THEORIES OF INSTABILITY IN DENSE HIGHWAY TRAFFIC

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ABSTRACT

It is shown that the continuum theories and stable car-following theories are not capable of describing the spontaneous generation of stoppages observed in the Holland Tunnel. An unstable car-following theory with a large reaction time has some desirable features but was abandoned because of mathematical difficulties. Most of the paper deals with a theory in which there are two velocity-headway curves one of which applies during acceleration and the other during deceleration. Cars maintain larger spacings during acceleration. When the acceleration changes sign, cars make the transition from one curve to the other with only a small change in velocity. This theory furnishes possible explanations or descriptions of all phenomena observed so far on the gross aspects of traffic flow that are known to be in conflict with previous theories. Among other things the theory describes the existence of acceleration shocks, instability, stoppages behind a bottleneck and the variation of the flow-density curves with the nature of the flow pattern. A more complete summary of results is contained in the concluding section.

INTRODUCTION

One of the most interesting phenomena that has been described in recent studies of highway traffic is the continual recurrence of stoppages in the Holland Tunnel [1-3]. These stoppages have been nearly eliminated by controls on the flow of traffic entering the tunnel but this was accomplished without a very clear understanding of why they occurred in the first place. Although this specific trouble has been corrected in the tunnel, similar phenomena are known to exist on other highways and a satisfactory explanation of these stoppages represents one of the main obstacles to the further development of a satisfactory theory of traffic flow at high densities. The purpose of the following is to investigate some types of theories that might succeed in describing the stoppages, as well as other observed phenomena and some theories that will not.

Extensive experiments conducted by the Port of New York Authority [1-4] suggest that there is a bottleneck in the Holland Tunnel at the foot of the upgrade. The flow behind this bottleneck, instead of adjusting itself to a constant value which can just pass the bottleneck as would be expected from any of the existing theories, oscillates between a value that exceeds the capacity of the bottleneck and a value of essentially zero (the stoppage). These stoppages originate at or near the bottleneck and propogate back through the tunnel with an average velocity of about 10 m/h. There is on the average, about one stoppage every four minutes.

In previous studies of traffic flow in tunnels, comparisons have been made with either the continuum theories [5, 6] or the car following theories [7-13]. Despite the successes of these theories, however, in describing such things as the propogation of waves, the conditions for maximum flow, etc., they do not, in their present form, seem adequate to describe the occurrance of these stoppages.

CONTINUUM THEORIES

In the continuum theories it is postulated that the relation between the flow q and the concentration (density) k, determined under steady flow conditions applies also under time dependent conditions. In the tunnels there are no physical changes in the highway that should cause the q vs. k relation to depend explicitly on the time but there is reason to believe that it varies with position x. Thus $q=q(k, x)$. The basic equation of motion derives from the conservation condition

$$\frac{\partial k}{\partial t}+\frac{\partial q}{\partial x}=0. \qquad (1)$$

Accurate experimental data to support the conjecture that the q vs. k relation is independent of time variations in the flow are difficult to obtain because of very large statistical fluctuations in measurements of relevant quantities. To within the accuracy of the available data, however, this assumption seems to be approximately correct at least in some average sense. In particular, the q vs. k relations deduced by Herman and Potts [10] from analysing car following data under time dependent conditions agrees well with q vs. k data obtained under supposedly steady conditions. Also the stoppage waves or shocks observed in the Holland Tunnel have a velocity that is at least in qualitative agreement with the velocities described by the continuum theory.

Despite this, the continuum theory represented by (1) seems inadequate to describe the stoppages for two reasons. First this theory provides no mechanism for the generation of stoppages. It is true that one can temporarily feed traffic into an empty tunnel at the capacity of the entrance and when this high flow reaches a point of lower capacity a shock wave will form. According to the continuum theory, however, the shock will travel back to the entrance and limit the subsequent rate of flow to just that value which can pass the bottleneck. The shock will certainly not involve a complete stoppage and furthermore the flow will remain steady once the shock has passed through the tunnel. Secondly, even if by some mechanism not explicitly described in the theory, one could generate a temporary stoppage which would necessarily be followed by an acceleration wave, the continuum theory predicts that the two disturbances would annihilate each other. This derives from the fact that the stopping wave will form a shock with a velocity intermediate between the wave velocities associated with the states of flow on either side of the shock. The starting wave which follows the stoppage will, therefore, travel backwards with a higher velocity than the shock. It will overtake the shock and start eating it away.

Actually if the q vs. k relation varies with x, one can imagine a situation in which the stoppage has sufficient spacial width "initially" that the shock velocity evaluated at its location is less than the wave velocity for acceleration evaluated at its position due to variations of the velocities with x. Although the stoppage and acceleration waves would not annihilate each other in this case, it is difficult to imagine how one could repeatedly generate stoppages large enough to initiate this behavior.

There has also been some speculation that the stoppages may be due to some anomalous driver or a low powered truck that had trouble on the upgrade. One could also treat such a situation with the continuum theory by considering the truck as a traveling bottleneck. Even a traveling bottleneck will not, however, according to the continuum theory, cause a complete stoppage but only a readjustment of the flow to that dictated by the bottleneck.

This failure of the continuum theory to describe stoppages is not entirely inconsistent with its success in describing other phenomena. There are many ways in which one can modify (1) in such a way that the equations of motion are not stable. For example one might add to (1) a second derivative term with the "wrong" sign so that it is analogous to the heat conduction equation but with a negative conductivity. By adding only a very small term of this sort to (1), one should be able to retain many of the realistic features of the unperturbed equation but at the same time produce oscillations with a long period (long compared with other time constants in the theory such as time headways, reaction times, etc. which are measured in seconds) due to the instability.

CAR FOLLOWING THEORIES

The car following theories have been subjected to a more detailed experimental study than the continuum theories and have already undergone several stages of refinements. There still exist, however, certain pieces of experimental data that do not fit well with present theories and which may have some connection with the stoppages in the tunnels.

Two slightly different approaches have been taken in the development of car following theories. In one version [8, 13] it is postulated that a jth driver chooses a velocity $v_j(t+T)$ at time $t+T$ as some function of the spacial headway at time t, i. e.

$$v_j(t+T)=G_j[x_{j-1}(t)-x_j(t)] \tag{2}$$

for some suitable function G_j. The time T is interpreted as some effective reaction time and the position of the jth car at time t is represented by $x_j(t)$. In the other version, first suggested by Chandler, Herman, and Montroll (C. H. M.) [7], it is assumed that a jth driver chooses an acceleration $a_j(t)$ that depends upon such observables as the velocity $v_j(\tau)$, the velocity difference $v_j(\tau)-v_{j-1}(\tau)$ and the spacing $x_{j-1}(\tau)-x_j(\tau)$, all evaluated at various past times $\tau<t$. Although there may be many reaction times and dependences upon other than nearest neighbor cars, the

object is not to construct the most general or most accurate theory but the simplest model that contains the main elements of the real world. We shall therefore immediately restrict the models of this second category to have a form

$$a_j(t+T)=H_j[v_j(t+T),\ v_{j-1}(t)-v_j(t),\ x_{j-1}(t)-x_j(t)]. \qquad (3)$$

Even this is too general for detailed mathematical or experimental study and will require further specialization but it includes essentially all specific models considered so far.

Equation (2) has the advantage of mathematical simplicity. It is essentially a discrete version of the continuum theory except possibly for the presence of the reaction time T. The function G_j is chosen to give the observed steady state relation between velocity and headway. Consequently there is the implication here also that the behavior of drivers under time dependent conditions is determined by their behavior under conditions of steady flow.

The development of the theory described by (3) has followed a pattern in which the simplest functional forms were considered first and then modifications were introduced as more data became available. C. H. M. first attempted only to consider small deviations from some steady state flow pattern and proposed an expansion of H to linear terms in 1. $v_j(t)-v_{j-1}(t)$ and 2. the diference between $x_{j-1}(t)-x_j(t)$ and some averaged fixed spacing. Their experiments showed, however, that the acceleration was much more sensitive to velocity differences than to the spacing and so the latter dependence was neglected. The first successful theory of the form (3) thus took the simple form

$$a_j(t+T)=\lambda_j[v_{j-1}(t)-v_j(t)]. \qquad (4)$$

Since (4) is a perfect differential in t, it can be integrated to give an equation of the form (2), namely

$$v_j(t+T)=\lambda_j[x_{j-1}(t)-x_j(t)]-C \qquad (5)$$

for some integration constant C.

C. H. M. investigated stability questions associated with (4), whereas Kometani and Sasaki (K. S.) [8] investigated the equivalent conditions for (5). They showed that for $\lambda T<\pi/2$ the jth car will react in a stable way to any motion of the $(j-1)$th car in the sense that any transients will decay in time. If, however, $1/2<\lambda T$ a disturbance will grow in amplitude as it propogates through a long line of cars. Values of λ and T determined by C. H. M. from a correlation analysis of car following experiments showed wide variations from driver to driver but led to an average value of λT very close to the stability limit $\lambda T=1/2$. Helly [10] analysed data taken from several other experiments and also found values of λT surprisingly close to 1/2. Values obtained by K. S. were closer to 1 but their experiments were done under rather unnatural conditions. The lead driver was instructed to make hard accelerations and decelerations periodically.

The question of whether or not the equations of motion are stable for the propogation of small disturbances is of vital importance to the gross aspects of traffic flow but as yet the car following experiments have not given an unequivocal answer. They do suggest, however, that traffic motion is unstable at least part of the time or under certain conditions.

Recently, Gazis, Herman and Potts [12] extended (4) by allowing λ to be a function of the spacing. A conservative interpretation of this is to say that (4) still applies only for small amplitude disturbances but the coefficients depend upon the unperturbed state of flow. Another interpretation is to conjecture that with this modification (4) is valid even for large disturbances or a continuous change of state. If one accepts the latter, (4) can be integrated to give an equation equivalent to (2) with G equal to the integral of λ. Since G has a natural interpretation as the steady state velocity-headway relation one can compare values of λ obtained from car following experiments with values of the derivative of G determined from steady state flow data. To within the accuracy of the experiments, this relation has been confirmed in some average sense.

Further modification of (4) by Edie [14] and Gazis, Herman and Rothery [15] have permitted λ to have a form

$$\lambda=A[v_j(t+T)]B[x_{j-1}(t)-x_j(t)] \tag{6}$$

for suitable functions A and B. If one substitutes this into (4) and divides by the function A, one again obtains an equation which is a perfect differential in t. This latter equation when integrated and solved for $v_j(t+T)$ once more gives an equation equivalent to (2). There is some redundancy, therefore, in consider both A and B unspecified functions. By substituting (2) into either A or B, we may consider A as a function of spacing or B a function of velocity and consequently λ can be represented as a function of the spacing only or of the velocity only.

Despite the greater generality of (3) as compared with (2), no one has yet considered any form for (3) that was not equivalent to (2). Furthermore if one differentiates (2) with respect to t, it is clear that (4), with λ a function of spacing only or of the form (6) is the only equation of the type (3) which can be derived from (2).

Although there is substantial evidence that (4) is approximately correct, there is other evidence that indicates some systematic deviations. In some old experiments conducted by the Port of New York Authority runs were made through the tunnels by cars equipted to record velocity and spacing as a function of time. At any given time the instantaneous velocity and spacing can be represented by a point in Fig. 1. The locus of such points describes a trajectory in the velocity-headway space. If velocity were a function of the instantaneous spacing only then this trajectory should always be on a curve giving velocity as a single valued function of spacing.

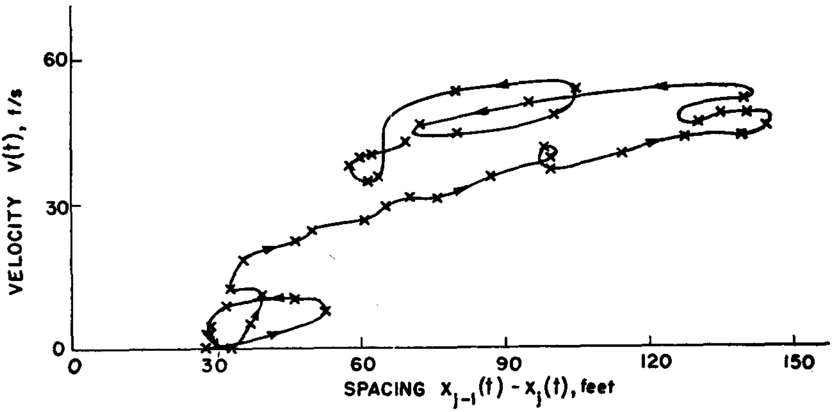

Fig. 1

The arrows in Fig. 1 indicate the direction of increasing time and the x's are placed at five second intervals. The graph shows only a portion of the trajectory for one run. If one tries to draw a long trajectory the graph becomes covered with a maze of lines. The pattern shown in Fig. 1 however illustrates the type of behavior shown in many runs. Although the trajectory loops around in a rather chaotic fashion (usually more so than shown here) it is almost always true that for any given velocity the spacing is larger during accelerations than during decelerations i. e. the trajectories loop in a counterclockwise direction.

The above effect could be due in part to a non-zero reaction time. If there were a unique relation between velocity at time $t+T$ and spacing at time t as postulated in (2), then during acceleration $v(t)$ would be less than $v(t+T)$. The instantaneous velocity vs. headway curve drawn in Fig. 1 would therefore lie below a curve of $v(t+T)$ vs. $x_{j-1}(t)-x_j(t)$ during acceleration and above it during deceleration. It is true that by a suitable choice of T one can reduce the size of the loops in Fig. 1 by plotting $v(t+T)$ instead of $v(t)$ but this is usually accomplished at the expense of making single loops into more complicated figure eights or worse.

Above all these experiments show that drivers do not follow any completely deterministic behavior and that any of the models considered here are valid at best only in some average sense. It is difficult to separate any systematic trend from a background of violent fluctuations and particularly difficult to distinguish between two theories with qualitatively similar properties. It is also very easy to see in any experiment almost anything that one wants to see.

There is however, another possible explanation for these loops and some trajectories, or parts thereof, indicate a tendency for drivers to behave in the following way. During acceleration, a driver allows relatively large spacings for a given velocity. When the acceleration ceases the headway is still large but the following driver is content to retain the excess gap almost indefinitely. If the lead car now decelerates, the following driver does not react immediately but waits until the excess gap has been dissipated before making any serious attempt to decelerate. One might imagine an idealized behavior of this sort by postulating that there are two relations like (2) between velocity and spacing one of which applies during acceleration and the other during deceleration as shown in Fig. 2 by the solid lines.* If the motion changes from one to the other

* A. Daou has independently proposed that one consider different velocity-headway curves for acceleration and deceleration. He has pursued the theory mainly as it applies to platoons, but there are many points of similarity between his work and that described later in this paper. He is also accumulating more convincing quantitative experimental evidence for this

the spacial gap is decreased or increased following along the broken lines of nearly constant velocity. Quite apart from any reaction time T, a trajectory that results from an oscillation between two velocities v_1 and v_2 would describe a loop in the graph as indicated by the arrows.

A hint that some such phenomena may exist is also suggested by some data obtained by Herman's group at General Motors. They suspected that since decelerations and accelerations are accomplished by different procedures, the coefficient in (4) may be different for accelerations than for decelerations. They did, in fact, find that λ was on the average about 15% larger for decelerations than accelerations [16]. They concluded [11] that "the effect was not of such importance, however, as to justify the added complication in including this refinement to the car-following law."

On the basis of this one might be tempted to consider a model in which

$$a_j(t+T)=\lambda[v_{j-1}(t)-v_j(t)]-\delta\mid v_{j-1}(t)-v_j(t)\mid \qquad (7)$$

with $0<\delta<\lambda$. This, in effect, would give a coefficient $\lambda+\delta$ for deceleration and $\lambda-\delta$ for acceleration. The difficulty with such a model, however, is that if a car is given some velocity at time t and after some maneuver returns to the same velocity at a later time, one finds by integrating (7) that the spacings between cars is always increased. There is no way by which the headway and velocity can ever be brought back to their initial values. Furthermore if the maneuver is repeated indefinitely, the spacing will increase without bound.

The model represented by Fig. 2 corrects this deficiency by introducing the broken line segments of the trajectory whereby a car does not decelerate until the spacing has been reduced to some minimum safe spacing. There are also other ways of correcting this failure to (7), for example, one could add a small term to the right side of (7) that is proportional to the deviation in spacing from some equilibrium value. A term of this sort was in the original theory C. H. M. but was discarded because it was too small to measure. It was again considered by Helly. Even a small term, however, in this modified theory would have the effect of producing a slow drift back to some equilibrium spacing whenever the

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theory. Since the experimental aspects of this are very tedious, the publication of Daou's work will be delayed although it was started at approximately the same time as the work described here.

asymmetry in (7) perturbed the system away from this equilibrium.

INSTABILITY FROM REACTION TIMES

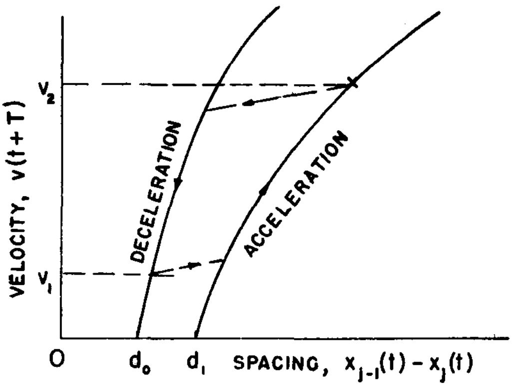

Fig. 2

Although previous theoretical studies of traffic centered around the hope that traffic behaved in a stable way, the weight of experimental evidence seems to be forcing us to consider theories with some sort of instability. In this section we consider some features of equations (2) or (4) with T large enough to produce instability and in the following sections we consider some consequences of the model represented in Fig. 2.

Although λ in (4) is a function of the spacing and is related to G in (2) by

$$\lambda = G'[x_{j-1}(t) - x_j(t)],$$

for small amplitude disturbances λ may be considered essentially constant. According to the conditions of stability of C. H. M. and K. S., (4) is stable for the propogation of small disturbances if

$$\lambda < 1/2\,T. \qquad (8)$$

Experiments indicate that G should have a shape as shown in Fig. 3 with G' monotone decreasing for all spacings greater than d_0. Although T should also depend upon the

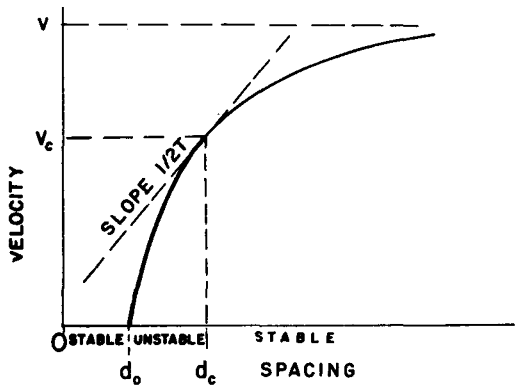

Fig. 3

spacing let us assume that it is constant. If T is so large that (8) is violated at the spacing d_o, the fact that $G' \longrightarrow O$ monotonically as the spacing increases means that there is a unique critical velocity v_c and spacing d_c where $\lambda T=1/2$. Equations (2) and (4) are unstable for velocities or headways less than v_c, d_c but stable for higher velocities. If G is defined to give zero velocity for spacing less than d_o, the equations are also stable for zero velocity and spacing less than d_o. The unstable region is shown in Fig. 3 by the heavy line where the slope of the curve exceeds $1/2T$.

An exact solution of (2) under the conditions described above is very complicated but one can predict some qualitative properties. Despite the instability for the propogation of small disturbances the equations are reasonably stable for large amplitude disturbances. If a sequence of cars travels at a velocity in the unstable region any small perturbation will be amplified as it propogates. Its amplitude can not grow indefinitely, however as in a linear theory because the form of G confines the velocities to the range between 0 and the free speed V. The accelerations are also bounded because in (4) the velocity difference $|v_{j-1}(t)-v_j(t)|$ can be at most V and λ is alway less than $\lambda(d_o)$, thus $|a|<\lambda(d_o)V$.

Previously [13] it was shown that if velocities are slowly varying with respect to the car number j and time, (2) is approximately equivalent to

$$v_j(t) \sim v_{j-1}(t-1/\lambda)+(2\lambda^2)^{-1}(1-2\,T\lambda)d^2v_{j-1}(t)/dt^2 \qquad (9)$$

which gives the velocity of the jth car in terms of the behavior of the $(j-1)$th car. This equation is also a discrete approximation to the heat conduction equation with a conductivity proportional to $1-2\,T\lambda$. The λ is a function of spacing or equivalently of the velocity and $1-2\,T\lambda$ is positive for velocities in the stable range but negative for velocities in the unstable range.

If v_{j-1} acquires a maximum value at some time $t-1/\lambda$ and has a slowly varying second derivative, then this second derivative will be negative at time t as well as $t-1/\lambda$. Equation (9) shows that $v_j(t)$ will be larger or smaller than the maximum of v_{j-1} accordingly as $1-2\,T\lambda<0$ or $1-2\,T\lambda>0$. Similarly if $1-2\,T\lambda<0$, $v_j(t)$ will assume smaller values than the minimum of $v_{j-1}(t)$.

If one has a stable theory, two of the simplest flow patterns to describe are those (a) behind a car moving at a constant velocity and

(b) behind a bottleneck of fixed capacity. In the former case, if the lead car moves at a speed less than the free speed of any of the following cars, the velocities of all cars will eventually adjust to that of the lead car and the spacings will approach those consistent with (2) for the given velocity of the lead car. In the latter case, if the flow approaching the bottleneck exceeds its capacity, a shock will travel back from the bottleneck leaving a state in which the flow is everywhere equal to the capacity of the bottleneck.

This situation will still obtain for the semi-stable theory above provided that the steady state velocities all lie in the stable range. If, however, in case (a) the lead car chooses to drive at a velocity v in the unstable region, the following cars will not be able to maintain a steady velocity v. If any jth car has a maximum (minimum) velocity in the unstable region, the $(j+1)$ th car according to (9) should experience a still larger maximum (smaller minimum). The amplification of the amplitude of any disturbance continues until the maximum velocity of some car lands in the stable region and the minimum velocity is zero. If the maximum velocity of some car exceeds v_c, however, it follows also from (9) that the next car will have a smaller maximum. It would seem, therefore, that far behind the lead car the velocity of a car will oscillate between 0 and v_c.

The spacing between the jth car and the leader will be bounded for all times and therefore

$$\frac{x_j(t)-x_0(t)}{t}=\frac{1}{t}\int_0^t v_j(t)dt-v\to 0 \text{ for } t\to\infty.$$

Thus the time average velocity of the jth car is v for all j. If for large j, $v_j(t)$ oscillates between 0 and v_c, it must do so in such a way as to maintain a fixed average speed.

A more detailed description of how these oscillations behave is difficult to determine from (2) and even if one could solve (2) exactly, the nature of the solution would probably be sensitive to certain features of the equations that one might not take seriously anyway, for example, the existence of a single "effective" reaction time T. The above description suggests, however, that either (2) or some slight modification of it could produce solutions in which a typical car will have a velocity vs time curve that is approximately a step function oscillation between velocities 0 and v_c, i. e. the time spent in accelerations or decelerations between the two extreme velocities is small compared with the time spent at or near these velocities. These oscillations would probably not be periodic in time. One should expect that because of independent fluctuations in the rate of propogation of consecutive stopping waves, that the time intervals between stoppages would be completely random.

A description of what should happen behind a bottleneck is less clear because it is not obvious what conditions should be imposed at the bottleneck. If, however, a stable theory would give rise to a velocity behind the bottleneck that for the semi-stable theory lies in the region of instability, one would expect the velocity again to oscillate between 0 and v_c. This is the sort of behavior observed in the Holland Tunnel, but even if one could conclude for certain that this semi-stable theory gives an oscillatory flow, it is difficult to see how one could explain an average time between stoppages as large as four minutes.

THE TWO-STATE MODEL

Since both the model of traffic flow represented in Fig. 2 by the two velocity-headway curves and the model discussed in the last section lead to loops for the trajectories when plotted as in Fig. 1, one would expect these two theories to be similar in other respects. One big advantage of the two-state model is that one can give more detailed descriptions of what should happen under various conditions where in the reaction time model we could only guess at the consequences.

To see the sort of conclusions one can expect from this two-state model we consider below some examples of hypothetical flow patterns.

A Starting Wave

Suppose initially one has a series of cars spaced at the maximum allowed spacing for velocity zero (spacing d_1 in Fig. 2). The lead car then starts to accelerate as from a traffic light. All cars will acquire non-negative accelerations and the spacings will be confined to the acceleration curve of Fig. 2. The present theory predicts the same acceleration pattern as given by previous discrete or continuum theories with a single velocity-headway curve namely the acceleration curve of Fig. 2.

If, however, cars start from zero velocity but a spacing less than d_1, for example d_0 in Figs. 2 or 4, no car will move until the car in front of it has moved a distance $d_1 - d_0$. If the lead car accelerates suddenly to a small velocity v_1, it will take a time at least $(d_1 - d_0)/v_1$ for the jth car to open the gap between it and the $(j+1)$th car to the spacing d_1. For sufficiently small v_1, this time will be large compared with the time required for a car to actually accelerate from velocity 0 to v_1. The starting wave, therefore, travels back through the line of cars with a time lag of approximately $(d_1 - d_0)/v_1$ per car or a velocity of about $(d_0 v_1)/(d_1 - d_0)$.

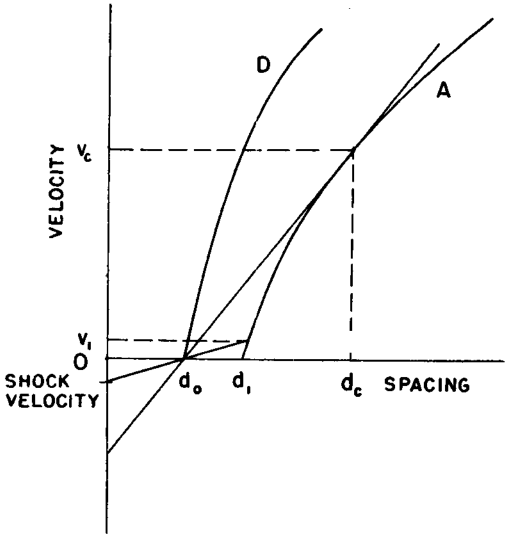

Fig. 4

This wave velocity is represented graphically in Fig. 4 as the intercept on the velocity axis of a line through the points $(0, d_0)$ and (v_1, d_1). The construction here is similar to that used to find shock velocities of deceleration waves in previous theories [13]. The starting wave is in fact a shock. If one considers an effective velocity-headway curve for this acceleration as consisting of the line between d_0 and d_1 at zero velocity and then the portion of the acceleration curve up to velocity v_1, this curve is convex. One obtains a shock for accelerations on a convex portion of the velocity-headway curve for exactly the same reasons that one obtains a shock for decelerations if the velocity-headway curve is concave. A shock has the property that the disturbance propagates with no change in shape as contrasted with an expanding wave the duration of which increases as it propagates.

One obtains the wave velocity for an infinitesimal disturbance by drawing the tangent to the velocity-headway curve and noting where this tangent intercepts the velocity axis. For the effective velocity-headway curve described above, the wave velocity at d_0 is zero while the wave velocity at velocity v_1 is significantly negative. A shock results whenever the wave velocity at the rear of the disturbance is larger than the wave velocity at the front.

As one increases the final velocity v_1, the shock velocity decreases (becomes more negative) whereas the wave velocity at v_1 increases until one reaches the velocity v_c of Fig. 4 where the shock line from $(0, d_o)$ to the velocity v_c is tangent to the velocity-headway curve. Here the shock velocity and the final wave velocity are equal. If one increases the final car velocity to a value $v_3 > v_c$ the final wave velocity will be larger than the shock velocity. In this case the velocities between 0 and v_c will still propogate as a shock but each velocity between v_c and v_3 will propogate with its own wave velocities. A driver far back from the leader will accelerate rapidly to the velocity v_c as the shock passes him but then accelerate slowly to the velocity v_3 as the expanding waves of velocities between v_c and v_3 pass.

A similar phenomena also occurs if one has somehow created a situation in which a series of cars are traveling at a constant velocity $v_o > 0$ and at a spacing intermediate between the two velocity-headway curves, as represented by the x in Fig. 5. If the lead car now accelerates to a velocity v_1 the following cars will use a path in the velocity-headway curve consisting of the connecting path through x to the acceeration curve and then up the acceleration curve to v_1 as indicated by the arrows in Fig. 5. Again one can have acceleration shocks and/or waves depending upon the velocity v_1. One will have a shock any time one can connect two points on this path from v_o to v_1 by a shock line which lies entirely above the path, as shown in Fig. 5.

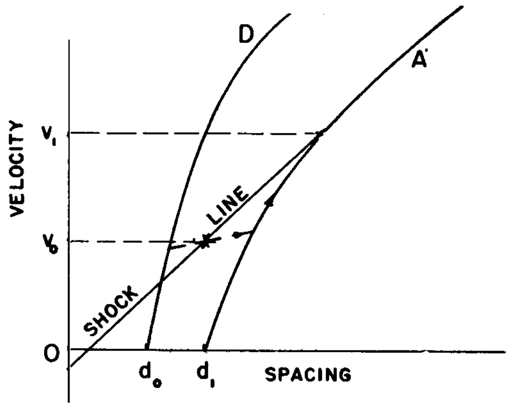

Fig. 5

Deceleration Waves.

Suppose one has a line of cars moving at a constant velocity and at a spacing between or on one of the two velocity-headway curves, for example the point x in Fig. 2 at velocity v_2. If the lead car decelerates to a velocity v_1, the following cars will decelerate as if the velocity-headway curve consisted of the connecting segment to the deceleration curve through x and then down the deceleration curve to v_1 along the arrows of Fig. 2. In all cases the deceleration wave forms a shock because this velocity-headway curve is everywhere concave. The pattern of deceleration is the same as that given by a single state theory with the effective velocity-headway curve described above.

Instability on the Acceleration Curve.

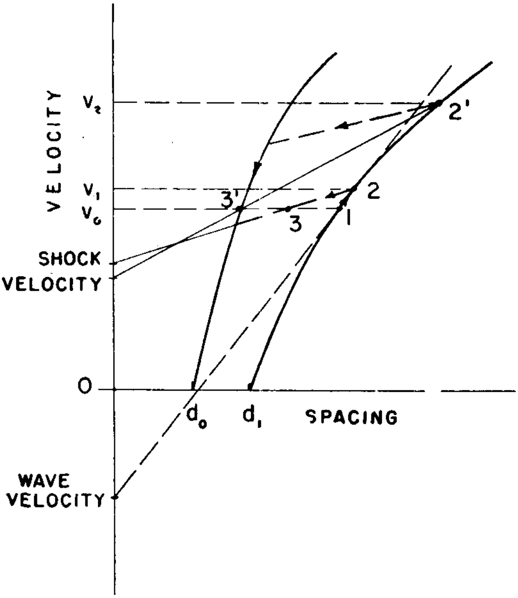

Fig. 6

Suppose we have a line of cars moving at a constant velocity v_0 and spacing on the acceleration curve (point 1 of Fing. 6). The lead car now accelerates slightly to a velocity v_1 and then decelerates back to the velocity v_0. The acceleration wave travels back through the cars at the wave velocity of Fig. 6 which one obtains by marking the intercept of the tangent line to the velocity-headway curve at v_0. After the wave has passed, the cars have the velocity and spacing of point 2 in Fig. 6. When the cars decelerate they do not follow the same path back to point 1 but follow along the connecting curve through point 2 to the deceleration curve. If $v_1 - v_0$ is sufficiently small and the connecting curve has a positive slope, the final state will be on the connecting curve through point 2 at the velocity v_0 (point 3 of Fig. 6). For larger values of $v_1 - v_0$ the final state will be on the deceleration curve, point 3′.

In either case, however, the deceleration shock travels with a larger velocity than the acceleration wave. A car far behind the leader will, therefore, experience the acceleration wave first but then must wait a significant length of time before the deceleration wave reaches him. Figure 7 gives an exaggerated picture of the trajectories for such a sequence of

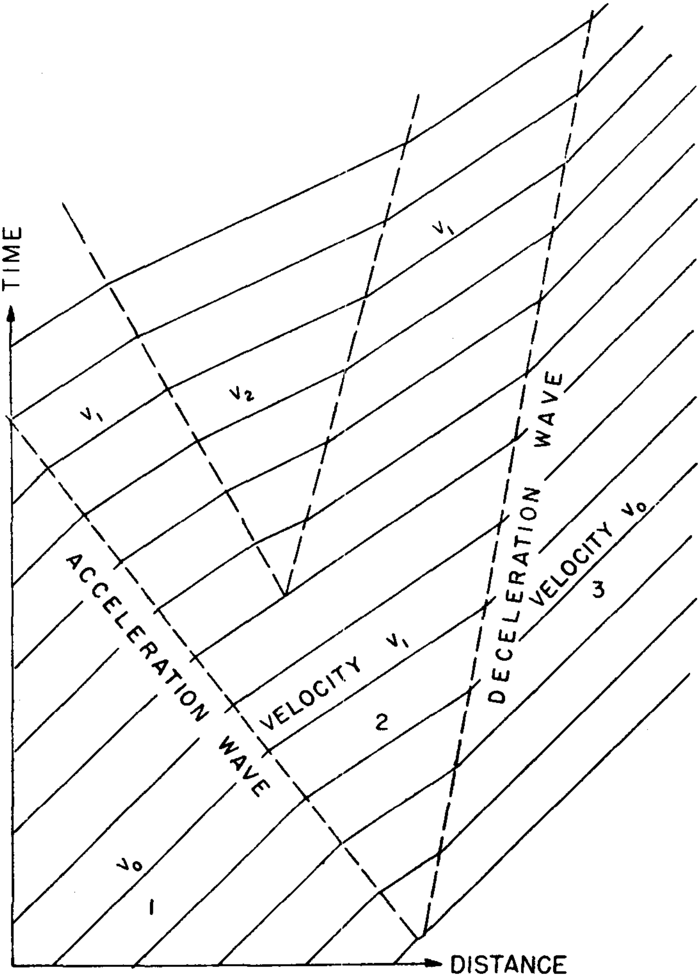

Fig. 7

cars. What, in effect, happens is that each driver initially has a large spacing corresponding to that of the acceleration curve and he stays on this curve as the acceleration wave passes him. Before he is forced to decelerate, however, he waits until he has used up part of this cushion and thus retards the propogation of the decelerating wave to the next car.

This phenomena gives rise to a certain type of instability. Suppose the motion attributed to the lead car above was just some random fluctuation in velocity. The effects of this fluctuation, however, do not die out and in Fig. 7 we see that it produces an expanding region in which cars are left in the state corresponding to point 2 of Fig. 6. We can now repeat the argument. Any car in this state can be considered a lead car for those behind it. If it undergoes a random fluctuation in which it accelerates from v_1 to a higher speed v_2 and then perhaps decelerates back to v_1, it will create another permanent and expanding region in Fig. 7 where the velocity is v_2. In this way one can produce higher and higher velocities.

This rise in velocity occurs only when a car accelerates first and then decelerates. If the pattern is reversed, however, nothing unusual happens and, particularly, a small deceleration followed by an acceleration will not annihilate the effects of a corresponding acceleration followed by a deceleration. If a car decelerates first, say from the state 2 to 3 in Fig. 6 and then accelerates back to the velocity v_1, the state returns to 2. The acceleration and deceleration waves also travel at nearly the same velocity. Whether or not they annihilate each other depends upon finer details of the connecting path from 2 to 3 than we would care to speculate about but the behavior certainly does not create an effect comparable with that described above for an acceleration followed by a deceleration.

The above instability dealt only with small amplitude disturbances. Before we investigate what limits the growth of these fluctuations, we consider an analogous instability for states on the deceleration curve.

Instability on the Deceleration Curve

Suppose a line of cars is moving at a constant velocity v_o on the deceleration curve (point 4 of Fig. 8). If the lead car decelerates slightly to a velocity v_3 and then accelerates back to v_o, the following cars first assume a spacing and velocity corresponding to point 5 of Fig. 8 as the deceleration wave passes. When the acceleration wave passes a car the spacing does not return to that of point 4 but to point 6 which lies either on the connecting curve through point 5 or on the acceleration curve. The velocity of the deceleration shock, however, is smaller than that of the acceleration wave and we have a situation very similar to that represented in Fig. 7 except that the trajectories would be traversed in the opposite direction. The space between the two waves grows with time and cars must retain the velocity v_3 during the time interval between the arrivals of the two waves. This again give rise to an instability because any disturbance among the cars moving at velocity v_3 that creates a still lower velocity v_4 will also yield a growing region in which cars travel at velocity v_4. The lowest velocity will thus continue to become lower and lower. If the lowest velocity becomes as low as shown in Fig. 8 by v_4, the acceleration wave from v_4 back to v_o will form an acceleration shock.

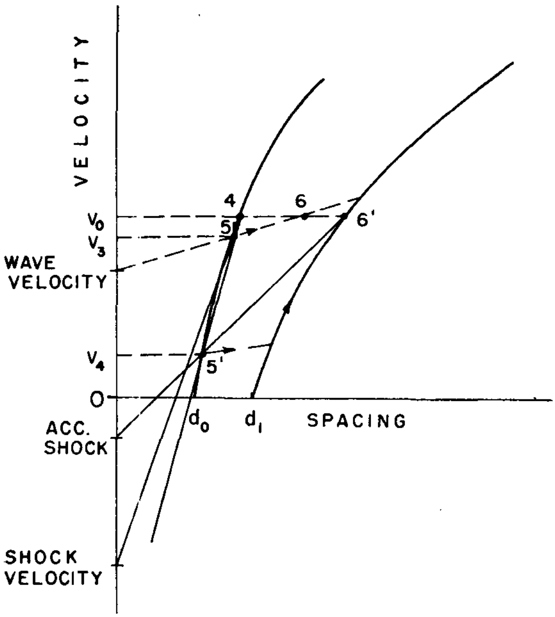

Fig. 8

Amplitude restrictions

Let us return to the situation represented in Figures 6 and 7. We have noticed already that the peak velocity will increase with time from v_1 to v_2 to still higher values, i. e. point 2 of Fig. 6 drifts higher and higher. As this happens, however, point 3 shifts to the left until it hits the deceleration curve at 3′. Still later we have an expanding wave of velocities from point 1 to point 2′, for example, and then a shock to point

3′. In Fig. 7 the two acceleration waves on the left represented by the broken lines fan out whereas the deceleration waves on the right coalesce into a single shock. The higher the point 2′, the faster is the wave velocity at 2′. Eventually as point 2′ moves upward the shock line from 3′

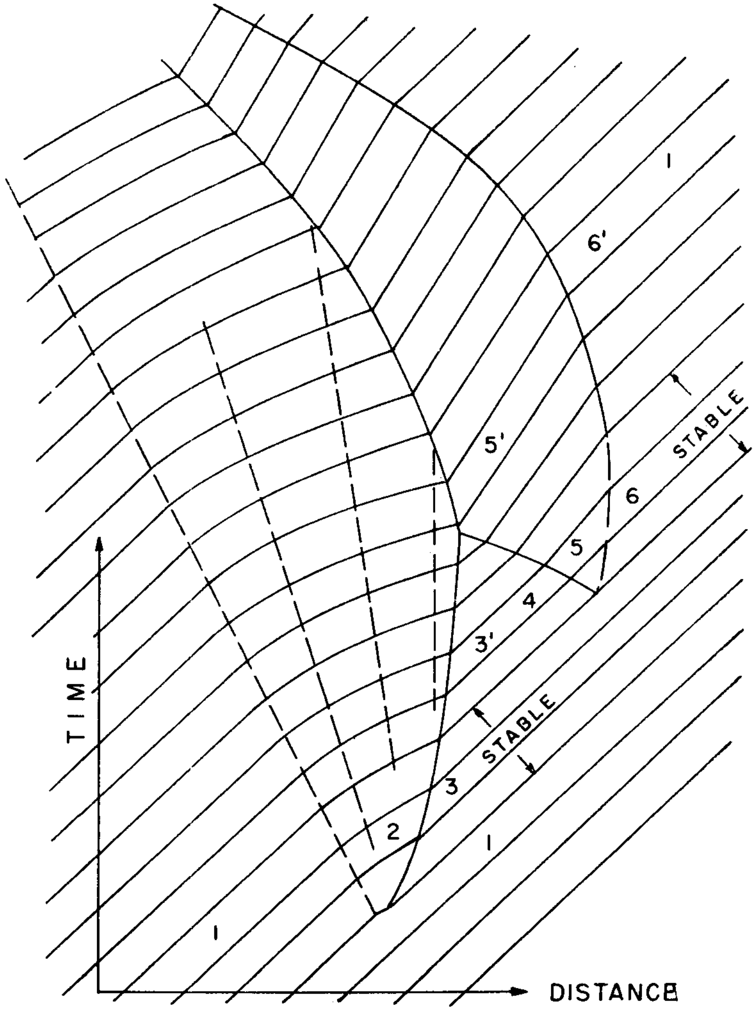

Fig. 9

to 2′ will become tangent to the acceleration curve and if one tries to push the maximum velocity still higher, the shock will travel slower than the fastest wave velocity. All waves traveling faster than the shock, however, will run into the shock and be absorbed thus bringing the peak velocity back to this critical value at which the wave velocity equals the shock velocity. The growth of the maximum velocity thus comes to an end.

Cars that pass through this acceleration-deceleration maneuver in its early stages of growth are left in a state 3 of Fig. 6 on the connecting curve. In this state the motion is considered to be stable against small perturbations. Cars that pass through this maneuver in its later stages, however, after the point 3 has shifted over to the deceleration curve at 3′, are left in another unstable state (of the second type). Regions 1, 2, 2′ and 3, 3′ of Fig. 9 show on a coarser scale the subsequent developments started in Fig. 7. The broken lines in 2, 2′ represent the fan of acceleration waves and the line between regions 2, 2′ and 3, 3′ is the decelerations shock that is developing on the right side of Fig. 7.

Since state 3′ is unstable, sooner or later a disturbance will be generated in region 3′ of Fig. 9 that gives rise to the developments represented in Fig. 8 where the point 4 of Fig. 8 is the same as point 3′ of Fig. 6. This disturbance is shown on the right hand side of Fig. 9. At first cars decelerate slightly to point 5 of Fig. 8 and accelerate back to point 6 which leaves them in a stable state similar to point 3 of Fig. 6. The shock velocity from states 4 to 5 of Fig. 8 is slower than that between 2′ and 3′ of Fig. 6, however, so as time goes on region 3′ 4 of Fig. 9 becomes narrower until the two shocks collide and form a single shock from point 2′ of Fig. 6 directly to point 5 or 5′ of Fig. 8. This combined shock will now start to overtake and annihilate any waves in region 2′ of Fig. 9 with wave velocities faster than the new shock. The fast waves are those for the highest car velocities so the peak velocity in region 2′ now decreases with time as these waves are absorbed.

Meanwhile the low velocity in region 5, 5′ drifts downward and point 6 of Fig. 8 moves to the right until it hits the acceleration curve at 6′ which is the same state as state 1 of Fig. 6 where this all started. As point 5 of Fig. 8 moves further down, the acceleration wave from 5 to 6 develops into an acceleration shock from 5′ to 6′.

As state 5′ continues to drift downward, the shock between states 2′ and 5′ of Fig. 9 becomes slower. It continues to overtake and absorb the fastest waves in region 2′ causing the velocity of state 2′ to drift down.

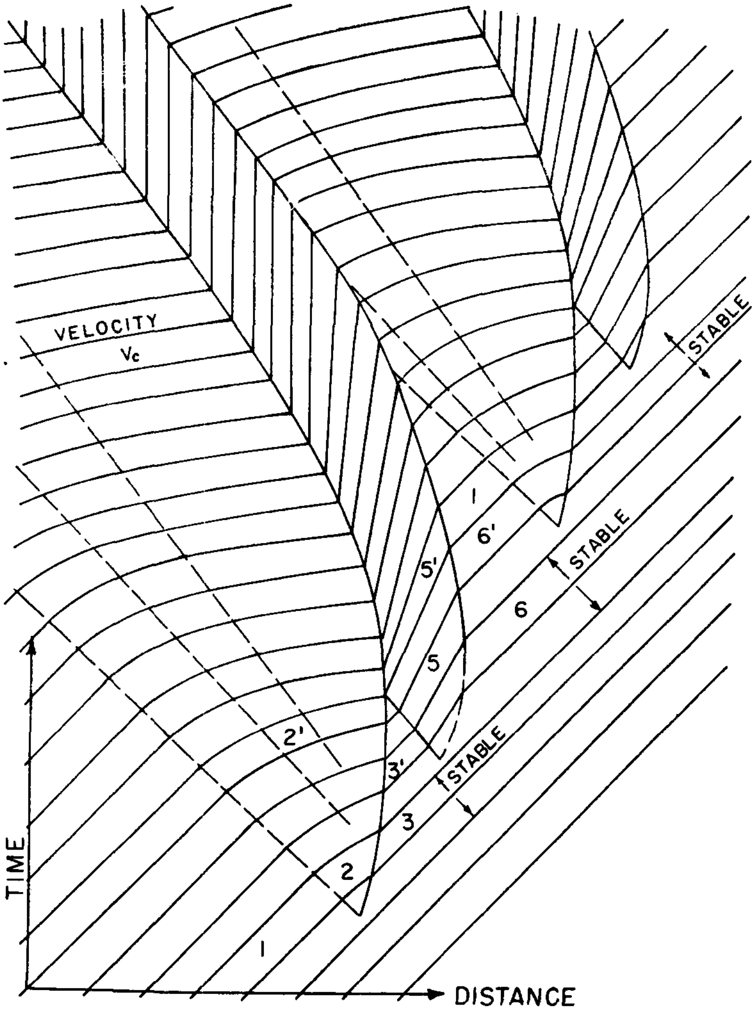

Fig. 10

Since the shock can overtake any waves traveling at a faster speed than the shock, there is a tendency for state 2′ to drift downward in such a way as to keep the shock line from 2′ to 5′ tangent to the acceleration curve. The acceleration shock from states 5′ to 6′ also becomes slower as the velocity of state 5′ drifts down. This trend continues until state 5′ has either reached zero velocity or caused the velocity of state 2′ in Fig. 6 to drift back to point 1.

If the velocity v_o of state 1 is larger than the velocity v_c in Fig. 4, state 2′ will drift back to state 1 before the velocity of state 5′ has reached zero. As point 2′ approaches point 1 of Fig. 6, both shock velocities, from 2′ to 5′ and from 5′ to 6′ decrease to the wave velocity of state 1 and the entire disturbance shown in Fig. 9 which up to now has been growing will cease to expand. If the velocity in region 5′ of Fig. 9 tries to drift still lower, which it will, the shock velocities become smaller than the wave velocity of state 1 and the boundaries of the disturbance in Fig. 9 now begin to converge as shown in the upper left corner of this figure. Eventually the entire disturbance will collapse but in doing so it leaves a large region in the upper right hand corner of Fig. 9 where the state (state 1) is unstable as it was initially. New disturbances will be generated and die but each time they leave some cars in a stable state such as states 3 or 6. The net effect of all this is that the disturbances increase the average velocities of cars slightly so that they can close the large gaps and achieve a state between the two velocity-headway curves which is stable for the given velocity of the lead car. Since all cars in Fig. 9 start at the left with velocity v_o and eventually return to velocity v_o but at a closer spacing, the flow behind the lead car is increasing as a result of these maneuvers.

If the velocity v_o of point 1 is less than v_c of Fig. 4, point 5′ drifts to zero velocity before it can pull the velocity of state 2′ back to that of state 1. Since the velocity cannot become negative, we are temporarily left with a situation in which point 2 ′ has been brought down to the velocity v_c and region 5′ of Fig. 9 has become a complete stoppage. Since the shock velocity between car velocities v_c and 0 is less than the shock velocity from car velocity 0 to v_o but larger than the wave velocity at state 1, regions 5′ and 2′ both continue to expand.

Cars that pass throught this disturbance, however, end up in an unstable state at point 6′ of Fig. 8 and here the velocity has a tendency to drift upward again. Figure 10 shows on the left hand side the developments analogous to those shown in Fig. 9 and then on the right hand side a repetition of the same kind of disturbance. Since the acceleration shock from state 5′ to 6′ is faster than the low velocity waves in this new disturbance, the shock overtakes and annihilates the waves of low car velocity. It continues to do this until the acceleration shock runs from velocity 0 to velocity v_c and the velocity of this shock has at the same time been reduced to that of the deceleration shock from v_c to 0 that preceded it. The stoppage that lies between these two shocks neither expands nor collapses.

As in the previous case, $v_o > v_c$, illustrated in Fig. 9, each new disturbance leaves some cars in a stable state with velocity v_o and eventually each car will achieve this state, but far behind the lead car we see the development of oscillations between velocity 0 and v_c. Unlike the previous case, these disturbances do not dissipate themselves.

An oscillatory flow between velocity 0 on the deceleration curve and velocity v_c on the acceleration curve is relatively stable. It is impossible for cars in the stoppage to do anything, so this is certainly stable. A group of cars moving at velocity v_c between two stoppages can create disturbances of the type represented in Fig. 7 in which the velocity drifts to higher values. The slowest wave velocity of either an acceleration or deceleration wave in this disturbance however, is greater than the velocity of the stopping shock. The entire disturbance, therefore, runs into the same shock (on the front end of the pack) and is absorbed piece by piece. As various parts of the disturbance are being absorbed the amplitude and velocity of the shock change somewhat but the shock maintains a fixed average speed. In contrast with this, region 5′ of Fig. 9 collapsed because any disturbance created in this region fanned out in such a way that the back side of the disturbance moved toward the back side of region 5′ while the front side of the disturbance moved toward the front side of region 5′. Eventually the disturbance covers the entire region and also forces the shocks to converge.

Although we have described above a mechanism for the creation of oscillations and shown that they should persist, there is one important ingredient of the theory that is still missing. we have not shown how one can calculate the time intervals between stoppage. This, it appears, depends upon the manner in which they were created initially since, once created, there is no tendency for them either to multiply or die. The rate at which the stoppages are created, however, depends upon the rate at which the peak velocity drifts higher or the minimum velocity lower. This, in turn, has been attributed to chance flucuations in velocity about which we know very little. By using arguments about the propogation of waves, we have implied, however, that these rates of drift are slow enough that velocities propogate essentially as in a continuum theory or a stable car following theory

BOTTLENECK FLOW

Since the above model was created mainly to explain oscillatory flow in the Holland Tunnel, we must now consider what type of flow pattern one can obtain from this model behind a bottleneck.

Let us imagine that the bottleneck consists of a short stretch of highway for which there are two velocity-headway curves similar to those of Fig. 4 but shifted to the right relative to those for the highway behind the bottleneck. The two curves for the bottleneck section of highway are shown in Fig. 11 by curves *A′* and *D′* while those for the highway behind the bottleneck are represented by curves *A* and *D*. We assume that the bottleneck and the road behind it are each homogeneous throughout their length and that curve *D′* intersects *A*

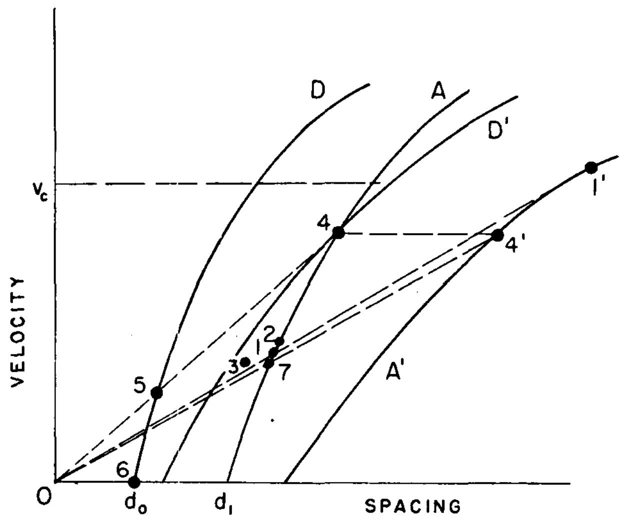

Fig. 11

in Fig. 11 at a velocity less than v_c, the critical velocity behind the bottleneck.

If cars travel at a constant velocity v and spacing d, the flow q is v/d. Geometrically the state (v, d) is a point in Fig. 11 and q is the slope

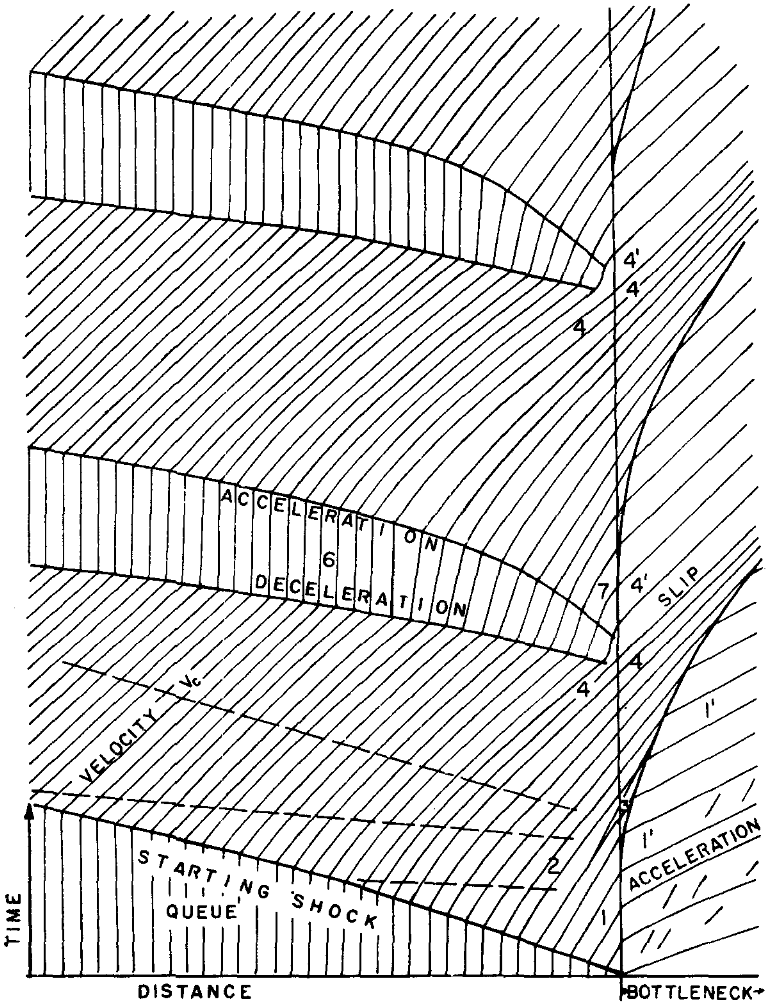

Fig. 12

of the line from the origin to the point (v, d). All points on the line from $(0, 0)$ to (v, d) represents states with the same flow. The state of maximum flow along the curve A' is the point $1'$ of Fig. 11 where the tangent line from the origin to A' meets A'.

To illustrate the temporal development of a specific initial flow pattern, let us imagine that we close the bottleneck completely and allow the cars to line up behind the bottleneck. Then at time zero we release the cars. All cars must accelerate initially so the flow pattern is determined at first by curves A and A' alone. The first car that enters the bottleneck can accelerate to its free speed in the bottleneck. Behind this car there will be a fan of acceleration waves as shown in the lower right hand corner of Fig. 12. Each car velocity above that of point $1'$ propogates into the bottleneck with the positive wave velocity associated with the point on A' for this car velocity. Since the tangent line to A' at point $1'$ has intercept at velocity zero, this car velocity propogates with zero wave velocity. It is stationary at the bottleneck. As in previous theories, the flow entering the bottleneck quickly adjusts to the maximum flow for the given velocity-headway curve, in the present case, curve A'.

Behind the bottleneck an acceleration wave will propogate back into the queue. Since the flow immediately behind the bottleneck must match that of state $1'$ just inside the bottleneck, the state here must lie on the straight line from the origin to point $1'$ of Fig. 11. Since cars reach this state by an acceleration, this state must also lie on curve A, thus at point 1 of Fig. 11. The jumps from state 1 to $1'$ at the entrance to the bottleneck is a shock in the usual sense with a shock line which passes through the origin and therefore has velocity zero. The acceleration waves from velocity zero to point 1 will be a fan of waves, a shock, or both depending upon the spacing in the queue according to the conditions represented in Fig. 4. In Fig. 12, it is represented by a shock. In any case the fastest wave will have a negative velocity. All cars in the expanding region of Fig. 12 between the fastest starting wave and the bottleneck will be in state 1.

Both states 1 and $1'$ are unstable and the velocities tend to drift upward as in Figures 6 and 7. Any disturbance in state $1'$ that produces higher velocities inside the bottleneck propogates to the right, however, and has no immediate effect on the cars entering the bottleneck. We will assume that the bottleneck is only a short stretch of road that empties into a highway of larger capacity so that such waves will not come back through the bottleneck at some later time and interfere with waves that enter the bottleneck.

An acceleration wave associated with a small disturbance in region 1 of Fig. 12, however, will propogate back into the stream while a deceleration wave will move forward toward the bottleneck. Cars that have passed through a small acceleration-deceleration maneuver as in Figs. 6 and 7 find themselves approaching the bottleneck in state 3 of Figs. 6 or 11. If point 3 lies between curves D' and A' as in Fig. 11, these cars can actually enter the bottleneck in state 3. They must, however, accelerate to point 1′ inside the bottleneck since they are following cars which accelerate from state 1 to 1′ upon entering the bottleneck. The shock line from points 3 to 1′ intersects the velocity axis at a positive velocity so the shock that was stationary from states 1 to 1′ now goes from state 3 to 1′ and starts to move into the bottleneck. A small region labeled 3 in Fig. 12 illustrates this.

After the cars in state 3 have entered the bottleneck, the deceleration wave bringing cars in state 2 hits the bottleneck. If point 2 lies between curves D' and A', these cars can also enter the bottleneck undisturbed and furthermore the wave associated with them will overtake the shock that has since started to move into the bottleneck. As this wave is absorbed by the shock the shock goes from state 2 to 1′ but this shock also has a positive velocity and so continues to move through the bottleneck. The region where cars are in state 2 now expands on both side of the entrance to the bottleneck. Since state 2 has a higher flow than states 1 and 1′we are now passing cars through the bottleneck at a faster rate than was possible under conditions of acceleration.

State 2 is stable inside the bottleneck but not behind it. Any new disturbance involving an acceleration and deceleration behind the bottleneck will cause the velocity to drift upward following the same sequence of events as described above for the drift from state 1 to state 2. The details of these disturbances are not shown in Fig. 12, only the net result which is a drift of the velocity from state 2 to states of higher velocity on curve A of Fig. 11. A fan of acceleration waves from these disturbances propogates back from the bottleneck as shown by the broken lines in the lower left corner of Fig. 12. The state of flow remains continuous across the entrance to the bottleneck and as these faster cars overtake the shock traveling into region 1′ of Fig. 12 inside the bottleneck, the shock continues to move faster.

This drift continues until the state at the entrance to the bottleneck has drifted up curve A to the intersection with curve D' at point 4 of Fig. 11. Behind the bottleneck disturbances are still being generated which try to push the velocity still higher but when these waves hit the bottleneck the cars are traveling too fast or too close together to be admitted into the bottleneck. These cars will be forced to decelerate in order to pass through the bottleneck, but since they are following cars in state 4, they must eventually accelerate back to the velocity of state 4.

State 4 is also unstable inside the battleneck because it lies on curve D', If any car in the bottleneck decelerates slightly and then accelerates, the acceleration wave will propogate rapidly to the right while the deceleration wave will propogate with the wave velocity of curve D' at point 4, The deceleration wave has a positive or negative velocity accordingly as point 4 lies above or below the point of maximum flow on curve D'. In the former case, this instability has no effect on the subsequent flow, but in the latter case the deceleration wave moves into the entrance and causes the velocity to drop below point 4. The latter case is shown in Fig. 11 and we shall consider this case first.

What happens now is quite sensitive to the magnitude and location of the deceleration-acceleration maneuver made by the first few cars that are repelled by the bottleneck. If we allow a car to enter the bottleneck temporarily in a state above point 4 and then decelerate inside the bottleneck to an acceptable state for the bottleneck, it would decelerate to another point on curve D'. If the wave velocity on D' at point 4 is negative, as we have assumed above so will the shock velocity to any point on D' below point 4. This deceleration will, therefore, produce a wave which is immediately thrown out of the bottleneck forcing later cars to decelerate before the bottleneck.

A car that decelerates behind the bottleneck follows a path in Fig. 11 from point 4 over to curve D which at first gives an increased flow and only makes the situation worse. In order to reduce the flow below that of point 4 and also give a negative shock velocity, the deceleration must reduce the car velocity below that of point 5 (the intersection of the constant flow line through point 4 and curve D). Cars in this state cannot enter the bottleneck either but from here they can accelerate to a state on curve A below point 4 from which they can enter the bottleneck. This acceleration must, however, also occur behind the bottleneck and to prevent a repetition of the rejection of cars by the bottleneck, the wave velocity for this acceleration must be negative. Since the cars that experience this maneuver are following cars that pass through the bottleneck in state 4, they must continue to accelerate inside the bottleneck to the velocity of point 4. They will finally reach a state in the bottleneck on the line of constant velocity between points 4 and 4′ of Fig. 11.

The above desoription does not uniquely define the magnitude of the deceleration-acceleration maneuver but only imposes certain restrictions which if violated will cause other cars to enter the bottleneck at an unacceptable spacing and be forced to decelerate even harder to correct the error. The smallest disturbance arises if the first car repelled by the bottleneck decelerates from its state above point 4 of Fig. 11 to a point on curve D just barely below point 5 and then accelerates back to a point on curve A just below point 4. The final acceleration inside the bottleneck then carries the state to a point just to the right of point 4. Behind the bottleneck the deceleration and acceleration waves both form shccks of small negative velocity with the velocity of the acceleration wave slightly larger (less negative) than the deceleration shock. All waves run away from the entrance to the bottleneck leaving an expanding region around the entrance where cars are left in the state on curve A just below point 4. Since this is unstable again, the velocity will drift upward until it tries to pass point 4 again. Then the traffic will jam once move.

The maximum disturbance is created if a car rejected by the bottleneck stops completely and then enters the bottleneck at a very low velocity. This car must then accelerate back to the velocity of point 4 but in so doing will reach state 4′ on curve A' of Fig. 11. If this final acceleration inside the bottleneck has a negative wave velocity, however, this acceleration wave will move back to the bottleneck entrance and pull cars in faster until the velocity at the entrance has increased at least to point 7 on curve A where the flow matches that at point $4'$. we thus create a deceleration shock from point 4 to point 6 on curve D and an acceleration shock from point 6 to point 7 both behind the bottleneck moving with negative velocities. We also create a stationary shock just inside the bottleneck from point 7 to $4'$. The expanding region behind the bottleneck in state 7 is unstable, however, and will tend to drift up along curve A to point 4 where the traffic will jam again. The main difference between the two extreme cases described in this and the preceding paragraph is that it will take longer for the state at the bottleneck to drift back to state 4 in the present case. The large disturbance shown in Fig. 12 around the regions labeled 4 and $4'$ is intermediate between the two extremes described above.

Regardless of the magnitude of the disturbance created at the bottleneck, the deceleration shock from curve A to D of Fig. 1 travels slower than the acceleration shock from D back to A. The region of low velocity between the two shocks, therefore, expands as it moves back from the bottleneck as shown in Fig. 12. States on D and A are all unstable, however, so as we move back along these shocks the low velocity drifts lower and the high velocity drifts higher until the low velocity is zero and the high velocity is v_c. The development of the stoppage shown in Fig. 12 is similar to that of Fig. 10. As the car velocities drift, the shock velocities decrease, particularly the acceleration shock until both shocks are traveling at the same velocity. One obtains far behind the bottleneck an oscillation between velocities 0 and v_c as shown on the left side of Fig. 12.

Waves also run through the bottleneck to the right. Just before a shock is created behind the bottleneck, cars are traveling through the bottleneck in state 4. As soon as a shock is created a car runs through the bottleneck in state $4'$ (or between 4 and $4'$) giving a sudden drop in flow and increase in spacing. This drop follows the trajectory of the car at the common velocity of points 4 and $4'$ and is not a wave in the usual sense. In fluid dynamics an analogous phenomena would be called a "slip stream". The discontinity shows very clearly in Fig. 12 along the trajectory labeled "slip". Similar discontinuities have appeared previously in Figs. 9 and 10 on the lower boundary of the regions labeled "stable" but the change in spacing along these trajectories was much smaller than in Fig. 12.

At the same time this slip stream is created, an acceleration shock is also created which at first remains stationary at the entrance of the bottleneck but then starts to move toward the right as the velocity of cars behind it increases. The velocity of the shock continues to increase and its amplitude decreases as it propogates. Region 4′ of Fig. 12 is unstable and the velocity drifts upward. If the bottleneck empties into a highway of high capacity, however, an acceleration wave will be created when cars in state 4′ hit the exit. This wave will pull the velocity up to point 1′ again.

The flow pattern described above is not the only type of flow that can exist through a bottleneck. Several critical assumptions have been made regarding the relative shapes of the velocity-headway curves of Fig. 11 and in addition we have assumed that the flow is congested at all times. (It started from and was maintained by a queue that was never dissipated.). This flow pattern, however, seems to conform best with the uncontrolled flow observed in the Holland Tunnel. Before considering the other types of flow patterns, we shall make a comparison of this theory with experimental data presently available from the Holland Tunnel.

EXPERIMENTAL EVIDENCE

In most experiments performed in the tunnels around New York City velocity-headway data have not been separated according to whether the cars were accelerating or decelerating although some work in this direction is in progress now. Preliminary experiments show higher flows during decelerations than during acceleration [17]but as yet there is not enough data to determine more than crude properties of the curves shown in Fig. 11. Most of the evidence to support the theory, beyond that described already as motivation for the model, is therefore circumstantial. The main success of the theory is simply that it predicts the existence of stoppages, an accomplishment that was particularly difficult to obtain from previous theories. It also explains several other peculiar qualitative properties of traffic flow.

The most spectacular experiments are those organized by Edie and Foote [1, 2] who placed observers at several locations in the tunnel to record the time at which consecutive cars passed. These experiments showed the following properties described by the theory in the last section. Disturbances originate at the bottleneck with relatively small amplitude and short duration. As they propagate back toward the entrance to the tunnel both the amplitude and duration increase until the low velocity reaches zero and the average peak velocity reaches some fixed value after whch the stoppage propagates with very little change. At the same time that a disturbance moves backward from the bottleneck another moves forward giving a temporary drop in flow due to an increase in the spacing between cars.

In previous theories, a deceleration wave would form a shock but an acceleration wave would fan out. In the present theory an acceleration wave can also form a shock. The experiments do not give accurate information on the widths of the acceleration or deceleration waves but they certainly show no very marked asymmetry. If the acceleration wave did fan out, the width of the wave after a mile run through the tunnel should have shown very clearly.

Extensive experiments have been made of the average velocity of cars as they pass various positions in the tunnel [18]. Large variations were found from one day to the next but in the most common pattern the average velocity decreased gradually from the entrance of the tunnel to the bottleneck and then increased more rapidly toward the exit. The bottleneck occurs at a point where the grade changes abruptly and there is a curve. From the geometry of the tunnel one would expect a rather sudden change in the velocity-headway curves here. In previous theories a sudden change in the highway would produce sudden changes in the average velocity. The present theory, however, gives a gradual change in the average velocity with position on the highway despite an assumption that the velocity-headway curves changed discontinuously at the entrance to the bottleneck. This occurs because most of the time the state of flow on either side of the discontinuity is the same and lies in a region of the velocity-headway space that is acceptable to both the bottleneck and the highway behind it simultaneously. This could not happen in the single state theories because the acceptable states at any position in the tunnel are confined to a single velocity-headway curve and on cannot find a state acceptable to different highway sections unless their velocity-highway curves intersect. The present theory also gives a minimum average velocity at the entrance to the bottleneck.

The most extensive set of data obtained in the tunnels has been used to determine the relation between average flow q and average density k at various positions in the tunnel. One can always define averages in such a way that q, k and the average velocity v are related by

$$q=kv$$

and k is related to an average spacing d by

$$k=1/d.$$

Thus an empirical relation between q and k implies a relation between v and d or any other of the quantities k, v, d and q.

If one assumes that the relation between q and k is unique at each position in the tunnel, then it is logical to define the capacity at each position as the maximum of q with respect to k. By evaluating the capacity as a function of position, one can then define a bottleneck as the position of minimum capacity. The foot of the upgrade in the Holland tunnel was found to be such a bottleneck but there were also other reasons for identifying this as the bottleneck. It was the point where stoppages seemed to originate and it was also suspect because of the geometry of the roadway.

There are several features of these experimental curves that were mysterious. The graph of capacity as a function of position showed a very gradual decrease in capacity from the entrance to the bottleneck [1, 2] and then a more rapid increase toward the exit. It was very similar to the graph of average velocity vs. position. From the geometry of the tunnel one might again have expected a more sudden drop in capacity near the bottleneck. Also if the continuum theory were correct one would have expected an increase in capacity behind the botteneck to produce a decrease in average velocity.

The curves of q vs. k which determined the capacity showed a sudden drop in q as k increased past the point where q has its maximum value. Edie [14] has even suggested that the curve has a discontinuity. If one assumes the existence of a unique relation between velocity and headway for each car and then postulates that the q vs. k curve is derived from an average of the curves for individual cars, the existence of a sudden drop in the q vs. k curve implies the existence of a sharp twist in the relation between velocity and headway. It certainly is not plausible, however, that drivers who have the freedom to choose their own safe driving distance should be conscious of some critical velocity near which their spacing is on the average very sensitive to small changes in velocity.

According to the continuum theory, the flow at any point behind a congested bottleneck should adjust to a density which is larger than that which produces the maximum flow. Much of the data used to construct the q vs. k curves, however. was at densities below that for the maximum q even though the data were obtained under what should have been congested conditions. Finally, after controls were imposed upon the traffic entering the tunnel, the q vs. k curves changed considerably at the bottleneck [3], so much so that one must seriously question any arguments based upon the assumption that the q vs. k curve is a unique property of the highway independent of other properties of the traffic itself.

In the present theory there is no single velocity-head way relation and conseqently no unique q vs. k relation either. The theory should, however, describe the type of empirical q vs. k relation one would observe in any given flow pattern.

The experiments described above were not controlled in the sense that one actually forced a particular value of q or k and then varied this as a parameter. Observations were made on the flows which existed naturally under congested conditions. The usual procedure was to average the flow and velocity or spacing over one (or more) minute time intervals. Because of fluctuations, these averages varied from one minute to the next so one could plot the one minute average q vs. the corresponding one minute average k. The range of k was thus limited to the range of natural fluctuations.

According to the present theory, the velocity far behind a bottleneck should oscillate between 0 and approximately v_c. The velocity v_c, however, is unstable in the sense that it is subject to drift although it is stable in that all fluctuations are eventually absorbed in the same shock. One must, therefore, imagine what sort of averages one would obtain by selecting from Fig. 12 those cars that pass some fixed point far behind the bottleneck during a random one minute time interval.

We consider first those time intervals which do not contain a stoppage or part of a stopping wave front, If from any series of observations we select the one minute interval in which the average velocity is largest, most cars observed in this miunte must have reached their velocity after an acceleration (if not, higher average velocity would be observed at a neighboring but otherwise identical point on the highway). The average spacing will, therefore, lie close to curve A of Fig. 11 at a velocity somewhat above v_c. If on selects a time interval with a relatively low average velocity, the cars may have reached this velocity either after an acceleration or after a deceleration. In the former case the spacings will again lie close to curve A but in the latter case they will either lie between curves A and D or on curve D depending upon the magnitude of the deceleration. The lower the observed average velocity the more likely it is that the spacing lies close to curve D and the wider is the range of fluctuations in the average spacing. Most observations of one minute average velocities and spacings (excluding stoppages) should, therefore, lie in a cone shaped region such as shown in Fig. 13 by the shaded area.

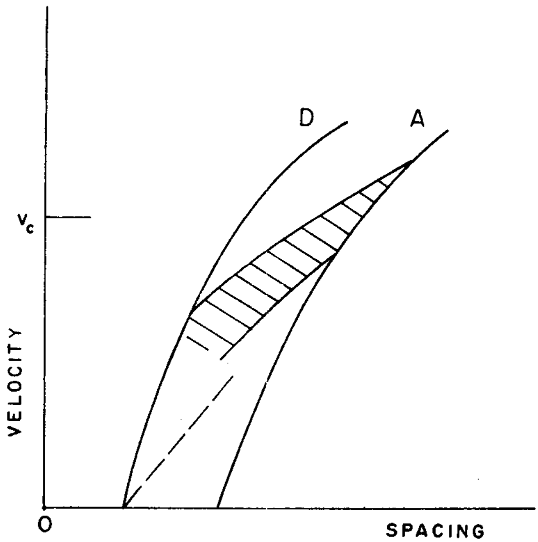

Fig. 13

If a time interval contains part of a stoppage or part of the shock front of a stoppages, the computed averages will be mixtures of two states one on D at zero velocity and the other in the shaded region of Fig. 13, most likely near its right end. If the stoppage covers a significant part of the time interval, the average velocities and spacings will scatter around the broken line of Fig. 13. This is likely to be the case if the observed average density is very high or the velocity very low.

If one converts Fig 13 into a q vs. k plot, the fluctuations in q will increase as k increases in agreement with most experimental observations. Despite the fact that the velocities and spacings in question here lie below those which give maximum flow on curve A and possibly curve D also, indicative of congested traffic, the maximum flow within the shaded region of Fig. 13 is likely to be near its left side. Most observed densities in the shaded region are therefore below that which produces the maximum observed flow, thus giving the illusion that the flow is not congested. Finally, if one tries to fit an empirical curve through the shaded region of Fig. 13 and the observations scattered around the broken line, it is conceivable that the transition between these two would be quite sharp and give the observed drop in q as k increased past the point of maximum flow.

The above description related primarily to the flow pattern far behind the bottleneck. As one moves toward the bottleneck, the peak velocity decreases and the low velocities are not necessarily zero. The average velocities and spacings will still form a picture similar to Fig. 13 but the shaded region will shift downward and the broken line moves up. If there was a sharp transition between the two types of observations before, it will become less pronounced as one moves toward the bottleneck. As the shaded region shifts down, the maximum flow in that region also decreases. Thus the indicated capacity of the highway varies with position in the same way as the average velocity even though we have assumed that the highway is homogeneous behind the bottleneck.

OTHER BOTTLENECK FLOWS

We shall investigate briefly here some other types of flow patterns which one can obtain behind a bottleneck from the present theory.

First we consider why by limiting the peak rate at which cars enter a tunnel one should eliminate the stoppages and increase the average flow. The Port of New York Authority has accomplished this by stopping the traffic at the tunnel entrance whenever more than some specified number of cars have entered during one minute. The immediate affects of this are that (a) the traffic travels through the tunnel in platoons and (b) cars approach the bottleneck at much higher velocities, above

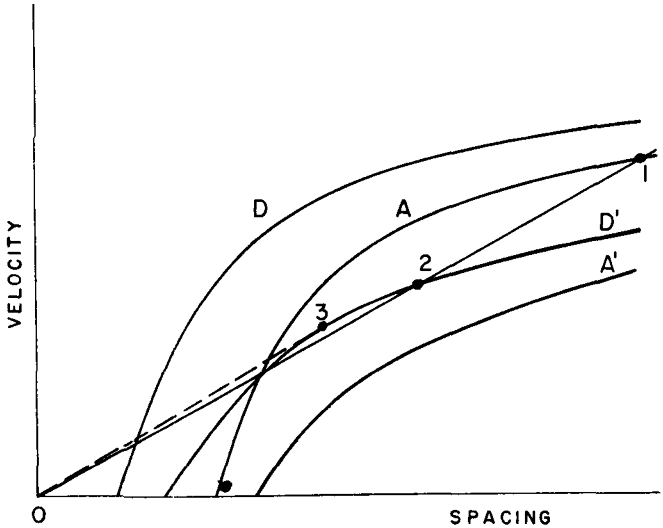

Fig. 14

the velocity of maximum flow on curve A of Fig. 11, which in turn is above v_c. This we assume is done in such a way that the average flow entering the tunnel is less than the maximum flow on curve D'. Thus in Fig. 14, if the cars approach the bottleneck in a state represented by point 1, the constant flow line from the origin to point 1 intersects curve D'.

When a fast car hits the bottleneck it must decelerate because its state lies above curve D'. Suppose, however, we allow the car to enter the bottleneck in this state temporarily provided that it immediately decelerates to an acceptable state on curve D'. Unlike the situation represented in Fig. 11 where the velocities were below the velocity which gives the maximum flow on curve D', it is possible now to obtain a deceleration shock to curve D' which has a non-negative velocity and will therefore stay inside the bottleneck. If a car decelerates to point 2 of Fig. 14, the intersection of curve D' with the constant flow line through point 1, the deceleration shock will have zero velocity and therefore stay at the bottleneck.

A shock from point 1 to point 2 of Fig. 14 thus represents a possible flow pattern into the bottleneck though perhaps not a unique one. Actually the theory described here is not quite complete. It describes the possible patterns of flow on either side of the bottleneck entrance and there are certain continuity conditions that must be satisfied across the entrance but the theory does not specify the velocity-headway relations which a car must follow temporarily when it is behind the bottleneck entrance and the car in front of it is already inside the bottleneck. Neither does the theory specify the nature of the velocity-headway relations along a highway whose properties are changing continuously with position. If there happens to be more than one way of matching solutions on either side of the bottleneck entrance, the present theory does not specify which solutions one should use.

The above flow pattern, however, should be more stable than that represented in Figs. 11 and 12. Inside the bottleneck, the wave velocities for both acceleration and deceleration are positive. Thus any fluctuations inside the bottleneck would travel away from the entrance. Since the velocities behind the bottleneck are above v_c, any instabilities here would produce a pattern of flow such as shown in Fig. 9 rather than Fig. 10. One might have fluctuations but certainly no stoppages. Actually even these milder fluctuations could not grow very well because the gaps between platoons hinder the propogation of waves. Furthermore what waves that might exist are moving forward so fast and with such small variations in speed that they do not have much time to interact with each other before they have run the length of the tunnel.

This flow with the shock at the bottleneck entrance could occur as long as the flow entering the tunnel is sufficiently low that even with fluctuations the instantaneous flow approaching the bottleneck does not exceed that of point 3 in Fig. 14, the maximum flow that the bottleneck can accommodate under any conditons. With proper controls, however, one should be able to produce a flow close to this maximum whereas under the congested conditions described in Figs. 11 and 12, the bottleneck flow never exceeded that of point 4 in Fig. 11 which was distinctly below the maximum.

If one can establish a flow which exceeds that of point 4 in Fig. 11 but is less than that of point 3 of Fig. 14, this flow will be slightly unstable. If some driver should suddenly decelerate before entering the bottleneck, the whole flow pattern will collapse. Subsequent cars will enter the bottleneck after an acceleration, the flow through the bottleneck will drop, and the flow pattern will revert to that shown in Fig. 12. Once this has happened, however, one can restore the high velocity flow by introducing a large gap in the traffic. This gap as it propogates into the tunnel will absorb the shocks traveling back from the bottleneck and increase the velocity of cars behind the gaps. It thus appears that the controls which cause platoons to form achieves results which could not be realized by any type of continuous control at the entrance to the tunnel.

So far in the discussion of bottleneck flow, we have assumed that curves A and D of Fig. 11 or 14 intersect and that the point of intersection has a velocity a) below v_c which in turn is less than the velocity of maximum flow on curve A, b) below the velocity of maximum flow on D', c) below the velocity of maximum flow on A' (point 4′ is below 1′) and d) has a flow above the maximum flow on A' (point 4 is above point 1). Whether or not any of these conditions are satisfied depends upon the relative shapes of the curves A', D' and A. Some of these assumptions are also rather critical.

In the discussion of congested bottleneck flow, these assumptions were made for the following reasons. Assumption b) guaranteed that if a car was forced to decelerate in order to enter the bottleneck, it must do so behind the bottleneck. This in turn meant that subsequent cars would enter the bottleneck with no change in state giving the continuity of the average velocity across the bottle entrance. This along with assumption a) guaranteed that the velocity just behind the bottleneck would not exceed v_c which meant that the fluctuations would develop into complete stoppages.

Assumption c) was less significant. If point 4′ were above 1′ in Fig. 11 one would obtain a slightly different behavior just inside the bottleneck behind the slip stream of Fig. 12. The first car that passes into the bottleneck after a shock will accelerate to state 4′, just as in Fig. 12. Since the wave velocity at state 4′ will now be positive, one will generate a fan of waves behind the slip stream with positive wave velocities and car velocities between those of state 1′ (with zero wave velocity) and state 4′. The shock at the bottleneck entrance will run from states 1 to 1′ and the subsequent flow will look more like that which developed at time zero when the first car suddenly accelerated to a velocity above that of state 1′. Either of these behavior patterns would be consistent with what is presently known about the flow in the tunnel.

Assumption d) was introduced because it seemed necessary to have some mechanism to explain the long (four minute) average time interval between stoppages. This was attributed to a slow drift of the velocity from state 1 or 7 up to state 4 in Fig. 11 which would not exist if state 4 were below state 1 or 7.

For a weak bottleneck, curve D' should lie close to curve D and A' close to A. Curves A and D' would probably still intersect but at a velocity close to the free speed, contrary to assumptions a), b), and c), above. Traffic flow through such a bottleneck should be fairly stable. If one starts the flow from a queue as in Fig. 12, the state at the bottleneck will still drift up along curve A through points 1, 2, etc. as in Fig. 11. Now, however, one reaches the point of maximum flow on A before the intersection with curve D'. After one passes this point, the wave velocities are all positive behind the bottleneck. The tendency for the velocity to drift higher now essentially stops because fluctuations which would cause the drift do not travel back to pull cars into the bottleneck at higher velocities but pass through the bottleneck and are lost. The final state of flow will depend upon how cars are fed into the highway far upstream but one should not expect anything unusual to happen at the bottleneck unless one can force the cars to approach the bottleneck at a flow exceeding the capacity on curve D'. This, however, cannot be done if the state of flow lies on the acceleration curve A.

As one makes the bottleneck more severe, the flow becomes less stable. If the velocity of point 4 in Fig. 11 lies above the velocity of maximum flow on D' (contrary to assumption b above) but below the velocity of maximum flow on A, the flow pattern from a queue will again start as in Fig. 12. When the velocity behind the bottleneck passes point 4, however, it is possible for cars to decelerate to curve D' inside the bottleneck as described in this section for the controlled flow. The velocity behind the bottleneck can continue to drift upward along A but there will be a deceleration shock to curve D' at the entrance to the bottleneck. This drift continues until either the velocity reaches the point of maximum flow on A or the flow behind the bottleneck exceeds the maximum flow on D', accordingly as the maximum flow on A is less or greater than the maximum flow on D'. In the former case the flow should stabilize. In the latter case, however, a shock will form when the flow approaching the bottleneck tries to exceed the maximum flow D'. The subsequent flow will then oscillate as in Fig. 12 except that a) the shocks form when the flow behind the bottleneck reaches the maximum flow on D' instead of point 4 of Fig. 11, b) the average velocity behind the bottleneck will be higher than that inside the bottleneck and c) if the velocities behind the bottleneck exceed v_c, the shocks may not develop into complete stoppages but give a pattern far behind the bottleneck similar to Fig. 9.

For a severe bottleneck it may be that either assumption d) above is violated (curves A and D' intersect but at such a low velocity that the flow rate at the intersection is below the maximum flow on A') or curve D' lies entirely to the right of curve A with no intersection at all. In either case the flow generated from a queue behind the bottleneck is highly unstable. Cars inside the bottleneck accelerate and try to achieve a flow equal to the maximum flow on curve A', point $1'$ of Fig. 11. The state behind the bottleneck starts at point 1 except that now this point is above curve D'. The state at point 1 of Fig. 11 is unstable and the velocity will try to drift upwards. This causes the flow to increase behind the bottleneck. But the bottleneck will not accept a higher flow because an increase in flow causes the shock at the bottleneck to move forward and pull cars into the bottleneck at a closer spacing than the bottleneck will allow. Cars are, therefore, forced to decelerate before the bottleneck to a state on curve D and then accelerate again. The bottleneck will then try again to pull the flow up to that of point 1 and the oscillation immediately starts again. Whereas in Fig. 11 one has an oscillation the period of which was essentially the time it takes for the velocity to drift from point 1 past point 4, in the present case point 4 is below point 1 and this drift time is zero. The oscillations should, therefore, be much faster.

In all of the above cases one should be able to achieve a relatively stable flow through the bottleneck if by controlling the flow approaching the bottleneck one can keep the velocity above that for the maximum flow on D' and the flow itself below the maximum flow on D'. This would be particularly advantageous for the strong bottleneck of the preceding paragraph. By keeping the velocities high enough one can achieve a flow approaching the maximum flow on curve D' whereas under congested conditions the highest flow is the maximum flow for curve A'.

Besides the data obtained in the tunnels around New York City, there is one other interesting collection of experiments that were done on the traffic approaching a temporary bridge along the Merritt Parkway in Connecticut [18]. The latter experiments are not as extensive as those in the tunnels and the data were collected only at various points within a few hundred feet behind the bridge, which is obviously a bottleneck of considerably smaller capacity than the parkway. Observations of speeds and spacings were made during about a half hour in which time the flow approaching the bridge started at a low level and increased until it overloaded the bridge and caused a queue.

The flow-density curves obtained by Palmer had a shape similar to those behind the bottleneck in the tunnel except that the drop in flow when the bottleneck became congested was much more severe. Furthermore it appeared that as k increased, q reached its maximum, dropped suddenly and then seemed to be increasing again toward a second maximum at a high density.

A possible explanation for this in terms of the severe bottleneck described above is that when the bottleneck was uncongested at low flow, below the maximum flow of curve D', the flow increased as the density increased in the usual way for moderate or low density traffic. As soon as the flow reached the maximum on curve D', however, a queue is formed and cars accelerate through the bottleneck. The flow then drops to the maximum on curve A'. Once one has reached this congested state, it is not unreasonable that a further increase in density should produce a slightly higher flow, for the same reason that a decrease in the average observed spacing in Fig. 13 also produced an increase in the average flow.

Palmer also showed that the flow behind the bottleneck after it had become congested oscillated with a period of about one minute. This is certainly much faster than the average period of oscillation in the tunnel as the theory would predict although it is still rather slow compared with other time constants of the system.

CONCLUSIONS

The purpose here has been to construct a theory which is as simple as possible yet would describe qualitatively the various phenomena that have been observed experimentally. We showed first that the stable continuum and car-following theories previously considered are not capable of describing such things as the stoppages observed in the Holland Tunnel. Although a car-following theory with a large reaction time seemed attractive from the point of view of individual driver behavior, the mathematical analysis of the equations was very difficult. Certain features of the solutions appeared encouraging but there were too many questions regarding the solutions which could not as yet be answered.

A theory based upon two velocity-headway curves as represented in Fig. 2 seemed to be sufficiently flexible as to furnish plausible explanations of all the known phenomena that had been in conflict with previous theories. Although the explanation given here may not all be correct, it is unlikely that they are all wrong and furthermore there are other possible explanations that one could give for some of these phenomena without changing the basic structure of the theory.

The new theory contains most of the ingredients of previous theories and is, therefore, capable of describing anything which older theories could describe correctly. In addition it furnishes possible descriptions or explanations for 1. the tendency for drivers to leave larger spacings during accelerations than decelerations, 2. the stronger response of drivers to changes during decelerations than during accelerations, 3. the existence of acceleration shocks as well as deceleration shocks, 4. the instability of traffic flow at high densities, 5. the existence of stoppages behind a bottleneck, 6. the slow variation of the average velocity of cars with position in the tunnel, 7. the fact that the indicated capacity of the tunnel at various positions in the tunnel changes in the same way as the average velocity, 8. the frequent observation of densities below the indicated optimal density under congested flow behind a bottleneck, 9. the long length of time between stoppages in the tunnel, 10. the shape of the observed q vs. k curves in the tunnels, 11. the eliminatlon of stoppages and the increase in flow that results from the introduction of gaps in the traffic stream, 12. the variation of the observed q vs. k curves with the nature of the flow, 13. the faster oscillation in congested flow behind a severe bottleneck, and 14. the possible double hump in the q vs k curves observed behind a severe bottleneck.

We have not attempted to obtain quantitative results but only have indicated the structure of the theory. Considerably more data than is presently available will be needed to test the correctness of the theory and to determine whether or not the theory is capable of giving quantitatively accurate results. We have also neglected reaction times and variations in behavior from one driver to the next. These things obviously exist but they are not necessarily critical to the gross aspects of the traffic flow.

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REFERENCES

1. L. C. Edie and R. S. Foote, Effect of Shock Waves on Tunnel Traffic Flow, Proc. Highway Research Board 39, 492–505 (1960).
2. L. C. Edie and R. S. Foote, Experiments on Single-Lane Flow in Tunnels, Theory of Traffic Flow, pp. 193–206, Elsevier, Amsterdam, 1961.
3. H. Greenberg and A. Daou, The Control of Traffic Flow to Increase the Flow, Opns. Res. 8, 524–532 (1960).
4. L. C. Edie and R. S. Foote, Traffic Flow in Tunnels, Proc. Highway Research Board 37, 334–344 (1958).
5. M. J. Lighthill and G. B. Whitham, On Kinematic Waves II: A Theory of Traffic Flow on Long Crowded Roads, Proc. Royal Soc, A 229, 317–345 (1955).
6. P. I. Richards, Shock Waves on the Highway, Opns. Res. 4, 42–51 (1956).
7. R. E. Chandler, R. Herman, and E. W. Montroll, Traffic Dynamics: Studies in Car Following, Opns. Res. 6, 165–184, (1958).

8. E. Kometani and T. Sasaki, On the Stability of Traffic Flow, J. Opns. Res. Japan, 2, 11–26 (1958).
9. R. Herman. E. W. Montroll, R. B. Potts, and R. W. Rothery, Traffic Dynamics: Analysis of Stability in Car Following. Opns. Res. 7, 86–106 (1959).
10. W. Helly, Simulation of Bottlenecks in Single-Lane Traffic Flow, Theory of Traffic Flow, pp. 207–238, Elsevier, Amsterdam, 1961.
11. R. Herman and R. B. Potts, Single-Lane Traffic Theory and Experiment, Theory of Traffic Flow, pp. 120–146, Elsevier, Amsterdam (1961).
12. D. C. Gazis, R. Herman, and R. B. Potts, Car Following Theory of Steady State Traffic Flow, Opns. Res. 7, 499–505 (1959).
13. G. F. Newell, Nonlinear Effects in the Dynamics of Car Following, Opns. Res. 9, 209–229 (1961).
14. L. C. Edie, Car-Following and Steady-State Theory for Noncongested Traffic, Opns. Res. 9, 66–76 (1961).
15. D. C. Gazis, R. Herman, and R. W. Rothery, Non-Linear Follow-the-leader Models of Traffic Flow, Opns. Res. 9, 546—567 (1961)
16. R. W. Rothery and R. Herman, Private Communication.
17. A. Daou, Private Communication.
18. The Port of New York Authority, Holland and Lincoln Tunnel Traffic Survey 1954 (unpublished).
19. M. R. Palmer, The Development of Traffic Congestion, Bureau of Highway Traffic, Yale University, May 1959.

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Masao Iri

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