OF POLICY III

In order to show the robustness of our policy in a case when μ variates, we shall compare the limiting efficiencies for Policy II and III. All values except μ (or α) in Example 5.1 are used here. We have for

$k_0=7,$

$$(6.1) \quad Eff_{\infty}^{(3)} = \frac{\mu\beta}{\mu\beta + B\left(k, \frac{1}{\beta}\right)\{(k-1)T_m + T_s\}} = \frac{1}{1 + \frac{2.4}{\mu}}$$

and for $t^*=8.5$

$$(6.2) \quad Eff'_{\infty}^{(2)} = \frac{t^*}{t^* + T_m\alpha t^{*\beta} + T_s} = \frac{1}{1.47 + \frac{3.33}{\mu^2}}$$

In the above formulas, after we have settled our policies under the conditions of Example 5.1, the changes of μ will affect to the limiting efficiency for Policy II or Policy III. But, we can keep the policy of type III to be optimal, because in this type we can decide k_0 independently of μ . Following Figure 6.1 and Table 6.1 show the behaviors of $Eff'_{\infty}^{(2)}$ and $Eff_{\infty}^{(3)}$ for various μ .

Table 6.1

μ	0.1	0.5	1	2	3	5	8	10	∞
$Eff_{\infty}^{(3)}$	0.04	0.17	0.29	0.45	0.56	0.68	0.77	0.81	1
$Eff'_{\infty}^{(2)}$	0.003	0.07	0.21	0.43	0.54	0.63	0.66	0.67	0.68

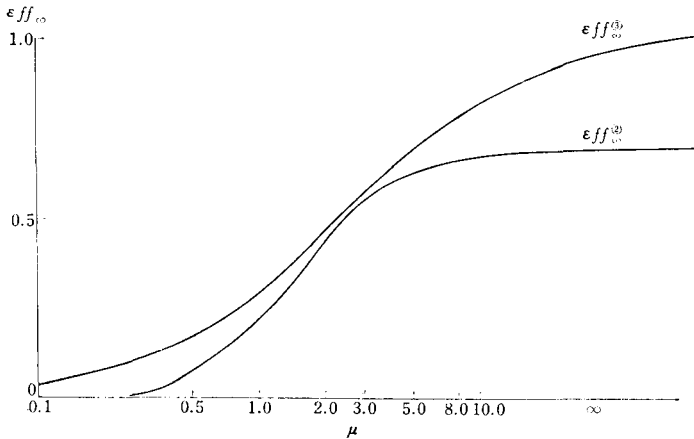


Fig. 6.1

From (6.1) and (6.2) we have in general

$$(6.3) \quad Eff_{\infty}^{(3)} = \frac{1}{1 + \frac{B\left(k, \frac{1}{\beta}\right)}{\beta} - ((k-1)T_m + T_s) \frac{1}{\mu}}$$

and

$$(6.4) \quad Eff'_{\infty}^{(2)} = \frac{1}{\left(1 + \frac{T_s}{t^*}\right) + t^{*\beta-1} T_m \left\{ \Gamma\left(1 + \frac{1}{\beta}\right) \right\}^2 \frac{1}{\mu^2}}$$

Hence for sufficiently large μ we can see

$$(6.5) \quad Eff_{\infty}^{(3)} \approx 1$$

but

$$(6.6) \quad Eff'_{\infty}^{(2)} \approx \frac{t^*}{t^* + T_s} < 1.$$

Remark 6.1 As stated in §2, there exists some losses for Policy II even when the mean life times of the systems are identical. In the following we shall show numerically this fact.

For the case of Example 5.3 where $T_s=2$, $T_m=1/2$, $\beta=3$, we have

$$(6.7) \quad Eff_{\infty}^{(3)} = \frac{1}{1 + 1.875/\mu}$$

and

$$(6.8) \quad Eff'_{\infty}^{(2)} = \frac{1}{1 + 2.125/\mu}$$

From this numerical results we can see that

$$(6.9) \quad Eff_{\infty}^{(3)} > Eff'_{\infty}^{(2)}$$

in this case.

This cause is that one may throw away a running system at the time point of each overhaul under the policy of type II, but under Policy III, it may be operated till its next failure. Thus, we can say that the predominance of Policy III appears explicitly in the case of short period for maintenance.

Remark 6.2 We stated in the above that our Policy III is most useful in maintaining large systems consisted with a number of equipments of same type. Here, we shall remark more precisely on this point.

Suppose a system consisted with a number of equipments such as a computer which has a number of packages. Or, suppose many machines of same type arranged in a room. Usually, a part of the system(subsys-

tem) will be used more frequently than the other. Thus, we may regard the life time distributions of these subsystems as different each other. Then, if we take the maintenance policy that we deal with these subsystems as individual systems and determine the maintenance period for each subsystem individually, it may be an efficient policy. However, in many practical cases, since this policy is too complex to adopt, one may use a simpler policy which is optimal in the case that all life time distributions of subsystems are identical.

By the way, when the life time distributions of subsystems are Weibull type, it is a natural assumption that the shape parameters (β) are identical and the scale parameters (α) are different corresponding as the frequencies of uses. This situation is identical with our case discussed in this paper. Theorem 4.1 shows that the optimal policy of type III is determined only by the shape parameter. In the other words, we can use an optimal policy over whole system in this case based on Policy III. But, if we adopt Policy II, we cannot consider any maintenance procedure to be optimal. This fact shows the robustness of Policy III in a different sense from the above.

REFERENCES

- [1] R. Barlow and L. Hunter, "Optimum preventive maintenance policies", *Oper. Res.*, Vol. 8, pp. 90—100, 1960
- [2] H. Hatori, "Some theorems in an extended renewal theory, I", *Kodai Math. Sem. Rep.*, Vol. 11, pp. 136—146, 1959