Journal of the
Operations Research
Society of Japan

VOLUME 5 June 1963 NUMBER 3

SEQUENCING ON TWO MACHINES WITH START LAG AND STOP LAG

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(Presented at the 10th meeting, November 4, 1961. Received January 9, 1962)

1. INTRODUCTION

Mitten [1, 2] and Johnson [3] have considered a problem involving the sequencing of n jobs on two machines with start lag and stop lag. In this paper we shall consider a problem of the same type which is slightly generalized form and give a different formulation and some results which cover the provious results, by using the functional-equation approach formulated in our previous paper [4].

2. PROBLEM

Let two machines be named by I, II, and each job i of n jobs be consist of two parts i_1 and i_2 where i_1 must be processed on machin I before i_2,. Let A_i, F_i be the time required to process i_1, i_2 respectively on the machine I and, after the processing of i_1 on machine I, i_1 must be processed immediately on machine II with the processing time B_i, unless i_2 is being processed on II, where the order of n jobs for each machines must be the same.

Moreover we assume that job i_1 is started on machine II, D_i time units (start lag) after it has been started on machine I and that job i_1 may not be completed on machine II sooner than E_i time units(stop lag) after its completion on machine I. (Fig. 1)

In this model, each part i_1 represents the main part of the job i and

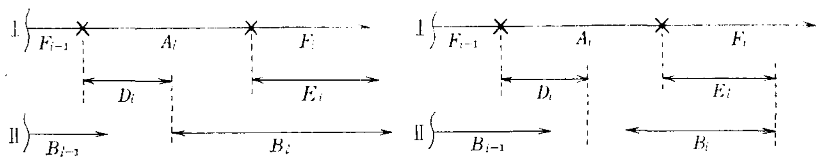

Fig. 1

each part i_2 represents the simple part completed only by one machine I and for each part i_1, the two machines I, II, create the most serious bottleneck machines and the start lag D_i and the stop lag E_i not only represent the processing time on the intermediate (non-bottleneck) machines before being started on machine II, but also represent the transportation time between machines. Also this model considers the overlapping production procedure.

Then the problem is to deciding the order in which n jobs should be processed by two machines I, II in order to minimize the time required to complete all the operations.

3. SOLUTION

When an optimal scheduling procedure is employed and after the processing of some definite sequence S of jobs, the machine II is committed t hours ahead for the machine I.

If job i is processed first after the squence S of jobs, then by defining

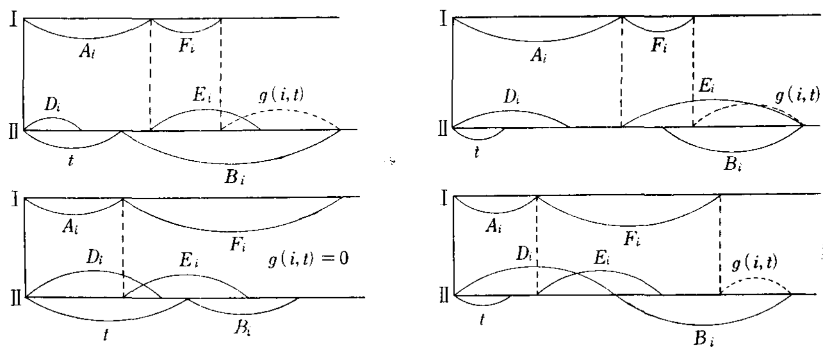

Fig. 2

$f_1(i, t)$=the time consumed in processing the job i on I and II, we have (Fig. 2)

$$f_1(i, t)=A_i+F_i+g(i, t) \qquad (1)$$

where
$$g(i, t)=\max[\max[A_i+E_i, \max(t, D_i)+B_i]-(A_i+F_i), 0]$$
$$=-A_i-F_i+\max[A_i+E_i, \max(t, D_i)+B_i, A_i+F_i]$$
$$=-A_i+B_i-F_i+\max[t, D_i, A_i-B_i+E_i, A_i-B_i+F_i] \qquad (2)$$
$$=-A_i+B_i+\max[t-F_i, D_i-F_i, A_i-B_i+E_i-F_i, A_i-B_i] \qquad (3)$$

By substituting (2) into (1), we have

$$f_1(i, t)=B_i+\max[t, D_i, A_i-B_i+E_i, A_i-B_i+F_i] \qquad (4)$$

If we choose the job j to follow, then by defining

$f_2(i, j, t)$=the time consumed in processing both the job i and the job j in this order on the machines I, II after the sequence S of jobs,

we obtain from (1), (3) and (4)

$$f_2(i, j, t)=A_i+F_i+f_1(j, g(i, t))$$
$$=A_i+F_i+B_j+\max[-A_i+B_i+\max[t-F_i, D_i-F_i, A_i-B_i+E_i-F_i, A_i-B_i], D_j, A_j-B_j+E_j, A_j-B_j+F_j]$$
$$=B_j+B_i+\max[t, D_i, A_i-B_i+E_i, A_i-B_i+F_i, D_j+A_i-B_i+F_i, A_i+A_j-B_i-B_j+F_i+E_j, A_i+A_j-B_i-B_j+F_i+F_j] \qquad (5)$$

On the other hand, if we interchange the order of the job i and the job j, we have similarly the formula of

$$f_2(j, i, t)=A_j+F_j+f_1(i, g(j, t))$$

obtained by exchanging j for i and i for j in (5).
So that, if

$$f_2(i, j, t)<f_2(j, i, t) \qquad (6)$$

holds, the order in which job i precedes job j minimizes the time required to complete all the operations.

In the following we shall derive some criterion from (6).
First, by putting

$$e_i=A_i-B_i, \ T_i=\max[D_i, e_i+E_i]>0, \qquad (7)$$

we obtain from (6)

$$\max[t, T_i, e_i+F_i, T_j+e_i+F_i, e_i+e_j+F_i+F_j]$$
$$<\max[t, T_j, e_j+F_j, T_i+e_j+F_j, e_j+e_i+F_j+F_i] \qquad (8)$$

So that, by dropping t and $e_i+e_j+F_i+F_j$ from both side of (8), if

$$\max[T_i, e_i+F_i, T_j+e_i+F_i]<\max[T_j, e_j+F_j, T_i+e_j+F_j] \qquad (9)$$

holds, the left hand side of (8) is not larger than the right hand side of

(8).

As $e_i+F_i<T_j+e_i+F_i,\ e_j+F_j<T_i+e_j+F_j$, we have from (9),

$$\max[T_i,\ T_j+e_i+F_i]<\max[T_j,\ T_i+e_j+F_j] \qquad (10)$$

By subtracting T_i+T_j from both sides of (10), we easily obtain (11).

$$\min[T_j,\ T_i-(e_i+F_i)]>\min[T_i,\ T_j-(e_j+F_j)] \qquad (11)$$

Hence we obtain the next theorem

Theorem 1. An optimal ordering is determined by the following rule: Job i precedes job j if

$$\min[T_i,\ T_j-(e_j+F_j)]<\min[T_j,\ T_i-(e_i+F_i)]$$

holds, where $e_i=A_i-B_i,\ T_i=\max[D_i,\ e_i+E_i]$

If there is equality, either ordering is optimal.

Next we shall derive Mitten's type criterion from (10). Let the set of n jobs be partitioned into two disjoint subsets,

$$\mathfrak{M}_1=\{i|e_i+F_i<0\},\ \mathfrak{M}_2=\{i|e_i+F_i\geqq 0\},$$

then we consider the following three cases.

(I) For $i\in\mathfrak{M}_1,\ j\in\mathfrak{M}_2$, being $e_i+F_i+T_j>T_j,\ \ T_i\leqq e_j+F_j+T_i$, (10) holds. So that job i precedes job j.

(II) For $i\in\mathfrak{M}_1,\ j\in\mathfrak{M}_1$, being $T_i>e_j+F_j+T_i,\ e_i+F_i+T_j<T_j$, (10) is identical with $T_i<T_j$. So that if $T_i>T_j$, job i precedes job j.

(III) For $i\in\mathfrak{M}_2,\ j\in\mathfrak{M}_2$ being $T_i\leqq e_j+F_j+T_i,\ e_i+F_i+T_j\geqq T_j$, (10) is identical with $e_i+F_i+T_j<e_j+F_j+T_i$and also with $T_j-(e_j+F_j)>T_i-(e_i+F_i)$. So that if $T_j-(e_j+F_j)<T_i-(e_i+F_i)$, job i precedes job j.

Hence we obtain the next theorem.

Theorem 2. Let $e_i=A_i-B_i,\ T_i=\max[D_i,\ e_i+E_i]$ and let the set of n jobs be partitioned into two disjoint subsets,

$$\mathfrak{M}_1=\{i|e_i+F_i<0\},\ \mathfrak{M}_2=\{i|e_i+F_i\geqq 0\}.$$

An optimal ordering is determined by the following rule:

(I) An optimal ordering is then $\mathfrak{M}_1$ followed by $\mathfrak{M}_2$.

(II) In $\mathfrak{M}_1$, job i precedes job j if $T_i<T_j$.

(III) In $\mathfrak{M}_2$, job i precedes job j if $T_i-(e_i+F_i)>\mathrm{T}_j-(e_j+F_j)$.

If there is equality in (II), (III), either ordering is optimal.

Another criterion similar to that of theorem 2 can be obtained by putting

$$m_i=\max[D_i-A_i,\ E_i-B_i], \tag{12}$$

then
$$m_i+A_i=\max[D_i,\ A_i-B_i+E_i]=T_i \tag{13}$$

$$\begin{aligned} m_i+B_i-F_i &=\max[D_i-A_i+B_i-F_i,\ E_i-F_i] \\ &=\max[D_i-e_i-F_i,\ E_i-F_i] \\ &=\max[D_i,\ e_i+E_i]-e_i-F_i \\ &=T_i-(e_i+F_i) \end{aligned} \tag{14}$$

Hence we obtain from theorem 2 the next theorem.

Therem 3. Let $m_i=\max[D_i-A_i,\ E_i-B_i]$ and let the set of n jobs be partitioned into two disjoint subsets,

$$\mathfrak{M}_1=\{i|(A_i+F_i)-B_i<0\},\ \mathfrak{M}_2=\{i|(A_i+F_i)-B_i\geqq 0\}.$$

An optimal ordering is determined by the following rule:

(I) An optimal ordering is then $\mathfrak{M}_1$ followed by $\mathfrak{M}_2$.

(II) In $\mathfrak{M}_1$, job i precedes job j if $m_i+A_i<m_j+A_j$.

(III) In $\mathfrak{M}_2$ job i precedes job j if $m_i+B_i-F_i>m_j+B_j-F_j$.

If there is equality in (II), (III), either ordering is optimal.

When $F_i\equiv 0$ for all i, theorem 3 coincides with the Mitten's criterion.

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