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ON THE MULTIPLE EXPONENTIAL CHANNEL QUEUING SYSTEM WITH HYPER-POISSON ARRIVALS

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1. INTRODUCTION

The notion of hyper-Poisson distribution was introduced to simulate the more random type than the Poisson distribution. (cf. Morse [2]). In the case of the hyper-Poisson arrivals, the arrival-timing channel consists of two branches and an arriving customer is assigned to one or the other branch at random with the fraction of σ and $(1-\sigma)$ respectively, where $0<\sigma\leqq\frac{1}{2}$. The mean rates of arrivals of these two branches are $2\sigma\lambda$ and $2(1-\sigma)\lambda$ respectively. When $\sigma=\frac{1}{2}$, this becomes Poisson arrivals. Morse [2] solved the queuing system with hyper-Poisson arrivals and single exponential channel.

Here we shall treat the case of hyper-Poisson arrivals and multiple exponential channels. Queue discipline is assumed to be first come, first served. We shall consider the steady state case only. The steady state probabilities are denoted by $p_{n,i}(i=1,2)$ where the second subscript expresses the branch of the arrival-timing channel that is occupied ($i=1$ if the $2\sigma\lambda$ branch is occupied, $i=2$ if the $2(1-\sigma)\lambda$ branch is busy). The probabilities that there are n customers in the queue and service channels are $p_n=p_{n,1}+p_{n,2}$. The number of channels is denoted by c. These c exponential channels are assumed to be arranged in parallel and have the same service rate μ. The utilization factor for the whole facility will be denoted by $\rho=\dfrac{\lambda}{c\mu}$.

In the next section, we shall solve this system for the case of infinite queue. And then we give the solution for the case of no queue. In the last section, some of the numerical tables for $c=2$ are given.

2. INFINITE QUEUE CASE

We can easily see that the equations of detailed balance for the steady-state conditions for this case are

$$\mu p_{1,1}=2\sigma\lambda p_{0,1},\quad \mu p_{1,2}=2(1-\sigma)\lambda p_{0,2} \tag{1}$$

$$\left.\begin{aligned}&2\sigma^2\lambda p_{n-1,1}+2\sigma(1-\sigma)\lambda p_{n-1,2}+(n+1)\mu p_{n+1,1}\\&\quad=(n\mu+2\sigma\lambda)p_{n,1}\\&2\sigma(1-\sigma)\lambda P_{n-1,1}+2(1-\sigma)^2\lambda p_{n-1,2}+(n+1)\mu p_{n+1,2}\\&\quad=(n\mu+2(1-\sigma)\lambda)p_{n,2}\qquad(1\leqq n<c)\end{aligned}\right\} \tag{2}$$

$$\left.\begin{aligned}&2\sigma^2\lambda p_{n-1,1}+2\sigma(1-\sigma)\lambda p_{n-1,2}+c\mu p_{n+1,1}\\&\quad=(c\mu+2\sigma\lambda)p_{n,1}\\&2\sigma(1-\sigma)\lambda p_{n-1,1}+2(1-\sigma)^2\lambda p_{n-1,2}+c\mu p_{n+1,2}\\&\quad=(c\mu+2(1-\sigma)\lambda)p_{n,2}\qquad(n\geqq c).\end{aligned}\right\} \tag{3}$$

From (3), analogously to the case of hyper-Poisson arrivals and single exponential channel, we have

$$p_n=v_0^{n-1}p_c\qquad(n\geqq c) \tag{4}$$

where

$$v_o=\frac{1}{2}+\rho-\sqrt{\frac{1}{4}-\rho(1-\rho)(1-2\sigma)^2}.$$

Adding the corresponding two equations of (1), (2) and (3), we obtain recurssively

$$\left.\begin{aligned}&n\mu p_n=2\sigma\lambda p_{n-1,1}+2(1-\sigma)\lambda p_{n-1,2},\ (c\geqq n\geqq 1)\\&c\mu p_n=2\sigma\lambda p_{n-1,1}+2(1-\sigma)\lambda p_{n-1,2},\ (n\geqq c+1).\end{aligned}\right\} \tag{5}$$

Multiplying the first equations of (1), (2) and (3) by $(1-\sigma)$ and the second by σ and adding the corresponding these two equations and using (5), we get the following recurssion formulae:

$$\left.\begin{aligned}&2c\rho[n\{1-2\sigma(1-\sigma)\}+2\sigma(1-\sigma)c\rho]p_n\\&\quad=(n+1)[(n+2c\rho)p_{n+1}-(n+2)p_{n+2}](c-2\geqq n\geqq 0)\\&2c\rho[(c-1)\{1-2\sigma(1-\sigma)\}+2\sigma(1-\sigma)c\rho]p_{c-1}\\&\quad=c[(c-1)+c(2\rho-v_o)]p_c.\end{aligned}\right\} \tag{6}$$

From these formulae, we can solve p_n ($c \geqq n \geqq 0$) recurssively. Thus, referring (4), all the state probablities p_n can be expressed by p_c. Using the condition

$$\sum_{n=0}^{\infty} p_n = 1,$$

we can determine the value of p_c and then all the values of p_n can be determined completely. But this procedure is rather tedious when c is large.

Now, we define the generating functions to be

$$F_1(z) = \sum_{n=0}^{c-1} z^n p_n$$

$$F_2(z) = \sum_{n=c}^{\infty} z^n p_n,$$

and

$$F(z) = F_1(z) + F_2(z) = \sum_{n=0}^{\infty} z^n p_n.$$

Then, from (4), we have

$$F_2(z) = \frac{z^c p_c}{1 - v_0 z}. \tag{7}$$

Let Q_c be the probability that there are c or more customers in the system (queue plus services). Then

$$Q_c = \sum_{n=c}^{\infty} p_n = F_2(1) = \frac{p_c}{1 - v_o}. \tag{8}$$

Multiplying the first equations of (6) by z^n and the second by z^{c-1} and adding them all, we obtain

$$(1-z)F_1''(z) - 2c\rho[(1-z) + 2\sigma(1-\sigma)z]F_1'(z) + 2c\rho 2\sigma(1-\sigma)c\rho F_1(z) = cz^{c-2}[c(2\rho - v_o)z - (c-1)(1-z)]p_c. \tag{9}$$

This is the differential equation for the generating function $F_1(z)$. If we put $z=1$ in (9), we get

$$F_1'(1) = c\rho F_1(1) - \frac{c(1-\rho)p_c}{1 - v_o}, \tag{10}$$

here we used the relation

$$(1 - v_o)(2\rho - v_o) = 4\rho\sigma(1-\sigma)(1-\rho).$$

From the relation

$$F(1)=F_1(1)+F_2(1)=\sum_{n=0}^{\infty} p_n=1,$$

using (8), we obtain

$$F_1(1)=1-\frac{p_c}{1-v_o}.$$

Inserting this value into (10), we have

$$F_1'(1)=c\rho-\frac{cp_c}{1-v_o}. \tag{11}$$

We shall denote the mean number in the system (in queue and in services) by L. This is expressed by

$$L=\sum_{n=0}^{\infty} np_n=F_1'(1)+F_2'(1).$$

From (7), (11) and (8), we obtain

$$\begin{aligned}L&=c\rho+\frac{v_o p_c}{(1-v_o)^2}\\ &=c\rho+\frac{v_o}{1-v_o}Q_c.\end{aligned} \tag{12}$$

Let V_L be the variance of L. Then

$$\begin{aligned}V_L&=\sum_{n=0}^{\infty}(n-L)^2 p_n\\ &=F''(1)+L-L^2.\end{aligned}$$

To obtain the value of $F''(1)$, we differentiate (9) with respect to z and put $z=1$. Using (11), we obtain

$$\begin{aligned}[1+2c\rho 2\sigma(1-\sigma)]F_1''(1)&=2c^2\rho^2[1+2\sigma(1-\sigma)(c\rho-1)]\\ &\quad -c[(c-1)(1-v_o)+2c\rho\{1+2\sigma(1-\sigma)(c+\rho-2)\}]Q_c.\end{aligned}$$

And from (7)

$$F_2''(1)=c(c-1)Q_c+2\left(c+\frac{v_o}{1-v_o}\right)(L-c\rho).$$

Thus, we have

$$\begin{aligned}V_L=&\frac{2c^2\rho^2[1+2\sigma(1-\sigma)(c\rho-1)]}{1+4c\rho\sigma(1-\sigma)}-2c\rho\left(c+\frac{v_o}{1-v_o}\right)\\ &+\frac{cv_o(cv_o-2c\rho-1)}{1+4c\rho\sigma(1-\sigma)}\cdot Q_c+\left(2c+1+\frac{2v_o}{1-v_o}\right)L-L^2.\end{aligned}$$

For the mean number in queue, we obtain

$$L_q=\sum_{n=c+1}^{\infty}(n-c)p_n=F_2'(1)-cQ_c$$

$$=\frac{v_o}{1-v_o}Q_c. \tag{13}$$

Comparing (13) with (12), we have

$$L=L_q+c\rho. \tag{14}$$

This is the same relation as the case for multiple exponential channels with Poisson arrivals.

Next, we shall find the waiting-time distribution in queue (delay-in-queue distribution). To do this, we have to find the probability q_n that an arriving unit finds n customers in the system. This probability must be proportional to $\sigma p_{n,1}+(1-\sigma)p_{n,2}$. Using the relations (5) and the condition

$$\sum_{n=0}^{\infty} q_n=1,$$

we obtain

$$q_n=\begin{cases}\dfrac{n+1}{c\rho}p_{n+1}, & (0\leqq n\leqq c-1)\\[2ex] \dfrac{v_0^{n-c+1}}{\rho}p_c. & (n\geqq c)\end{cases} \tag{15}$$

The probability that at most m customers are served in time t is

$$\sum_{n=0}^{m}\frac{(c\mu t)^n}{n!}e^{-c\mu t}.$$

So, the delay-in-queue distribution $G_q(t)$ is

$$\begin{aligned}G_q(t)&=\sum_{m=0}^{\infty} q_{c+m}\sum_{n=0}^{m}\frac{(c\mu t)^n}{n!}e^{-c\mu t}\\ &=\frac{p_c}{\rho}e^{-c\mu t}\sum_{n=0}^{\infty}\frac{(c\mu t)^n}{n!}\sum_{m=n}^{\infty} v_0^{m+1}\\ &=\frac{v_o}{\rho}Q_c e^{-c\mu(1-v_o)t}.\end{aligned}$$

Thus, the probability density function of waiting-time in queue is

$$w_q(t)=\frac{c\mu v_o(1-v_o)Q_c}{\rho}e^{-c\mu(1-v_o)t},$$

and the mean waiting time in queue is

$$\begin{aligned}W_q&=\int_0^{\infty} t w_q(t)dt\\ &=\frac{v_oQ_c}{\lambda(1-v_o)}.\end{aligned} \tag{16}$$

From (14) and (16), we have the relation

$$W_q=\frac{L_q}{\lambda}. \tag{17}$$

To find the total waiting time distribution, we have to combine the two processes, one of which is the process in the queue and the other is in the service channel. The probability that the arriving unit will not yet have emerged from the system a time t after it enters the queue is

$$\int_t^\infty dx\int_0^x w_q(y)\mu e^{-\mu(x-y)}dy=\frac{v_oQ_c}{\{c(1-v_o)-1\}\rho}\{c(1-v_o)e^{-\mu t}-e^{-c\mu(1-v_o)t}\}.$$

And the probability that the arriving unit can enter the one of the service channels immediately is

$$1-\sum_{n=c}^\infty q_n=1-\frac{v_o}{\rho}Q_c.$$

Then the total watiting time distribution $G(t)$ is

$$G(t)=\left(1-\frac{v_oQ_c}{\rho}\right)e^{-\mu t}+\frac{v_oQ_c}{\{c(1-v_o)-1\}\rho}\{c(1-v_o)e^{-\mu t}-e^{-c\mu(1-v_o)t}\}$$

$$=\frac{v_oQ_c}{\{c(1-v_o)-1\}\rho}[e^{-\mu t}-e^{-c\mu(1-v_o)t}]+e^{-\mu t}.$$

The mean waiting time is

$$W=\frac{1}{\mu}+\frac{1}{\{c(1-v_o)-1\}\rho}\left[\frac{v_oQ_c}{\mu}-\rho W_q\right]$$

$$=\frac{1}{\mu}+W_q.$$

Referring (14) and (17), we obtain

$$W=\frac{L}{\lambda}.$$

This relation had been obtained by Little [1] for the general case.

3. NO QUEUE CASE

In this case, the steady state balance equations are

$$\mu p_{1,1}=2\sigma\lambda p_{0,1},\qquad \mu p_{1,2}=2(1-\sigma)\lambda p_{0,2} \tag{18}$$

$$\left.\begin{aligned}&2\sigma^2\lambda p_{n-1,1}+2\sigma(1-\sigma)\lambda p_{n-1,2}+(n+1)\mu p_{n+1,1}=(n\mu+2\sigma\lambda)p_{n,1}\\&2\sigma(1-\sigma)\lambda p_{n-1,1}+2(1-\sigma)^2\lambda p_{n-1,2}+(n+1)\mu p_{n+1,2}\\&\quad=(n\mu+2(1-\sigma)\lambda)p_{n,2}\end{aligned}\right\} \tag{19}$$

$$\left.\begin{aligned}&2\sigma^2\lambda p_{c-1,1}+2\sigma(1-\sigma)\lambda p_{c-1,2}=c\mu p_{c,1}\\&2\sigma(1-\sigma)\lambda p_{c-1,1}+2(1-\sigma)^2\lambda p_{c-1,2}=c\mu p_{c,2}\end{aligned}\right\} \tag{20}$$

From these equations, as before, we have

$$\left.\begin{aligned} &n\mu p_n=2\sigma\lambda p_{n-1,1}+2(1-\sigma)\lambda p_{n-1,2} \qquad (1\leqq n\leqq c) \\ &2\sigma(1-\sigma)p_c=(1-\sigma)p_{c,1}+\sigma p_{c,2} \end{aligned}\right\} \tag{21}$$

Using (21), from (18) and (19), we obtain the fundamental equations:

$$\left.\begin{aligned} &2c\rho[n\{1-2\sigma(1-\sigma)\}+2\sigma(1-\sigma)c\rho]p_n \\ &\quad=(n+1)[(n+2c\rho)p_{n+1}-(n+2)p_{n+2}] \quad (0\leqq n\leqq c-2) \\ &2c\rho[(c-1)\{1-2\sigma(1-\sigma)\}+2\sigma(1-\sigma)c\rho]p_{c-1} \\ &\quad=c[2\sigma(1-\sigma)2c\rho+(c-1)]p_c. \end{aligned}\right\} \tag{22}$$

From this system of equations, we can solve all p_n $(0\leqq n\leqq c-1)$ by p_c.

We shall define the generating function to be

$$F(z)=\sum_{n=0}^{c} z^n p_n.$$

Then, from (22), we get the differential equation for the generating function $F(z)$:

$$\begin{aligned} &(1-z)F''(z)-2c\rho[(1-z)+2\sigma(1-\sigma)z]F'(z)+2c\rho 2\sigma(1-\sigma)c\rho F(z) \\ &\quad=2c\rho[2\sigma(1-\sigma)c\rho z-c(1-z)\{1-2\sigma(1-\sigma)\}]z^{c-1}p_c. \end{aligned} \tag{23}$$

By the definition of $F(z)$, we have

$$F(1)=1$$

and

$$F'(1)=\sum_{n=0}^{c} np_n=L.$$

Thus, putting $z=1$ in (23), the mean number in the system is

$$L=c\rho(1-p_c). \tag{24}$$

This L is the mean number of filled channels, because we are considering no queue case. The relation (24) is just same as the multiple exponential channel system with Poisson arrivals. Differentiating (23) with respect to z and putting $z=1$, we obtain the varince of L:

$$V_L=\frac{2c[1+(c\rho-1)2\sigma(1-\sigma)]}{1+2c\rho 2\sigma(1-\sigma)}[(1+\rho)L-c\rho]+L-L^2.$$

Now, let p_{lost} be the probability of customers lost and p_{served} be the probability that the arriving unit can enter one of the service channel. Then, p_{lost} is proportional to

$$2\sigma\lambda p_{c,1}+2(1-\sigma)\lambda p_{c,2},$$

and p_{served} is proportional to

$$2\sigma\lambda\sum_{n=0}^{c-1} p_{n,1}+2(1-\sigma)\lambda\sum_{n=0}^{c-1} p_{n,2}$$

$$=\sum_{n=1}^{c} n\mu p_n$$
$$=\mu L=\lambda(1-p_c).$$

Using the condition

$$p_{lost}+p_{served}=1,$$

we have

$$\left.\begin{aligned} p_{lost}&=\frac{2\{1-2\sigma(1-\sigma)\}p_c}{1+\{1-4\sigma(1-\sigma)\}p_c} \\ p_{served}&=\frac{1-p_c}{1+\{1-4\sigma(1-\sigma)\}p_c} \end{aligned}\right\} \qquad (25)$$

When $\sigma=\frac{1}{2}$, we get

$$p_{lost}=p_c, \qquad p_{served}=1-p_c.$$

4. TABLES FOR $c=2$

As the simplest case, we will work out the case where the number of channels is 2. First we shall consider the infinite queue case. We can solve the equations (6) directly and we have

$$p_0=\frac{1+2\rho-4\rho^2+(2\rho-1)v_o}{1+4\rho\{1+2\rho\sigma(1-\sigma)\}-v_o}$$
$$p_1=\frac{2\rho[1+8\rho\sigma(1-\sigma)(1-\rho)-v_o]}{1+4\rho\{1+2\rho\sigma(1-\sigma)\}-v_o}$$
$$p_2=\frac{4\rho^2(1-v_o)[1+2\sigma(1-\sigma)(2\rho-1)]}{1+4\rho\{1+2\rho\sigma(1-\sigma)\}-v_o}.$$

The numerical table of these probabilities and the mean number in the system for some values of ρ and σ will be given below:

ρ	σ	p_0	p_1	p_2	Q_2	L
0.25	0.1	0.6374	0.2252	0.0839	0.1374	0.5876
	0.2	0.6195	0.2610	0.0809	0.1195	0.5570
	0.3	0.6083	0.2834	0.0779	0.1083	0.5423
	0.4	0.6020	0.2960	0.0757	0.1020	0.5354
	0.5	0.6	0.3	0.075	0.1	0.5333
0.5	0.1	0.4032	0.1936	0.1210	0.4032	1.9410
	0.2	0.3677	0.2646	0.1471	0.3677	1.5515
	0.3	0.3474	0.3052	0.1592	0.3474	1.4107
	0.4	0.3367	0.3266	0.1650	0.3367	1.3506
	0.5	0.3333	0.3333	0.1667	0.3333	1.3333

0.75	0.1	0.1976	0.1048	0.0771	0.6976	7.1121
	0.2	0.1697	0.1606	0.1187	0.6697	6.1433
	0.3	0.1538	0.1924	0.1432	0.6538	3.8310
	0.4	0.1455	0.2091	0.1565	0.6454	3.5168
	0.5	0.1429	0.2143	0.1607	0.6428	3.4285

Next, we can solve the system of equations (22) for the no queue case, and we have

$$p_0 = \frac{1}{1+2\rho+8\sigma(1-\sigma)\rho^2}$$

$$p_1 = \frac{2\rho(1+8\sigma(1-\sigma)\rho)}{(1+2\rho)(1+2\rho+8\sigma(1-\sigma)\rho^2)}$$

$$p_2 = \frac{4\rho^2\{1-2\sigma(1-\sigma)(1-2\rho)\}}{(1+2\rho)(1+2\rho+8\sigma(1-\sigma)\rho^2)}.$$

Here we shall give the numerical table for these probabilities, L and p_{lost}.

ρ	σ	p_0	p_1	p_2	L	p_{lost}
0.25	0.1	0.6472	0.2545	0.0983	0.4509	0.1515
	0.2	0.6329	0.2785	0.0886	0.4557	0.1168
	0.3	0.6231	0.2949	0.0820	0.4590	0.0939
	0.4	0.6173	0.3045	0.0782	0.4609	0.0811
	0.5	0.6154	0.3077	0.0769	0.4616	0.0769
0.5	0.1	0.4587	0.3119	0.2294	0.7706	0.328
	0.2	0.4310	0.3535	0.2155	0.7845	0.272
	0.3	0.4132	0.3802	0.2066	0.7934	0.232
	0.4	0.4032	0.3952	0.2016	0.7984	0.208
	0.5	0.4	0.4	0.2	0.8	0.2
0.75	0.1	0.3442	0.3181	0.3377	0.9935	0.4554
	0.2	0.3106	0.3652	0.3242	1.0137	0.3949
	0.3	0.2903	0.3936	0.3161	1.0259	0.3490
	0.4	0.2793	0.4090	0.3117	1.0325	0.3202
	0.5	0.2759	0.4138	0.3103	1.0346	0.3103
1	0.1	0.2688	0.3083	0.4229	1.1542	0.5459
	0.2	0.2337	0.3551	0.4112	1.1776	0.4871
	0.3	0.2137	0.3818	0.4045	1.1910	0.4408
	0.4	0.2032	0.3957	0.4011	1.1978	0.4105
	0.5	0.2	0.4	0.4	1.2	0.4

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