A FUNCTIONAL EQUATION ARISING IN CONTROL PROCESS WITH CERTAIN PROBABILITY CRITERION

TOSHIO ODANAKA

Tokyo Metropolitan Technical College

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1. INTRODUCTION

An equation we have used is the following. Consider a system specified at any time t by a finite dimensional vector $x(t)$, satisfying a vector differential equation $\frac{dx}{dt}=g(x, y, z)$, $x(0)=c$, where c is the initial state, $y(t)$ is a randomforcing term possessing a known distribution, and $z(t)$ is a forcing chosen, via a feed back process, so as to minimize the functional defined by the probability that $\{\max_{0\le t\le T} x(t)\}$ or $\{\min_{0\le t\le T} x(t)\}$ exceeds a specified bound. The functional equation is introduced by the technique of dynamic programing which may be used to obtain analytic approach to problem of this genre.

The purpose of the present note is to discuss the character of the solution of the functional equation introduced. The method of successive approximations that we employ is a very powerful analytic tool for the treatment of functional equations of this general class.

2. MATHEMATICAL FORMULATION

We wish to determine the control vector $z(t)$, so as to minimize a prescribed functional of $x(t)$ and $y(t)$ which can be written

$$J\{z\}=\text{Prob}\{(\max_{0\le t\le T} x(t)>\alpha) \text{ or } (\min_{0\le t\le T} x(t)<\beta)\} \tag{2.1}$$

subject to constraint on $z(t)$. This constraint is expressible in the form

$$0\le z(t)\le A \tag{2.2}$$

In order to study the minimziation of the expression (2.1) with the technique of dynamic programing, it is desirable to formulate the problem in discrete form.

We assume that the differential equation $\frac{dx}{dt}=g(x, y, z)$, $x(0)=c$

replaced by the simple difference equation

$$x_{k+1}-x_k=z_k+y_k,\ x_0=c. \tag{2.3}$$

We define

$f_k(c)$=minimum probability over $(N-k)$ stage period starting with an initial quality C is $k-$stage and ending at stage $N-1$, using an optimal policy. $k=0, 1, 2, \cdots\cdots N-1$.

That is

$$f_k(c)=\min_{0\leq(z)\leq A} \text{Prob}[(\max_{k\leq n\leq N-1} x_n>\alpha) \text{ or } (\min_{k\leq n\leq N-1} x_n<\beta)] \tag{2.4}$$

where $x_k=c,\ (k=0, 1, 2, \cdots\cdots N-2)$.

Then we have

$$f_k(c)=1, \qquad (c>\alpha,\ c<\beta),$$

$$=\min_{c\leq m\leq A-c}\left[\int_{w-\beta}^{\infty}\phi_k(y)dy+\int_{-\infty}^{w-\alpha}\phi_k(y)py+\int_{w-\alpha}^{w-\beta}f_{k+1}(w-y)\phi(y)dy\right],$$

$$(\beta\leq c\leq\alpha)$$

$$(k=0, 1, 2, \cdots\cdots N-2), \tag{2.5}$$

with

$$f_{N-1}(c)=1,\ (c>\alpha,\ c<\beta),$$
$$=0,\ (\beta<c<\alpha), \tag{2.6}$$

where $\phi_k(y)$ is the known probability density function of random variable y_k.

3. THE CHARACTER OF THE SOLUTIONS

The result we wish to prove is

Theorem 1. Let us assume that

a. $\phi_k(y)\geq 0, \int_{-\infty}^{\infty}\phi_k(y)dy=1$, for all y, all k. (3.1)

b. $\int_{w-\alpha}^{w-\beta}\phi_k(y)dy\leq a<1$, for all w, all k. (3.2)

c. $\phi_k(y)$ is unimodal in an interval $[-\infty, \infty]$. (3.3)

d. $\phi_k'(b_k)=0,\ 0\leq b_k\leq\alpha-\beta,\ b_k\leq A\leq b_k+(\alpha-\beta)$ and b_k is unique.

Then the optimal policy has the following form:

For each k,

$$\text{(a)}\quad \text{for } c\leq\bar{x}_k,\ z_k=\bar{x}_k-c, \tag{3.4}$$
$$\text{(b)}\quad \text{for } c>\bar{x}_k,\ z_k=0, \tag{3.5}$$

where, the sequence $\bar{x}_n=b_n+\alpha-\theta(\alpha-\beta)$.

If $\phi_k(y)=\phi(y)$, then we have $\bar{x}_k=\bar{x}=b_0+\alpha-\theta(\alpha-\beta)$, where $\phi'(b_0)=0$, $0<\theta<1$.

Remark: Let us introduce the definition of unimodal.
A function $f(x)$ is unimodal in an interval $[0, b]$, if there is a number x_0, $0\leq x_0\leq b$, such that $f(x)$ is either strictly increasing for $x\leq x_0$ and strictly decreasing for $x\geq x_0$, or, else strictly increasing for $x<x_0$, and strictly decreasing for $x\geq x_0$. The most important example of functions of this nature are concave function.

Proof: The proof will be inductive. We have with $f_{N-2}(c)$ defined as the following

$$f_{N-2}(c)=1, \qquad (c>\alpha,\ c<\beta),$$
$$=\min_{0\leq z\leq A}\left[\int_{w-\beta}^{\infty}\phi_{k-2}(y)dy+\int_{-\infty}^{w-\alpha}\phi_{k-2}(y)dy\right],\ (\beta<c<\alpha). \quad (3.6)$$

as our critical value of w is attained by setting the partial derivative with respect to w equal to zero,

$$0=-\phi_{N-2}(w-\beta)+\phi_{N-2}(w-\alpha)=F_1(w). \quad (3.7)$$

By the mean value theorem, $F_1(w)$ has following

$$F_1(w)=(\alpha-\beta)\phi'(w-\alpha+\theta(\alpha+\beta))=0.$$

The solution if it exists, is unique, where $w=c+z$ this value does exist under assumption d. for the probability density function $\phi_k(y)$. Call this value $\bar{x}_{N-2}=b_{N-2}+\alpha-\theta(\alpha-\beta)$. It is clear thus that for $k=N-2$ the optimal policy is $z_{N-2}=\bar{x}_{N-2}-c$ for $c\leq\bar{x}_{N-2}$, and $z_{N-2}=0$ for $c\geq\bar{x}_{N-2}$.
When $c\leq\bar{x}_{N-2}$, we have $f_{N-2}'(c)=0$ and for $c\geq\bar{x}_{N-2}$, we have

$$f_{N-2}(c)=\left[\int_{c-\beta}^{\infty}\phi_{N-2}(y)dy+\int_{-\infty}^{c-\alpha}\phi_{N-2}(y)dy\right]=1-\int_{c-\alpha}^{c-\beta}\phi_{N-2}(y)dy, \quad (3.8)$$

and

$$f'_{N-2}(c)=\phi_{N-2}(c-\alpha)-\phi_{N-2}(c-\beta)>0, \quad (3.9)$$

Consider the case $k=N-3$. We have

$$f_{N-3}(c)=1 \qquad (c>\alpha,\ c<\beta),$$
$$=\min_{c\leq z\leq A-c}\left[\int_{w-\beta}^{\infty}\phi_{N-3}(y)dy+\int_{-\infty}^{w-\alpha}\phi_{N-3}(y)dy\right.$$
$$\left.+\int_{w-\alpha}^{w-\beta}f_{N-2}(w-y)\phi_{N-3}(y)dy\right],\ (\beta<c<\alpha), \quad (3.10)$$

The critical value of w is attained by setting the partical derivative with respect to w equal to zero.

$$0=-\phi_{N-3}(w-\beta)+\phi_{N-3}(w-\alpha)+\int_{w-\alpha}^{w-\beta} f_{N-2}(w-y)\phi'_{N-3}(y)dy=F_2(w). \tag{3.11}$$

As, $\phi'(y)>0$ in $A-(\alpha-\beta)>(\alpha-\beta)\geq b_0$, we have by the mean value theorem

$$F_2(w)=\phi(w-\alpha)-\phi(w-\beta)+f_{N-2}(\xi)\int_{w-\alpha}^{w-\beta}\phi'_{N-3}(y)dy$$
$$=(\alpha-\beta)(1-f_{N-2}(\xi))(\phi'_{N-3}(w-\alpha+\theta(\alpha-\beta)).$$

For, $c\leq w\leq A+c$, $\beta<c<\alpha$, the value of w that $w-\alpha+\theta(\alpha-\beta)=b_{N-3}$, exist for $0<\theta<1$. Call this root $\bar{x}_{N-3}$.

The policy is then

$$\begin{aligned} z_{N-3}&=\bar{x}_{N-3}-c, \quad c\leq\bar{x}_{N-3},\\ z_{N-3}&=0, \qquad\qquad c\geq\bar{x}_{N-3}. \end{aligned} \tag{3.18}$$

It remains to show that $\bar{x}_{N-3}=\bar{x}_{N-2}$. The quantity $\bar{x}_{N-2}$ is determined by equation (3. 7), while $\bar{x}_{N-3}$ is determined by (3. 11). Since it follows that if $\phi_k(y)=\phi(y)$, then we have $\bar{x}_{N-3}=\bar{x}_{N-2}=\bar{x}$. From this it is clear that if $\phi_k(y)=\phi(y)$ then $\bar{x}_{N-3}=\bar{x}_{N-2}$.

The case $\phi_k(y)=\phi(y)$, we show that

$$f'_{N-3}(c)\leq f'_{N-2}(c), \tag{3.13}$$

We have

$$\begin{aligned} f'_{N-2}(c)&=0, & c&\leq\bar{x},\\ &=\phi(c-\alpha)-\phi(c-\beta), & c&\geq\bar{x}, \end{aligned} \tag{3.14}$$

and

$$\begin{aligned} f'_{N-3}(c)&=0, \qquad c\leq\bar{x},\\ &=\phi(c-\alpha)-\phi(c-\beta)+\int_{c+z-\alpha}^{c+z-\beta} f'_{N-2}(c+z+y)\phi(y)dy, \qquad c\geq\bar{x}, \\ &=(1-f_{N-2}(\xi))(\phi(c-\alpha)-\phi(c-\beta)), \qquad \beta<\xi<\alpha. \end{aligned} \tag{3.15}$$

In the interval $[\beta, \alpha]$ the inequality is clear. We now have all the ingredients of an inductive proof.

4. UNBOUNDED PROCESS

The case of unbounded process, i. e. $N=\infty$, yields the set of functional equation

$$f_k(c)=\lim_{N\to\infty}\min \text{Prob}\,[(\max_{k\le n\le N-1} x_n>\alpha) \text{ or } (\min_{k\le n\le N-1} x_n<\beta)] \quad (4.1)$$

We obtain the some general character.

Theorem 2: Under the above assumption upon $\phi_k(y)$, the optimal policy has the following form:

a. $z_k=\bar{x}_k-c$, for $c\le\bar{x}_k$, (4.2)

b. $z_k=0$, for $c\ge\bar{x}_k$, (4.3)

where $\bar{x}_k=b_k-\alpha+\theta(\alpha-\beta)$.

Especially, if $\phi_k(y)=\phi(y)$, we have

a. $\bar{x}_k=\bar{x}$, for all k.

b. $f_k'(c)\le f'_{k+1}(c)\le\cdots\cdots\le f'_{N-2}(c)$.

c. $f_k(c)\ge f_{k+1}(c)\ge\cdots\cdots\ge f_{N-2}(c)$.

Since $f_k(c)$ converge to $f(c)$, $f_k'(c)$ to $f'(c)$, z_k to z and $\bar{x}_k=\bar{x}$, we see that the solution has the indicated form.

5. DISCUSSION

a) Terminal Control Process

Let us consider a terminal control process involving the minimization of a functional such as

$$J(y)=\text{Prob}\{(x_N>\alpha) \text{ or } (x_N<\beta)\}, \quad (5.1)$$

where

$$x_{k+1}=x_k+z_k-y_k,\ x_k=c,\ 0\le z_k\le A.$$

Write

$$\min J(z)=f_k(c)$$

Then we obtain the following functional equation, such as

$$\begin{aligned} f_{N-1}(c)&=1,\ (c<\alpha,\ c<\beta),\\ &=0,\ (\beta<c<\alpha), \end{aligned} \quad (5.2)$$

$$f_k(c)=\min_{y_k}\left\{\int f_{k+1}(c+x_k-y_k)\phi_k(y)dy\right\},\ \text{for all } c.$$

This equation is analogous to the equation (2. 5) and (2. 6).

(b) Constraint

Consider a control problem involving the minimization of the probability. Suppose the equation

$$x_{k+1}-x_k=z_k-y_k,\ x_0=c,$$

and we wish to choose z_k, subject to the constraint $\sum_{n=k}^{N-1} z_n^2<m$, so as to minimize the functional

$$J(y)=\text{Prob}\{(\max_{k\leq n\leq N-1} x_n<\alpha) \text{ or } (\min_{k\leq n\leq N-1} x_n<\beta)\}.$$

Consider the new problem of maximization the expression.

$$\min[\text{Prob}\{(\max_{k\leq n\leq N-1} x_n<\alpha) \text{ or } (\min_{k\leq n\leq N-1} x_n<\beta)\}e^{\lambda\Sigma Z^2{}_n}]=f_k(c), \quad (5.3)$$

$$\begin{aligned} f_k(c)&=\min(e^{\lambda\Sigma Z^2{}_n})=1, & (c>\alpha,\ c>\beta),\\ &=\min[e^{\lambda Z_k} f_{k+1}(c+z-y)\phi_k(y)dy], & (\beta<c<\alpha),\end{aligned}$$

with

$$\begin{aligned} f_{N-1}(c)&=\min(e^{\lambda Z^2{}_{N-1}})=1, & (c>\alpha,\ c<\beta), \quad (5.4)\\ &=0, & (\beta<c<\alpha),\end{aligned}$$

(c) The errors of first kind and of second kind.

In this case, we wish to introduce the errors of first kind and of second kind.

The error of the first kind may be stated mathematically as

$$P_1=\text{Prob}\{(\max_{k\leq n\leq N-1} x_n>\mu+B\sigma) \text{ or } (\min_{k\leq n\leq N-1} x_n<\mu-B\sigma)\} \quad (5.5)$$

where let us consider the polulation's $N(\mu,\ \sigma^2)$, if we now put

$$U_n=\frac{x_k-\mu}{\sigma}$$

then we have $E(U_k)=0$, Var. $(U_k)=1$,

and

$$P_1=\text{Prob}\{(\max_{k\leq n\leq N-1} U_n<B) \text{ or } (\min_{k\leq n\leq N-1} U_n>-B)\}, \quad (5.6)$$

Selected values of B are easily computed by use of a table

$$\lim_{n\to\infty} \text{Prob}\{\min_{1\leq k\leq n}|S_k|\leq\sqrt{n}\,\alpha\}=\frac{4}{\pi}\sum_{m=0}^{\infty}\frac{(-1)^m}{2m+1}\exp\left\{-\frac{(2m+1)}{8\alpha^2}\pi^2\right\}, \quad (5.7)$$

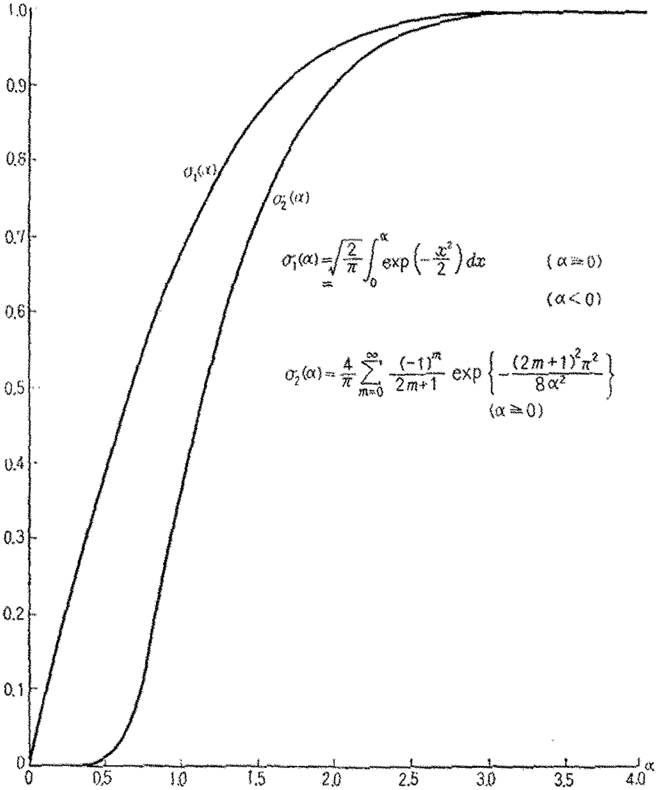
where $S_n=X_1+X_2+\cdots\cdots+X_n$ and $\{X_n\}$ are random variables with N (0, 1). (Fig. 1, Table 1)

The erors of the second kind may be stated as

$$P_2=\text{Prob}\{(\max_{k\leq n\leq N-1} U_n>B+\delta_k) \text{ or } (\min_{k\leq n\leq N-1} U_n<B-\delta_k)\},$$

where $\delta_k=\dfrac{\mu_k'-\mu_k}{\sigma}$ and let us cosider the pupulation $N(\mu_k',\ \sigma^2)$.

In our notations, Min P_2 is $F_k(c)$.


1.0
0.9
0.8
0.7
0.6
0.5
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0.2
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1.5
2.0
2.5
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4.0
α
$\sigma_1(\alpha)$
$\sigma_2(\alpha)$
$\sigma_1(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^{\alpha} \exp\left(-\frac{x^2}{2}\right) dx$ $(\alpha \geqq 0)$
$= 0$ $(\alpha < 0)$
$\sigma_2(\alpha) = \frac{4}{\pi} \sum_{m=0}^{\infty} \frac{(-1)^m}{2m+1} \exp\left\{-\frac{(2m+1)^2\pi^2}{8\alpha^2}\right\}$
$(\alpha \geqq 0)$

Fig. 1

Table 1.

α	$\sigma_1(\alpha)$	$\sigma_2(\alpha)$	α	$\sigma_1(\alpha)$	$\sigma_2(\alpha)$
0	0.0000	0.0000			
0.1	0.0797	0.0000	2.1	0.9643	0.9284
0.2	0.1585	0.0000	2.2	0.9722	0.9441
0.3	0.2358	0.0000	2.3	0.9786	0.9573
0.4	0.3108	0.0005	2.4	0.9836	0.9675
0.5	0.3829	0.0072	2.5	0.9876	0.9740
0.6	0.4515	0.0324	2.6	0.9907	0.9806
0.7	0.5161	0.0805	2.7	0.9907	0.9864
0.8	0.5763	0.1451	2.8	0.9949	0.9901
0.9	0.6319	0.2776	2.9	0.9963	0.9921
1.0	0.6827	0.3707	3.0	0.9973	0.9947
1.1	0.7287	0.4591	3.1	0.9981	0.9965
1.2	0.7899	0.5402	3.2	0.9986	0.9978
1.3	0.8064	0.6130	3.3	0.9990	0.9982
1.4	0.8385	0.6773	3.4	0.9993	0.9985
1.5	0.8664	0.7330	3.5	0.9995	0.9986
1.6	0.8904	0.7807	3.6	0.9997	0.9991
1.7	0.9109	0.8216	3.7	0.9998	0.9998
1.8	0.9281	0.8561	3.8	0.9999	1.0000
1.9	0.9425	0.8850	3.9	0.9999	1.0000
2.0	0.9545	0.9095	4.0	9.9999	1.0000

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