A QUEUEING SYSTEM WITH SERVICE DEPENDING ON QUEUE-LENGTH

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1. INTRODUCTION

The theory of queues is concerned with the development of mathematical models to predict the behavior of systems that provide services for randomly arising demands. Many types of systems that have been studied in the theory of queues deal with the following type of situation: The succesive service times of the customers are independent of the queue length.

The problems in which the service time is dependent on the queue length, however, arise from the properties of the customers and the capacity of the server. This paper is concerned with such a problem.

In § 2 we define the system with such a service mechanism and in § 3 we refer to the two lemmas due to Takács [3, 4] in order to investigate the stochastic properties of the queue length and of the busy period of the system. In § 4 we consider the transient behavior of the queue length and in § 5 we consider the limiting behavior of the queue length. In § 6 we consider the distribution of the busy period. In the last section, we consider a special case.

In [5] by applying the method of the imbedded Markov chain [1, 2] the author obtained some theorems which provide criteria for determining whether the system is ergodic, transient, or recurrent, and obtained the equilibrium distribution of the queue length and the waiting time.

In this paper the approach we employ is based on the above method.

2. DESCRIPTION OF THE SYSTEM

The description of the queueing process considered in this paper consists of three parts:

(a) Input Process: The input process is assumed to be a homogeneous Poisson process of density λ. Suppose customers arrive at the counter

at times $\tau_1, \tau_2, \cdots\cdots, \tau_n, \cdots\cdots$ $(0<\tau_1<\tau_2<\cdots\cdots<\tau_n<\cdots\cdots)$. Then the inter-arrival times $\tau_n-\tau_{n-1}$ $(n=1, 2, \cdots\cdots, \tau_0\equiv 0)$ are independently and identically distributed random variables with distribution function $P\{\tau_n-\tau_{n-1}\leqq x\}=F(x)$ where

$$(1) \qquad F(x)=\begin{cases} 1-e^{-\lambda x} & \text{for} \quad x\geqq 0 \\ 0 & \text{for} \quad x<0 \end{cases}$$

(b) Queue Discipline: The policy followed here is the "first come-first served" policy.

(c) Service Mechanism: There is only one server at the counter. A service period commences only when there is some customer waiting in the queue to be served.

Let $N_1, N, \cdots\cdots, N_l$ be integers such that $0<N_1<N_2<\cdots\cdots<N_l<\infty$. Define $N_0=0$ and $N_{l+1}=\infty$. Let us denote by $H_1(x), H_2(x), \cdots\cdots, H_{l+1}(x)$ the $l+1$ service time distributions and we shall suppose that such distribution $H_s(x)$ has a expected value $\mu_s=\int_0^\infty x dH_s(x)$ $(s=1, 2, \cdots\cdots, l+1)$. Let χ_n denote the service time of the nth customer $(n=1, 2, \cdots\cdots)$. The service times χ_n $(n=1, 2, \cdots\cdots)$ are assumed to be mutually independent positive random variables and also are independent of the sequence $\{\tau_n\}$ too.

Let $\xi(t)$ denote the queue length at the instant t i. e. the number of customers waiting in the queue or being served at the instant t.

Let $\tau_1', \tau_2', \cdots\cdots, \tau_n', \cdots\cdots$ denote the instants of the succesive departures. Define $\xi_n=\xi(\tau_n'+0)$ so shat ξ_n is the queue length immediately after the departure of the n th customer.

Let $\tau_1'', \tau_2'', \cdots\cdots, \tau_n'', \cdots\cdots$ denote the instants at which the services of customers commence. The service time χ_n of the nth customer is assumed to be distributed according to the distribution function $H_{s+1}(x)$ if $N_s+1\leqq\xi(\tau''_n)\leqq N_{s+1}$ $(s=0, 1, \cdots\cdots, l)$.

3. AUXILIARY LEMMAS

Throughout this paper, we shall use the following lemmas due to Takács [3, 4]. Let $H(x)$ denote the distribution function of a positive random variable and let $\mu=\int_0^\infty x dH(x)<\infty$.

Introduce the transform $\phi(s)=\int_0^\infty e^{-sx}dH(x)$ for $R(s)\geqq 0$.

Lemma 1. If (a) $R(s)\geqq 0$, $|w|<1$ or (b) $R(s)>0$, $|w|\leqq 1$ or (c) $\lambda\mu>1$, $R(s)\geqq 0$, $|w|\leqq 1$ where $0<\lambda<\infty$, then the equation

$$z=w\phi[s+\lambda(1-z)]$$

has exactly one root $z=\gamma(s, w)$ in the unit circle $|z|<1$. We have

$$\gamma(s, w)=w\sum_{j=1}^{\infty}\frac{(-\lambda w)^{j-1}}{j\quad !}\left(\frac{d^{j-1}[\phi(y)]^{j}}{dy^{j-1}}\right)_{y=\lambda+s} \tag{2}$$

If $\gamma(s, w)$ is defined by (2) for $R(s)\geqq 0$, $|w|\leqq 1$, then in this extended domain $\gamma(s, w)$ is a regular function of s and w, $|\gamma(s, w)|\leqq 1$ and $z=\gamma(s, w)$ satisfies the equation

$$z=w\phi(s+\lambda(1-z)).$$

We shall introduce the following abbreviations:

Let

$$\gamma(s)=\gamma(s, 1), \tag{3}$$

$$g(w)=\gamma(0, w), \tag{4}$$

and

$$\omega=\gamma(0, 1)=\gamma(0)=g(1). \tag{5}$$

Lemma 2. If $\lambda\mu>1$, then ω is the exact one positive real root of the equation

$$z=\phi[\lambda(1-z)]$$

and $\omega<1$. If $\lambda\mu\leqq 1$, then $\omega=1$. Further we have

$$\gamma'(0)=\frac{\mu}{\lambda\mu-1} \qquad \text{if } \lambda\mu<1 \tag{6}$$

$$=\infty \qquad \text{if } \lambda\mu=1,$$

and

$$g'(1)=\frac{1}{1-\lambda\mu} \qquad \text{if } \lambda\mu<1 \tag{7}$$

$$=\infty \qquad \text{if } \lambda\mu=1.$$

4. THE TRANSIENT BEHAVIOR OF $\{\xi_n\}$

Let $\nu_n^{(s)}$ denote the number of customers arriving at the counter during the nth service time in which the service time has the distribution function $H_s(x)$. Then we see that if $N_s+1\leqq\xi_n\leqq N_{s+1}$,

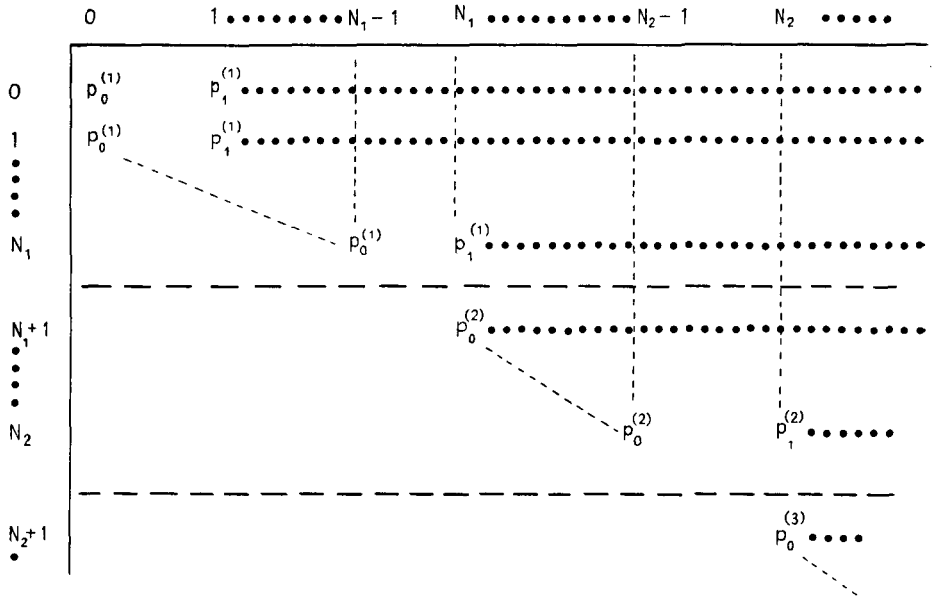
$$\xi_{n+1}=[\xi_n-1]^{+}+\nu_{n+1}^{(s)} \tag{8}$$

where $[a]^{+}=\text{Max}\ (a, 0)$. Hence the sequence $\{\xi_n\}$ of random variables forms a homogeous Markov chain. We shall say that the system is in state E_j at the nth step if $\xi_n=j$, so that the state space is $\{E_0, E_1, \cdots\cdots,$

$E_n, \cdots\cdots\}$. From (8) the matrix of transition probabilities $\|p_{ik}\|$ $(i, k=0, 1, 2, \cdots\cdots)$ where

$$p_{ik}=P\{\xi_{n+1}=k|\xi_n=i\} \qquad (n=0, 1, 2, \cdots\cdots)$$

has the following form.



Where $p_j^{(s)}=\int_0^\infty e^{-\lambda t}\frac{(\lambda t)^j}{j\,!}dH_s(t)$, $j=0, 1, 2, \cdots\cdots, s=1, \cdots\cdots, l+1$.

The chain is aperiodic and irreducible. Let us denote by $\|p_{ij}^{(n)}\|$ the nth power of the matrix $\|p_{ij}\|$. Then we have that $\lim_{n\to\infty} p_{ij}^{(n)}=\pi_j$ always exists and is independent of i; and further that either $\pi_j>0$ for any j or $\pi_j=0$ for any j. The chain is ergodic if and only if $\pi_j>0$ for any j. With respect to the states of the chain $\{\xi_n\}$ the following results (a) to (c) have been obtained in [5].

(a) The chain $\{\xi_n\}$ is ergodic if and only if $\lambda\mu_{l+1}<1$.

(b) The chain $\{\xi_n\}$ is recurrent if and only if $\lambda\mu_{l+1}\leqq 1$.

(c) The chain $\{\xi_n\}$ is transient if and only if $\lambda\mu_{l+1}>1$.

Let $\rho_s=\lambda\mu_s$ $(s=1, 2, \cdots\cdots, l+1)$. We introduce the generating fun-

ction of the sequence $\{\pi_j\}$, $P(z)=\sum_{n=0}^{\infty}\pi_n z^n$ for $|z|\leqq 1$ and further we define $Q_s(z)=\sum_{n=N_{s-1}+1}^{N_s}\pi_n z^n$ for $|z|\leqq 1$ $(s=2, 3, \cdots\cdots, l+1)$ and $Q_1(z)=\sum_{n=1}^{N_1}\pi_n z^n$ for $|z|\leqq 1$. Let $Q_s{}^*=\dfrac{Q_s(1)}{\pi_0}$ if $\pi_0>0$. Then if $\rho_{l+1}<1$, the following relation has been shown in [5]:

$$(9)\qquad 1-\rho_s=\pi_0\left[1+\sum_{i=1}^{s}Q_i{}^*(\rho_i-\rho_s)\right]\qquad (s=1, 2, \cdots\cdots, l+1).$$

If we put

$$\psi_s(\alpha)=\int_0^{\infty}e^{-\alpha x}dH_s(x)\qquad \text{for } R(\alpha)\geqq 0,$$

then we have the equation

$$(10)\qquad \sum_{j=0}^{\infty}p_j^{(s)}z^j=\psi_s[\lambda(1-z)]\qquad \text{for } |z|\leqq 1,\ s=1, 2, \cdots\cdots, l+1.$$

Assuming that the server is free at time $t=0$, ready to start the first service, we take throughout this paper $\tau_0'=0$ and $\xi_0=\xi(0)$.

We denote by $\sigma_n^m[V(z)]$ the partial sum $\sum_{i=n}^{m}v_i z^i$ in the power series expansion of a function $V(z)$.

The higher transition probabilities

$$p_{ik}^{(n)}=P\{\xi_n=k|\xi_0=i\}$$

are given by the following

Theorem 1. For $|z|\leqq 1$ and $|w|<1$ we define the following functions:

$$(11)\qquad A_1(z, w)=w(z-1)\psi_1[\lambda(1-z)],$$

$$(12)\qquad B_1(z)=z^{l+1},$$

$$(13)\qquad A_{s+1}(z, w)=A_1(z, w)+\sum_{j=1}^{s}wA_N^{(j)}(z, w)(\psi_j[\lambda(1-z)]-\psi_{j+1}[\lambda(1-z)])\qquad (s=1, 2, \cdots\cdots, l),$$

$$(14)\qquad B_{s+1}(z, w)=B_1(z)+\sum_{j=1}^{s}wB_N^{(j)}(z, w)(\psi_j[\lambda(1-z)]-\psi_{s+1}[\lambda(1-z)])\qquad (s=1, 2, \cdots\cdots, l),$$

$$(15)\qquad A_N^{(s)}(z, w)=\sigma_{N_{s-1}+1}^{N_s}\left[\frac{A_s(z, w)}{z-w\psi_s[\lambda(1-z)]}\right],$$

$$A_N^{(1)}(z,\ w)=\sigma_0^{N_1}\left[\frac{A_1(z,\ w)}{z-w\phi_1[\lambda(1-z)]}\right],$$

$$B_N^{(s)}(z,\ w)=\sigma_{N_{s-1}+1}^{N_s}\left[\frac{B_s(z,\ w)}{z-w\phi_s[\lambda(1-z)]}\right],$$

and

$$B_N^{(1)}(z,\ w)=\sigma_0^{N_1}\left[\frac{B_1(z,\ w)}{z-w\phi_1[\lambda(1-z)]}\right],$$

Let $g(w)$ be the root in z of the equation

$$z=w\phi_{l+1}[\lambda(1-z)]$$

in the unit circle $|z|<1$ and assume $A_{l+1}(g(w),\ w)=0$. Then

$$\Omega(z,\ w)=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}p_{ik}^{(n)}w^n z^k \tag{16}$$

$$=\frac{A_{l+1}(g(w),\ w)B_{l+1}(z,\ w)-A_{l+1}(z,\ w)B_{l+1}(g(w),\ w)}{(z-w\phi_{l+1}[\lambda(1-z)])A_{l+1}(g(w),\ w)}$$

for $|z|\leqq 1$ and $|w|<1$.

Proof. By the theorem of total probability, we can write that

$$p_{ik}^{(n+1)}=\sum_{j=0}^{\infty}p_{ij}^{(n)}p_{jk}. \tag{17}$$

Thus we have

$$p_{ik}^{(n+1)}=p_{i0}^{(n)}p_k^{(1)}+\sum_{j=1}^{k+1}p_{ij}^{(n)}p_{k-j+1}^{(1)} \qquad \text{if } 0\leqq k\leqq N_1-1, \tag{19}$$

and

$$p_{ik}^{(n+1)}=p_{i0}^{(n)}p_k^{(1)}+\sum_{j=1}^{N_1}p_{ij}^{(n)}p_{k-j+1}^{(1)}+\sum_{j=N_1+1}^{N_2}p_{ij}^{(n)}p_{k-j+1}^{(2)}+\cdots\cdots \tag{19}$$

$$+\sum_{j=N_s+1}^{k+1}p_{ij}^{(n)}p_{k-j+1}^{(s+1)} \qquad \text{if } N_s\leqq k\leqq N_{s+1}-1 \qquad (s=1,\ 2,\ \cdots\cdots,\ l).$$

Now for fixed i and $|z|\leqq 1$, introduce the generating function

$$U_i^{(n)}(z)=\sum_{k=0}^{\infty}p_{ik}^{(n)}z^k, \tag{20}$$

then by (18) and (19) we have

$$U_1^{(0)}(z)=z^i, \tag{21}$$

$$U_0^{(1)}(z)=U_1^{(1)}(z), \tag{22}$$

$$U_i^{(1)}(z)=z^{i-1}\phi_s[\lambda(1-z)] \qquad \text{if } N_{s-1}+1\leqq i\leqq N_s, \tag{23}$$

and

$$zU_i^{(n+1)}(z)=z\phi_1[\lambda(1-z)]p_{i0}^{(n)}+(\phi_1[\lambda(1-z)]-\phi_{l+1}[\lambda(1-z)]) \tag{24}$$

$$\sum_{j=1}^{N_1}z^j p_{ij}^{(n)}+\cdots\cdots+(\phi_l[\lambda(1-z)]-\phi_{l+1}[\lambda(1-z)])\sum_{j=N_{l-1}+1}^{N_l}z^j p_{ij}^{(n)}$$

$$+\psi_{l+1}[\lambda(1-z)]\{U_i^{(n)}(z)-p_{i0}^{(n)}\} \qquad (n=1,\ 2,\ \cdots\cdots).$$

Further if we introduce the generating function

$$\Omega(z,\ w)=\sum_{n=0}^{\infty} U_i^{(n)}w^n=\sum_{n=0}^{\infty}\sum_{k=1}^{\infty} p_{ik}^{(n)}w^n z^k, \tag{25}$$

which is convergent if $|z|\leqq 1$ and $|w<1$. Furthermore we introduce the generating function

$$Q_{ik}(w)=\sum_{n=0}^{\infty} p_{ik}^{(n)}w^n \qquad \text{for } |w|<1, \tag{26}$$

and we define

$$P_1(z,\ w)=\sum_{n=0}^{N_1} Q_{in}(w)z^n, \tag{27}$$

and

$$P_s(z,\ w)=\sum_{n=N_{s-1}+1}^{N_s} Q_{in}(w)z^n \qquad (s=2,\ 3,\ \cdots\cdots,\ l), \tag{28}$$

then by (21)~(24) we obtain that

$$\begin{aligned}\Omega(z,\ w)=\{&z^{i+1}+w\psi_1[\lambda(1-z)](z-1)Q_{i0}(w)+wP_1(z,\ w)\\ &(\psi_1[\lambda(1-z)]-\psi_{l+1}[\lambda(1-z)])+\cdots\cdots+wP_l(z,\ w)(\psi_l[\lambda(1-z)]\\ &-\psi_{l+1}[\lambda(1-z)])\}/\{z-w\psi_{l+1}[\lambda(1-z)])\}.\end{aligned} \tag{29}$$

Now $\Omega(z,\ w)$ is a regular function of z if $|z|\leqq 1$ and $|w|<1$. In this domain, the denominator of (29) has one and only one root $z=g(w)$ in the unit circle $|z|<1$ by lemma 1. Consequently $z=g(w)$ must be a root of the nominator of (29) too. Hence to determine $Q_{i0}(w)$ we have to observe the following relations between $Q_{i0}(w)$, $P_1(z,\ w)$, $\cdots\cdots$, and $P_l(z,\ w)$. By (18) and (19) we have

$$\left\{\begin{aligned}&Q_{i0}(w)-p_{i0}^{(0)}=p_1^{(1)}wQ_{i0}(w)+p_0^{(1)}wQ_{i1}(w),\\ &Q_{i1}(w)-p_{i1}^{(0)}=p_1^{(1)}wQ_{i0}(w)+p_1^{(1)}wQ_{i1}(w)+p_0^{(1)}wQ_{i2}(w),\\ &\vdots\\ &Q_{i,N_1-1}(w)-p_{i,N_1-1}^{(0)}=p_{N_1-1}^{(1)}wQ_{i0}(w)+p_{N_1-1}^{(1)}wQ_{i1}(w)\\ &\qquad\qquad +p_{N_1-2}^{(1)}wQ_{i2}(w)+\cdots\cdots+p_0^{(1)}wQ_{iN_1}(w),\end{aligned}\right. \tag{30}$$

$$\left\{\begin{aligned}&Q_{iN_1}(w)-p_{iN_1}^{(0)}=p_{N_1}^{(1)}wQ_{i0}(w)+p_{N_1}^{(1)}wQ_{i1}(w)+p_{N_1-1}^{(1)}wQ_{i2}(w)+\cdots\cdots\\ &\vdots\qquad\qquad +p_1^{(1)}wQ_{1N_1}(w)+p_0^{(2)}wQ_{i,N_1+1}(w),\\ &Q_{i,N_2-1}(w)-p_{i,N_2-1}^{(0)}=p_{N_2-1}^{(1)}wQ_{i0}(w)+p_{N_2-1}^{(1)}wQ_{i1}(w)+p_{N_2-2}^{(1)}\\ &\qquad\qquad wQ_{i2}(w)+\cdots\cdots+p_{N_2-N_1}^{(1)}wQ_{i,N_1}(w)+p_{N_2-N_1-1}^{(2)}\\ &\qquad\qquad wQ_{i,N_1+1}(w)+\cdots\cdots+p_0^{(2)}wQ_{i,N_2}(w),\end{aligned}\right. \tag{31}$$

and so on.

From the above relations we see that $Q_{in}(w)$ $(n=1,\ 2,\ \cdots\cdots)$ are

able to write down in linear forms of $Q_{i0}(w)$ and further the relation (30) is similar to the corresponding one in the queuing system M/G/1 (e. g. [1]):

$$\Omega_0(z, w)=\{z^{i+1}+w\phi_1[\lambda(1-z)](z-1)Q_{i0}(w)\}/\{z-w\phi_1[\lambda(1-z)]\}$$

In the other words, for $j=0, 1, \cdots\cdots, N_1$ we have

coefficient of z^j in $\Omega(z, w)$=coefficient of z^j in $\Omega_0(z, w)$.

Now we shall proceed to obtain the expression of $Q_{i0}(w)$. In the first we shall treat the case $l=1$ and denote $\Omega_0(z, w)$ by $\Omega_1(z, w)$ in this case.

Let us define the following functions

$$A_1(z, w)=w(z-1)\phi_1[\lambda(1-z)],$$

$$B_1(z)=z^{i+1},$$

$$A_N^{(1)}(z, w)=\sigma_0^{N_1}\left[\frac{A_1(z, w)}{z-w\phi_1[\lambda(1-z)]}\right]$$

and

$$B_N^{(1)}(z, w)=\sigma_0^{N_1}\left[\frac{B_1(z)}{z-w\phi_1[\lambda(1-z)]}\right],$$

Then we obtain that

$$P_1(z, w)=A_N^{(1)}(z, w)Q_{i0}(w)+B_N^{(1)}(z, w). \tag{32}$$

In the case $l=1$, we have

$$\Omega_1(z, w)=\{z^{i+1}+w\phi_1[\lambda(1-z)](z-1)Q_{i0}(w)+wP_1(z, w)(\phi_1[\lambda(1-z)]-\phi_2[\lambda(1-z)])\}/\{z-w\phi_2[1-z)]\}. \tag{33}$$

We now define the following functions

$$A_2(z, w)=A_1(z, w)+wA_N^{(1)}(z, w)(\phi_1[\lambda(1-z)]-\phi_2[\lambda(1-z)]),$$

$$B_2(z, w)=B_1(z)+wB_N^{(1)}(z, w)(\phi_1[\lambda(1-z)]-\phi_2[1-z)]),$$

$$A_N^{(2)}(z, w)=\sigma_{N_1+1}^{N_2}\left[\frac{A_2(z, w)}{z-w\phi_2[\lambda(1-z)]}\right],$$

and

$$B_N^{(2)}(z, w)=\sigma_{N_1+1}^{N_2}\left[\frac{B_2(z, w)}{z-w\phi_2[\lambda(1-z)]}\right].$$

Then we have

$$P_2(z, w)=A_N^{(2)}(z, w)Q_{i0}(w)+B_N^{(2)}(z, w). \tag{34}$$

Along the same lines as the above procedure, we define the following functions

$$A_s(z, w)=A_1(z, w)+w\sum_{j=1}^{s}A_N^{(j)}(z, w)(\phi_j[\lambda(1-z)]-\phi_s[\lambda(1-z)]),$$

$$B_s(z, w)=B_1(z)+w\sum_{j=1}^{s}B_N^{(j)}(z, w)(\phi_j[\lambda(1-z)]-\phi_s[\lambda(1-z)]),$$

$$A_N^{(s)}(z, w)=\sigma_{N_{s-1}+1}^{N_s}[A_s(z, w)/\{z-w\psi_s[\lambda(1-z)]\}],$$

and $$B_N^{(s)}(z, w)=\sigma_{N_{s-1}+1}^{N_s}[B_s(z, w)/\{z-w\psi_s[\lambda(1-z)]\}]$$

$$(s=3, 4, \cdots\cdots, l+1).$$

Then we have

$$(34)^* \quad P_s(z, w)=A_N^{(s)}(z, w)Q_{i0}(w)+B_N^{(s)}(z, w) \quad (s=3, 4, \cdots\cdots, l+1).$$

By (32), (34) and (34)* we can rewrite (29) in the following form.

$$(35) \quad \Omega(z, w)=\{A_{l+1}(z, w)Q_{i0}(w)+B_{l+1}(z, w)\}/\{z-w\psi_{l+1}[\lambda(1-z)]\}.$$

Hence if $A_{l+1}(g(w), w)\neq 0$ then we get

$$(36) \quad Q_{i0}(w)=-B_{l+1}(g(w), w)/A_{l+1}\, g(w(w), w).$$

Finally we obtain (16) if we insert (36) in (35).

5. THE LIMITING BEHAVIOR OF $\{\xi_n\}$

Now we consider the relation between the limiting behavior of $\{\xi_n\}$ and the result which has been obtained in [5].

Theorem 2. For $|z|\leqq 1$ we define the following functions:

$$(37) \quad Q_1(z)=(z-1)\psi_1[\lambda(1-z)],$$

$$(38) \quad Q_s(z)=Q_1(z)+\sum_{j=1}^{s} Q_N^{(j)}(z)(\psi_j[\lambda(1-z)]-\psi_s[\lambda(1-z])$$

$$(s=2, 3, \cdots\cdots, l+1),$$

$$(39) \quad Q_N^{(s)}(z)=\sigma_{N_{s-1}+1}^{N_s}[Q_s(z)/\{z-\psi_s[\lambda(1-z)]\}],$$

and $$Q_N^{(1)}(z)=\sigma_0^{N_1}[Q_1(z)/\{z-\psi_1[\lambda(1-z)]\}].$$

If $\rho_{l+1}<1$, the limiting distribution

$$\lim_{n\to\infty} P\{\xi_n=k\}=\pi_k \qquad (k=0, 1, 2, \cdots\cdots)$$

exists and is independent of the initial distribution and and for $|z|\leqq 1$ we have

$$(40) \quad \sum_{k=0}^{\infty}\pi_k z^k=\frac{1-\rho_{l+1}}{1+\sum_{i=1}^{l} Q_i^*(\rho_i-\rho_{l+1})}\cdot\frac{Q_{l+1}(z)}{z-\psi_{l+1}[\lambda(1-z)]}$$

$$=\pi_0\frac{Q_{l+1}(z)}{z-\psi_{l+1}[\lambda(1-z)]}.$$

Proof. Clearly we observe in § 4 that the sequence $\{\xi_n\}$ is an irreducible and aperiodic Markov chain and also we have $\lim_{n\to\infty} P\{\xi_n=k\}=\pi_k$ always exists and is independent of the initial distribution. Furthermore, either every $\pi_k>0$ and $\{\pi_k\}$ is a probability distribution or every $\pi_k=0$. Using the Abel's theorem we have

$$\sum_{k=0}^{\infty}\pi_k z^k=\lim_{w\to 1}(1-w)\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}p_{ik}^{(n)}w^n z^k \qquad \text{for any } i,$$

and in order to calculate the right side of the equation we have to use (16) are lemma 2.

Now we define the following functions:

$$Q_1(z)=(z-1)\psi_1[\lambda(1-z)],$$

and

$$Q_N^{(1)}(z)=\sigma_0^{N_1}[Q_1(z)/\{z-\psi_1[\lambda(1-z)]\}].$$

Then we have

$$Q_N^{(1)}(z)=\lim_{w\to 1}A_N^{(1)}(z,\ w)$$

and

$$Q_1^*=\lim_{z\to 1}Q_N^{(1)}(z).$$

The avoid a complicate calculation we shall consider the case $l=1$. The case $l\geqq 2$ will be treated in the same way. Then we have

$$\Omega(z,\ w)=\{z^{i+1}+wB_N^{(1)}(z,\ w)(\psi_1[\lambda(1-z)]-\psi_2[\lambda(1-z)])+(A_1(z,\ w)$$
$$+wA_N^{(1)}(z,\ w))(\psi_1[\lambda(1-z)]-\psi_2[\lambda(1-z)])Q_{i0}(w)\}/\{z-w\psi_2[\lambda(1-z)]\}$$

hence

$$\lim_{w\to 1}(1-w)\Omega(z,\ w)=\frac{\psi_1[\lambda(1-z)](z-1)+Q_N^{(1)}(z)(\psi_1[\lambda(1-z)]-\psi_2[\lambda(1-z)])}{z-\psi_2[\lambda(1-z)]}.$$

$$\lim_{w\to 1}(1-w)Q_{i0}(w)=\frac{Q_2(w)}{z-\psi_2[\lambda(1-z)]}\cdot\lim_{w\to 1}(1-w)Q_{i0}(w),$$

where $Q_2(z)=Q_1(z)+Q_N^{(1)}(z)(\psi_1[\lambda(1-z)]-\psi_2[\lambda(1-z)])$.

Using lemma 2, we have that if $\rho_2<1$

$$g(w)\to 1\ (w\to 1) \text{ and } g'(1)=\frac{1}{1-\rho_2}.$$

Noting that

$$Q_{i0}(w)=\{-[g(w)]^{i+1}-(w\psi_1[\lambda(1-g(w))]-g(w))\cdot B_N^{(1)}(g(w),\ w)\}/$$
$$\{w(g(w)-1)\psi_1[\lambda(1-g(w))]+(w\psi_1[\lambda(1-g(w))]-g(w))A_N^{(1)}(g(w),\ w)\},$$

we have

$$\lim_{w\to 1}(1-w)Q_{i0}(w)=\frac{1}{\dfrac{1}{1-\rho_2}+\left(1+\dfrac{\rho_1}{1-\rho_2}-\dfrac{1}{1-\rho_2}\right)Q_1^*}$$
$$=\frac{1-\rho_2}{1+(\rho_1-\rho_2)Q_1^*}=\pi_0.$$

We shall prove now a more general theorem than theorem 1.

Theorem 3. Let us define

(41) $$U_n(s,\ z)=E\{e^{-s(\tau_{n+1}'-\tau_n')}z^{\xi_{n+1}}\} \qquad (n\geqq 0) \text{ for } R(s)\geqq 0$$
and $|z|\leqq 1$. Then if $A_{l+1}(g(w),\ w)\neq 0$ we have that

(42) $$\begin{aligned} wzU(s,\ z,\ w)&=wz\sum_{n=0}^{\infty}U_n(s,\ z)w^n \\ &=[A_{l+1}(g(w),\ w)[(z-w\phi_{l+1}[\lambda(1-z)])B_{l+1}(s,\ z,\ w) \\ &\quad +w\phi_{l+1}[s+\lambda(1-z)]B_{l+1}(z,w)]-B_{l+1}(g(w),\ w) \\ &\quad [z-w\phi_{l+1}[\lambda(1-z)])\{wz\phi_1[s+\lambda(1-z)](\phi(s)-1) \\ &\quad +A_{l+1}(s,\ z,\ w)\}+w\phi_{l+1}[s+\lambda(1-z)]A_{l+1}(z,\ w)]] \\ &\quad /(z-w\phi_{l+1}[\lambda(1-z)])A_{l+1}(g(w),\ w)-z^{i+1} \end{aligned}$$
for $R(s)\geqq 0, |z|\leqq 1$ and $|w|<1$, where
$$\phi(s)=\int_0^{\infty}e^{-sx}dF(x),$$
$$\begin{aligned} A_{l+1}(s,\ z,\ w)&=w\phi_1[s+\lambda(1-z)](z-1)+\sum_{j=1}^{l}wA_N^{(j)}(z,\ w)(\phi_j[s+\lambda(1-z)] \\ &\quad -\phi_{l+1}[s+\lambda(1-z)]), \end{aligned}$$
and $$\begin{aligned} B_{l+1}(s,\ z,\ w)&=B_N^{(1)}(z)+\sum_{j=1}^{l}wB_N^{(j)}(z,\ w)(\phi_j[s+\lambda(1-z)] \\ &\quad -\phi_{l+1}[s+\lambda(1-z)]). \end{aligned}$$

Proof. We can write
$$\tau_{n+1}'=\tau_n'+\chi_{n+1}+\varepsilon_n\delta_n \qquad (n=1,\ 2,\ \cdots\cdots)$$
where $\varepsilon_n=\begin{cases}1 & \text{if} \quad \xi_n=0\\ 0 & \text{if} \quad \xi_n>0\end{cases}$

and $\{\chi_n\}$ $(n=1,\ 2,\cdots\cdots)$ and $\{\delta_n\}$ $(n=0,\ 1,\ 2,\cdots\cdots)$ are independent sequences of mutually independent random variables and $\{\delta_n\}$ are random variables with distribution function $P\{\delta_n\leqq x\}=F(x)$ given by (1). Using (8) for $n\geqq 0$ we get
$$\begin{aligned} U_n(s,\ z)&=E\{e^{-s(\tau_{n+1}'-\tau_n)}z^{\xi_{n+1}}\} \\ &=E\{e^{-s(\chi_{n+1}+\varepsilon_n\delta_n)}z^{\xi_{n+1}}\} \\ &=P\{\xi_n=0|\xi_0=i\}\sum_{k=0}^{\infty}\int_0^{\infty}\int_0^{\infty}e^{-s(x+\delta)}z^ke^{-\lambda x}\frac{(\lambda x)^k}{k\ !}dH_1(x)\lambda e^{-\delta\lambda}d\delta \\ &\quad +P\{\xi_n=1|\xi_0=i\}\sum_{k=0}^{\infty}\int_0^{\infty}e^{-sx}z^ke^{-\lambda x}\frac{(\lambda x)^k}{k\ !}dH_1(x) \\ &\quad +P\{\xi_n=N_1|\xi_0=i\}\sum_{k=0}^{\infty}\int_0^{\infty}e^{-sx}z^{N_1-1+k}e^{-\lambda x}\frac{(\lambda x)^k}{k\ !}dH_1(x) \\ &\quad +P\{\xi_n=N_1+1|\xi_0=i\}\sum_{k=0}^{\infty}\int_0^{\infty}e^{-sx}z^{N_1+k}e^{-\lambda x}\frac{(\lambda x)^k}{k\ !}dH_2(x) \\ &\quad \vdots \end{aligned}$$

$$
\begin{aligned}
&=\phi_1[s+\lambda(1-z)]\{p_{i_0}^{(n)}\phi(s)+p_{i_1}^{(n)}+p_{i_2}^{(n)}z+\cdots\cdots+p_{iN_1}^{(n)}z^{N_1-1}\\
&\quad+\phi_2[s+\lambda(1-z)]\{p_{i,N_1+1}^{(n)}z^{N_1}+\cdots\cdots+p_{i,N_2}^{(n)}z^{N_2-1}\}\\
&\quad\vdots\\
&\quad+\phi_l[s+\lambda(1-z)]\{p_{i,N_{l-1}+1}^{(n)}z^{N_{l-1}}+\cdots\cdots+p_{i,N_l}^{(n)}z^{N_l-1}\}\\
&\quad+\phi_{l+1}[s+\lambda(1-z)]\{p_{i,N_l+1}^{(n)}z^{N_l}+\cdots\cdots\}\\
&=\frac{1}{z}\{\phi_1[s+\lambda(1-z)]p_{i_0}^{(n)}(\phi(s)z-1)\}\\
&\quad+\phi_1[s+\lambda(1-z)]-\phi_{l+1}[s+\lambda(1-z)](p_{i_0}^{(n)}+p_{i_1}^{(n)}z+\cdots\cdots+p_{iN_1}^{(n)}z^{N_1})\\
&\quad\vdots\\
&\quad+\phi_l[s+\lambda(1-z)]-\phi_{l+1}[s+\lambda(1-z)](p_{i,N_{l-1}+1}^{(n)}z^{N_{l-1}+1}\cdots\cdots\\
&\quad+p_{i,N_l}^{(n)}z^{N_l})+\phi_{l+1}[s+\lambda(1-z)]U_i^{(n)}(z)\}
\end{aligned}
$$

where $U_i^{(n)}(z)$ is defined by (20). If we define

$$U(s,\ z,\ w)=\sum_{n=0}^{\infty}U_n(s,\ z)w^n \qquad \text{for } |w|<1,$$

we have that

$$
\begin{aligned}
zU(s,\ z,\ w)=&\phi_1[s+\lambda(1-z)](\phi(s)z-1)Q_{i0}(w)\\
&+(\phi_1[s+\lambda(1-z)]-\phi_{l+1}[s+\lambda(1-z)])P_1(z,\ w)\\
&\vdots\\
&+(\phi_l[s+\lambda(1-z)]-\phi_{l+1}[s+\lambda(1-z)])P_l(z,\ w)\\
&+\phi_{l+1}[s+\lambda(1-z)]\Omega(z,\ w)
\end{aligned}
$$

where $Q_{i0}(w)$, $P_1(z,\ w)$, $\cdots\cdots$, $P_l(z,\ w)$ and $\Omega(z,\ w)$ are defined by (26), (27), (28) and (25) respectively. Using (35), we get

$$
\begin{aligned}
zU(s,\ z,\ w)=&\{\phi_1(s+\lambda(1-z)](\phi(s)z-1)+A_N^{(1)}(z,\ w)(\phi_1[s+\lambda(1-z)]\\
&-\phi_{l+1}[s+\lambda(1-z)])+\cdots\cdots+A_N^{(l)}(z,\ w)(\phi_l[s+\lambda(1-z)]\\
&-\phi_{l+1}[s+\lambda(1-z)]\}Q_{i0}(w)+\{B_N^{(1)}(z)(\phi_1[s+\lambda(1-z)])\\
&-\phi_{l+1}[s+\lambda(1-z)])+\cdots\cdots+B_N^{(l)}(z,\ w)(\phi_l[s+\lambda(1-z)]\\
&-\phi_{l+1}[s+\lambda(1-z)])\}+\phi_{l+1}[s+\lambda(1-z)]\Omega(z,\ w).
\end{aligned}
$$

Now if we define

$$
\begin{aligned}
A_{l+1}(s,\ z,\ w)=&w\phi_1[s+\lambda(1-z)](z-1)+\sum_{j=1}^{l}wA_N^{(j)}(z,\ w)(\phi_j[s+\lambda(1-z)]\\
&-\phi_{l+1}[s+\lambda(1-z)]),
\end{aligned}
$$

and

$$
\begin{aligned}
B_{l+1}(s,\ z,\ w)=&B_N^{(1)}(z)+\sum_{j=1}^{l}wB_N^{(j)}(z,\ w)(\phi_j[s+\lambda(1-z)]\\
&-\phi_{l+1}[s+\lambda(1-z)]),
\end{aligned}
$$

then we get

$$
\begin{aligned}
wzU(s,\ z,\ w)=&\{w\phi_1[s+\lambda(1-z)](\phi(s)z-1)-w\phi_1[s+\lambda(1-z)](z-1)\\
&+A_{l+1}(s,\ z,\ w)\}Q_{i0}(w)+\{B_{l+1}(s,\ z,\ w)-B_N^{(1)}(z)\}\\
&+w\phi_{l+1}[s+\lambda(1-z)]\Omega(z,\ w)
\end{aligned}
$$

$$
\begin{aligned}
&=\{wz\phi_1[s+\lambda(1-z(](\phi(s)-1)+A_{l+1}(s,\ z,\ w)\}Q_{i0}(w)\\
&\quad+B_{l+1}(s,\ z,\ w)+w\phi_{l+1}[s+\lambda(1-z)]\Omega(z,\ w)-z^{l+1}\\
&=[A_{l+1}(g(w),\ w)[(z-w\phi_{l+1}[\lambda(1-z)])B_{l+1}(s,\ z,\ w)\\
&\quad+w\phi_{l+1}[s+\lambda(1-z)]B_{l+1}(z,\ w)]-B_{l+1}(g(w),\ w)\\
&\quad[(z-w\phi_{l+1}[\lambda(1-z)])\{wz\phi_1[s+\lambda(1-z)](\phi(s)-1)\\
&\quad+A_{l+1}(s,\ z,\ w)\}+w\phi_{l+1}[s+\lambda(1-z)]A_{l+1}(z,\ w)]]/\\
&\quad(z-w\phi_{l+1}[\lambda(1-z)])A_{l+1}(g(w),\ w)-z^{l+1}
\end{aligned}
$$

provided $A_{l+1}(g(w),\ w)\neq 0$.

Remark 1. If we put $s=0$ in the above theorem, we get theorem 1.

6. THE DISTRIBUTION OF THE BUSY PERIOD

Let us denote by $G_{nk}(x)$ the probability that the busy period is at most of length x, consists of at least n services and at the end of the nth service, k customers are present in the queue. Denote by $G_n(x)$ the probability that a busy period consists of n services and its length is at most x. Then we have that

$$G_n(x)=G_{n0}(x). \tag{43}$$

Write

$$\Gamma_{nk}(s)=\int_0^\infty e^{-sx}dG_{nk}(x) \qquad \text{if } R(s)\geqq 0 \tag{44}$$

and

$$\Gamma_n(s)=\int_0^\infty e^{-sx}dG_n(x) \qquad \text{if } R(s)\geqq 0, \tag{45}$$

then by (43)

$$\Gamma_n(s)=\Gamma_{n0}(s). \tag{46}$$

Theorem 4. For $R(s)\geqq 0$, $|z|\leqq 1$ and $|w|<1$, we define the following functions:

$$C_1(s,\ z,\ w)=-w\phi_1[s+\lambda(1-z)], \tag{47}$$

$$D_1(s,\ z,\ w)=-zC_1(s,\ z,\ w), \tag{48}$$

$$\begin{aligned}C_i(s,\ z,\ w)=C_1(s,\ z,\ w)+w\sum_{j=1}^{i}C_N^{(j)}(s,\ z,\ w)(\phi_j[s+\lambda(1-z)]\\ -\phi_i[s+\lambda(1-z)]). \qquad (i=2,\ 3,\cdots\cdots,\ l+1),\end{aligned} \tag{49}$$

$$\begin{aligned}D_i(s,\ z,\ w)=D_1(s,\ z,\ w)+w\sum_{i=1}^{i}D_N^{(j)}(s,\ z,\ w)(\phi_j[s+\lambda(1-z)]\\ -\phi_i[s+\lambda(1-z)]) \qquad (i=2,\ 3,\cdots\cdots,\ l+1),\end{aligned} \tag{50}$$

$$C_N^{(i)}(s,\ z,\ w)=\sigma_{N_{i-1}+1}^{N_i}[C_i(s,\ z,\ w)/\{z-w\phi_i[s+\lambda(1-z)]\}] \tag{51}$$

$$(i=2,\ 3,\cdots\cdots,\ l),$$

$$C_N^{(1)}(s,\ z,\ w)=\sigma_0^{N_1}[C_1(s,\ z,\ w)/\{z-w\psi_1[s+\lambda(1-z)]\}],$$

$$D_N^{(i)}(s,\ z,\ w)=\sigma_{N_{i-1}+1}^{N_i}[D_i(s,\ z,\ w)/\{z-w\psi_i[s+\lambda(1-z)]\}] \tag{52}$$

$$(i=2,\ 3,\cdots\cdots,\ l),$$

and $$D_N^{(1)}(s,\ z,\ w)=\sigma_0^{N_1}[D_1(s,\ z,\ w)/\{z-w\psi_1[s+\lambda(1-z)]\}].$$

Let $\gamma(s,\ w)$ be the root in z of the equation

$$z=w\psi_{l+1}[s+\lambda(1-z)]$$

in the unit circle $|z|<1$, and assume $C_{l+1}(s,\ \gamma(s,\ w),\ w)\neq 0$ for $|w|\leqq 1$. Then we have that for $R(s)\geqq 0$ and $|w|\leqq 1$

$$\sum_{n=1}^{\infty}\Gamma_n(s)w=-\frac{D_{l+1}(s,\ \gamma(s,\ w),\ w)}{C_{l+1}(s,\ \gamma(s,\ w),\ w)} \tag{53}$$

Hence we have for $R(s)\geqq 0$

$$\Gamma(s)=\sum_{n=1}^{\infty}\Gamma_n(s)=-\frac{D_{l+1}(s,\ \gamma(s),\ 1)}{C_{l+1}(s,\ \gamma(s),\ 1)}. \tag{54}$$

Proof. By the theorem of total probability, we can write that

$$G_{1k}(x)=\int_0^x e^{-\lambda y}\frac{(\lambda y)^k}{k\,!}dH_1(y) \qquad (k\geqq 0) \tag{55}$$

and

$$\begin{aligned} G_{nk}(x)=&\int_0^x\sum_{j=1}^{N_1}G_{n-1,j}(x-y)e^{-\lambda y}\frac{(\lambda y)^{k-j+1}}{(k-j+1)!}dH_1(y)\\ &+\int_0^x\sum_{j=N_1+1}^{N_2}G_{n-1,j}(x-y)e^{-\lambda y}\frac{(\lambda y)^{k-j+1}}{(k-j+1)!}dH_2(y)\\ &+\cdots\cdots\\ &+\int_0^x\sum_{j=N_{s-1}+1}^{k+1}G_{n-1,j}(x-y)e^{-\lambda y}\frac{(\lambda y)^{k-j+1}}{(k-j+1)!}dH_s(y) \end{aligned} \tag{56}$$

if $n\geqq 2$ and $N_{s-1}\leqq k\leqq N_s-1$.

Taking Laplace-Stieltjes transforms of (55) and (56), we get

$$\Gamma_{1k}(s)=\int_0^{\infty}e^{-(\lambda+s)y}\frac{(\lambda y)^k}{k!}dH_1(y) \qquad \text{for } k\geqq 0 \tag{57}$$

and if $n\geqq 2$ and $N_{i-1}\leqq k\leqq N_i-1$

$$\begin{aligned} \Gamma_{nk}(s)=&\sum_{j=1}^{N_1}\Gamma_{n-1,j}(s)\int_0^{\infty}e^{-(\lambda+s)y}\frac{(\lambda y)^{k-j+1}}{(k-j+1)!}dH_1(y)\\ &+\sum_{j=N_1+1}^{N_2}\Gamma_{n-1,j}(s)\int_0^{\infty}e^{-(\lambda+s)y}\frac{(\lambda y)^{k-j+1}}{(k-j+1)!}dH_2(y)\\ &+\cdots\cdots\cdots\cdots \end{aligned} \tag{58}$$

$$+\sum_{j=N_{i-1}+1}^{k+1}\Gamma_{n-1,j}(s)\int_0^\infty e^{-\lambda(1+s)y}\frac{(\lambda y)^{k-j+1}}{(k-j+1)!}dH_i(y).$$

Let us introduce the generating function,

$$V_n(s,\ z)=\sum_{k=0}^{\infty}\Gamma_{nk}(s)z^k, \tag{59}$$

which is convergent if $R(s)\geqq 0$ and $|z|\leqq 1$.

Then we have from (57) and (59) that

$$V_1(s,\ z)=\psi_1[s+\lambda(1-z)], \tag{60}$$

and

$$\begin{aligned}V_n(s,\ z)=&\psi_1[s+\lambda(1-z)]\{\Gamma_{n-1,1}(s)+z\Gamma_{n-1,2}(s)+\cdots\cdots\\&+z^{N_1-1}\Gamma_{n-1,N_1}(s)\}+\psi_2[s+\lambda(1-z)]\{z^{N_1}\Gamma_{n-1,N_1+1}(s)+\cdots\cdots\\&+z^{N_2-1}\Gamma_{n-1,N_2}(s)\}\\&+\cdots\cdots\\&+\psi_{l+1}[s+\lambda(1-z)]\{z^{N_l}\Gamma_{n-1,N_l+1}(s)+\cdots\cdots\}\\&\qquad\qquad(n=2,\ 3,\ \cdots\cdots).\end{aligned} \tag{61}$$

Also we can rewrite (61) in the following form,

$$\begin{aligned}zV_n(s,\ z)=&(\psi_1[s+\lambda(1-z)]-\psi_{l+1}[s+\lambda(1-z)])(z\Gamma_{n-1,1}(s)+\cdots\cdots\\&+z^{N_1}\Gamma_{n-1,N_1}(s))+(\psi_2[s+\lambda(1-z)]-\psi_{l+1}[s+\lambda(1-z)])\\&(z^{N_1+1}\Gamma_{n-1,N_1+1}(s)+\cdots\cdots+z^{N_2}\Gamma_{n-1,N_2}(s))\\&+\cdots\cdots\\&+(\psi_l[s+\lambda(1-z)]-\psi_{l+1}[s+\lambda(1-z)])(z^{N_{l-1}+1}\Gamma_{n-1,N_{l-1}+1}(s)\\&+\cdots\cdots+z^{N_l}\Gamma_{n-1,N_l}(s))+\psi_{l+1}[s+\lambda(1-z)](V_{n-1}(s,\ z)\\&-\Gamma_{n-1,0}(s))\qquad(n=2,\ 3,\ \cdots\cdots).\end{aligned} \tag{62}$$

Further we introduce the generating functions

$$R_k(s,\ w)=\sum_{n=1}^{\infty}\Gamma_{n.k}(s)w^n\qquad\text{for } |w|<1\qquad(k=0,\ 1,\ 2,\cdots\cdots), \tag{63}$$

and we define for $|z|\leqq 1$ and $|w|<1$,

$$T_1(s,\ z,\ w)=\sum_{k=0}^{N_1}R_k(s,\ w)z^k, \tag{64}$$

and

$$T_i(s,\ z,\ w)=\sum_{k=N_{i-1}+1}^{N_i}R_k(s,\ w)z^k\qquad(i=2,\ 3,\ \cdots\cdots,\ l+1). \tag{65}$$

Furthermore if we introduce the generating function

$$T(s,\ z,\ w)=\sum_{n=1}^{\infty}V_n(s,\ z)w^n\qquad\text{for } R(s)\geqq 0,\ |z|\leqq 1 \text{ and } |w|<1, \tag{66}$$

then by (60)~(62) we obtain that

$$
\begin{aligned}
(67)\qquad T(s,\ z,\ w)=\{&w\phi_1[s+\lambda(1-z)](z-R_0(s,\ w))\\
&+wT_1(s,\ z,\ w)(\phi_1[s+\lambda(1-z)]-\phi_{l+1}[s+\lambda(1-z)])\\
&+\cdots\cdots+wT_l(s,\ z,\ w)(\phi_l[s+\lambda(1-z)]\\
&-\phi_{l+1}[s+\lambda(1-z)])\}/\{z-w\phi_{l+1}[s+\lambda(1-z)]\}.
\end{aligned}
$$

The left side of (67) is a regular function of z if $|z|\leqq 1$, $R(s)\geqq 0$ and $|w|<1$. In this domain, the denominator of the right side has exactly one root $z=\gamma(s,\ w)$. This must also be the root of the numerator. Hence to determine $R_0\ (s,\ w)$ we have to observe the following relations between $R_0(s,\ w), T_1(s,\ z,\ w), \cdots\cdots$, and $T_l(s,\ z,\ w)$. By (57) and (58) we have

$$
(68)\quad \left\{
\begin{aligned}
&\frac{1}{w}wR_0(s,\ w)-\Gamma_{10}(s)=\Gamma_{10}(s)R_1(s,\ w),\\
&\frac{1}{w}R_1(s,\ w)-\Gamma_{11}(s)=\Gamma_{10}(s)R_2(s,\ w)+\Gamma_{11}(s)R_1(s,\ w),\\
&\frac{1}{w}R_2(s,\ w)-\Gamma_{12}(s)=\Gamma_{10}(s)R_3(s,\ w)+\Gamma_{11}(s)R_2(s,\ w)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad +\Gamma_{12}(s)R_1(s,\ w),\\
&\vdots\\
&\frac{1}{w}R_{N_1-1}(s,\ w)-\Gamma_{1,N_1-1}(s)=\Gamma_{10}(s)R_{N_1}(s,\ w)+\Gamma_{11}(s)R_{N_1-1}(s,\ w)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad +\cdots\cdots+\Gamma_{1,N_1-1}(s)R_1(s,\ w),
\end{aligned}
\right.
$$

$$
(69)\quad \left\{
\begin{aligned}
&\frac{1}{w}wR_{N_1}(s,\ w)-\Gamma_{1,N_1}(s)=\Gamma^*_{10}(s)R_{N_1+1}(s,\ w)+\Gamma_{11}(s)R_{N_1}(s,\ w)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad +\cdots\cdots+\Gamma_{1,N_1}(s)R_1(s,\ w),\\
&\frac{1}{w}R_{N_1+1}(s,w)-\Gamma_{1,N_1+1}(s)=\Gamma^*_{10}(s)R_{N_1+2}(s,w)+\Gamma_{11}(s)R_{N_1+1}(s,w)\\
&\vdots\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad +\cdots\cdots+\Gamma_{1,N_1+1}(s)R_1(s,\ w),
\end{aligned}
\right.
$$

where $\Gamma^*_{10}(s)=\int_0^\infty e^{-(\lambda+s)y}dH_2(y)$.

From the above relations we see that $R_k(s,\ w)$ $(k=1,\ 2,\ \cdots\cdots)$ are able to write down in linear forms of $R_0\ (s,\ w)$ and further the relation (68) is the similar one which corresponds to the well-known queue $M/G/1$, i. e. the corresponding one $T_0^*(s,\ z,\ w)$ to (67) in the queue $M/G/1$ is

$$T_0^*(s,\ z,\ w)=w\phi_1[s+\lambda(1-z)](z-R_0(s,\ w))/\{z-w\phi_1[s+\lambda(1-z)]\},$$

and for $j=0,\ 1,\cdots\cdots,\ N$ we have

coefficient of z^j in $T(s,\ z,\ w)$=coefficient of z^j in $T_0^*\ (s,\ z,\ w)$.

Let the corresponding one to (67) in the case $l=1$ be $T_1^*\ (s,\ z,\ w)$.

Let us define the following functions:

$$C_1(s,\ z,\ w)=-w\phi_1[s+\lambda(1-z)],$$
$$D_1(s,\ z,\ w)=wz\phi_1[s+\lambda(1-z)]=-zC_1(s,\ z,\ w),$$
$$C_N^{(1)}(s,\ z,\ w)=\sigma_0^{N_1}[C_1(s,\ z,\ w)/\{z-w\phi_1[s+\lambda(1-z)]\}],$$

and
$$D_N^{(1)}(s,\ z,\ w)=\sigma_0^{N_1}[D_1(s,\ z,\ w)/\{z-w\phi_1[s+\lambda(1-z)]\}].$$

Then we obtain that
$$T_1(s,\ z,\ w)=C_N^{(1)}(s,\ z,\ w)R_0(s,\ w)+D_N^{(1)}(s,\ z,\ w).$$

In the case $l=1$ we have
$$T_1^*(s,z,w)=\{w\phi_1[s+\lambda(1-z)](z-R_0(s,w))+wT_1(s,z,w)(\phi_1[s+\lambda(1-z)]-\phi_2[s+\lambda(1-z)])\}/\{z-w\phi_2[s+\lambda(1-z)]\}.$$

Then we define the following functions
$$C_2(s,\ z,\ w)=C_1(s,\ z,\ w)+wC_N^{(1)}(s,\ z,\ w)(\phi_1[s+\lambda(1-z)]-\phi_2[s+\lambda(1-z)]),$$
$$D_2(s,\ z,\ w)=D_1(s,\ z,\ w)+wD_N^{(1)}(s,\ z,\ w)(\phi_1[s+\lambda(1-z)]-\phi_2[s+\lambda(1-z)]),$$
$$C_N^{(2)}(s,z,w)=\sigma_{N_1+1}^{N_2}[C_2(s,\ z,\ w)/\{z-w\phi_2[s+\lambda(1-z)]\}],$$

and
$$D_N^{(2)}(s,z,w)=\sigma_{N_1+1}^{N_2}[D_2(s,\ z,\ w)/\{z-w\phi_2[s+\lambda(1-z)]\}].$$

Then we have
$$T_2(s,\ z,\ w)=C_N^{(2)}(s,\ z,\ w)R_0(s,\ w)+D_N^{(2)}(s,\ z,\ w).$$

Along the above line let us define the following functions:
$$C_i(s,\ z,\ w)=C_1(s,\ z,\ w)+w\sum_{j=1}^{i}C_N^{(j)}(s,\ z,\ w)(\phi_j[s+\lambda(1-z)]-\phi_i[s+\lambda(1-z)]),$$
$$D_i(s,\ z,\ w)=D_1(s,\ z,\ w)+w\sum_{j=1}^{i}D_N^{(j)}(s,\ z,\ w)(\phi_j[s+\lambda(1-z)]-\phi_i[s+\lambda(1-z)]),$$
$$C_N^{(i)}(s,\ z,\ w)=\sigma_{N_{i-1}+1}^{N_i}[C_i(s,\ z,\ w)/\{z-w\phi_i[s+\lambda(1-z)]\}],$$

and
$$D_N^{(i)}(s,\ z,\ w)=\sigma_{N_{i-1}+1}^{N_i}[D_i(s,\ z,\ w)/\{z-w\phi_i[s+\lambda(1-z)]\}]\quad(i=3,\ 4,\cdots\cdots,\ l+1).$$

Then we get

(70) $$T_i(s,\ z,\ w)=C_N^{(i)}(s,\ z,\ w)R_0(s,\ w)+D_N^{(i)}(s,\ z,\ w)\quad(i=1,\ 2,\cdots\cdots,\ l+1).$$

Consequently if we insert (70) in (67) we obtain

(71) $$T(s,\ z,\ w)=\{C_{l+1}(s,\ z,\ w)R_0(s,\ w)+D_{l+1}(s,\ z,\ w)\}/\{z-w\phi_{l+1}[s+\lambda(1-z)]\}.$$

Then if $C_{l+1}(s,\ \gamma(s,\ w),\ w)\neq 0$ for $|w|<1$ we have

$$R_0(s, w) = -\frac{D_{l+1}(s, \gamma(s, w), w)}{C_{l+1}(s, \gamma(s, w), w)} \tag{72}$$

From (46) we obtain (53). If further $C_{l+1}(s, \gamma(s, w), w) \neq 0$ for $|w|=1$, then the left side of (53) is a regular function of w for $R(s) \geqq 0$ and $|w| \leqq 1$ and then by lemma 1 we can show (72) to be true for $|w| \leqq 1$. Putting $w=1$ in (53) and defining $\Gamma(s) = \sum_{n=1}^{\infty} \Gamma_n(s)$, we get (54).

Throughout the following statements we shall keep the imposed assumptions in theorem 4.

Let $G(x)$ denote the distribution function of the length of the busy period.

Corollary 1. If $\rho_{l+1} \leqq 1$ we have

$$\lim_{x \to \infty} G(x) = 1. \tag{73}$$

Proof. Clearly we have

$$G(x) = \sum_{n=1}^{\infty} G_n(x),$$

then $\lim_{x \to \infty} G(x) = \lim_{s \to 0} \Gamma(s)$. Using (54) and lemma 2, we get (73).

Corollary 2. The expected length of the busy period, μ^*, is given by

$$\mu^* = \int_0^{\infty} x dG(x) \frac{\mu_{l+1}}{1-\rho_{l+1}} \left\{ 1 + \sum_{i=1}^{l} (\rho_i - \rho_{l+1}) Q_i^* \right\} \tag{73}$$

$$= \frac{\mu_{l+1}}{\pi_0} \text{ if } \rho_{l+1} < 1.$$

Proof. If $\rho_{l+1} < 1$, $G(x)$ is a proper distribution function and by definition

$$\mu^* = -\lim_{s \to 0} \frac{d\Gamma(s)}{ds}.$$

To avoid a complicate calculation we shall consider the case $l=1$. Then

$$\mu^* = -\left\{ \frac{\mu_2}{\rho_2 - 1} + \left(\frac{\mu_2}{\rho_2 - 1} - \frac{\mu_2}{\rho_2 - 1} \right) \lim_{s \to 0} [C_N^{(1)}(s, \gamma(s), 1) + D_N^{(1)}(s, \gamma(s), 1)] \right\}$$

$$= -\left\{ \frac{\mu_2}{\rho_2 - 1} + \frac{\mu_2}{\rho_2 - 1} (\rho_1 - \rho_2) Q_1^* \right\} = \frac{\mu_2}{1-\rho_2} \{1 + (\rho_1 - \rho_2) Q_1^*\} = \frac{\mu_2}{\pi_0}.$$

In this evaluation we have to use lemma 2 and

$$\lim_{s \to 0} [C_N^{(1)}(s, \gamma(s), 1) + D_N^{(1)}(s, \gamma(s), 1)] = Q_1^*.$$

To prove the above equation we proceed as follows

$$\lim_{s \to 0} [C_N^{(1)}(s, \gamma(s), 1) + D_N^{(1)}(s, \gamma(s), 1)]$$

$$=\lim_{\substack{s\to 0\\ w\to 1}}[C_N^{(1)}(s,\ \gamma(s,\ w),\ w)+D_N^{(1)}(s,\ \gamma(s,\ w),\ w)]$$
$$=\lim_{\substack{w\to 1\\ z\to 1}}A_N^{(1)}(z,\ w)$$
$$=Q_1^*.$$

We shall deduce the distribution of the number of services in a busy period as follows.

Let us denote by $f_0^{(n)}$ the probability that starting from state E_0 the system returns to the state E_0 for the first time at the nth step. In other words $f_0^{(n)}$ is the probability that a busy period consists of n services.

Corollary 3. We have for $|w|\leqq 1$ that

$$\sum_{n=1}^{\infty} f_0^{(n)} w^n = -\frac{D_{l+1}(0,\ g(w),\ w)}{C_{l+1}(0,\ g(w),\ w)}. \tag{75}$$

Proof. We have that

$$f_0^{(n)}=\int_0^{\infty} dG_n(x)=\Gamma_n(0).$$

Putting $s=0$ in (53), we get (75).

Corollary 4. The expected number of services during a busy period, μ^{**}, is given by

$$\mu^{**}=\frac{1}{1-\rho_{l+1}}\left\{1+\sum_{i=1}^{l}(\rho_i-\rho_{i+1})Q_i^*\right\}=\frac{1}{\pi_0} \qquad \text{if } \rho_{l+1}<1. \tag{76}$$

Proof. Define

$$f_0=\sum_{n=1}^{\infty} f_0^{(n)}.$$

If $f_0=1$, then $\{f_0^{(n)}\}$ is a probability distribution and we have

$$\mu^{**}=\sum_{n=1}^{\infty} n f_0^{(n)}.$$

Now $f_0=\lim_{w\to 1}\left(\sum_{n=1}^{\infty} f_0^{(n)} w^n\right)$ and using (75) and lemma 2, we get

$$f_0=1 \qquad \text{if } \rho_{l+1}<1.$$

Hence we have

$$\mu^{**}=\lim_{w\to 1}\frac{d}{dw}\left(\sum_{n=1}^{\infty} f_0^{(n)} w^n\right).$$

To avoid a complicate calculation we shall treate the case $l=1$, then

$$\mu^{**}=\lim_{w\to 1}\frac{d}{dw}R_0(0,\ w)$$

$$=\frac{1}{1-\rho_2}+\left(\frac{\rho_1}{1-\rho_2}-\frac{\rho_2}{1-\rho_2}\right)\lim_{w\to 1}[C_N^{(1)}(0,\ g(w),\ w)+D_N^{(1)}(0,\ g(w),\ w)]$$
$$=\frac{1}{1-\rho_2}\{1+(\rho_1-\rho_2)Q_1^*\}=\frac{1}{\pi_0}.$$

In this calculation we have to use that

$$\lim_{w\to 1}[C_N^{(1)}(0,\ g(w),\ w)+D_N^{(1)}(0,\ g(w),\ w]=Q_1^*.$$

Which is proved in the same way as that of corollary 2.

7. SPECIAL CASE

We consider the case $l=1$ and we shall assume that $H_i(x)$ $(i=1,\ 2)$ are exponential distributions. Then we have

$$\psi_i[\lambda(1-z)]=\frac{1}{1+\rho_i}\sum_{k=0}^{\infty}\left(\frac{\rho_i}{1+\rho_i}z\right)^k$$
$$A_1(z,\ w)=\frac{w(z-1)}{1+\rho_1-\rho_1 z},$$

and $B_1(z)=z^{i+1}$.

Therefore

$$\begin{aligned}
&A_1(z,\ w)/\{z-w\psi_1[\lambda(1-z)]\}\\
&=1+z\left\{\frac{1+\rho_1}{w}-1\right\}\\
&\quad+z^2\left\{\frac{-\rho_1}{w}+{}_2C_0\frac{(1+\rho_1)^2}{w^2}-\frac{1+\rho_1}{w}\right\}\\
&\quad+z^3\left\{{}_2C_1\frac{(1+\rho_1)(-\rho_1)}{w^2}+{}_3C_0\frac{(1+\rho_1)^3}{w^3}-\frac{-\rho_1}{w}-{}_2C_0\frac{(1+\rho_1)^2}{w^2}\right\}\\
&\quad+z^4\left\{{}_2C_2\frac{(-\rho_1)^2}{w_2}+{}_3C_1\frac{(1+\rho_1)^2(-\rho_1)}{w^3}+{}_4C_0\frac{(1+\rho_1)^4}{w^4}-{}_2C_1\frac{(1+\rho_1)(-\rho_1)}{w^2}\right.\\
&\quad\left.-{}_3C_0\frac{(1+\rho_1)^3}{w_3}\right\}+\cdots\cdots.
\end{aligned}$$

Generally if we put $1+\rho_1=a$ and $-\rho_1=b$, then the coefficient of z^{2n} is given by

$$\begin{aligned}
&{}_nC_n\frac{b^n}{w^n}+{}_{n+1}C_{n-1}\frac{a^2b^{n-1}}{w^{n+1}}+\cdots\cdots+{}_{n+r}C_{n-r}\frac{a^{2r}b^{n-r}}{w^{n+r}}+\cdots\cdots+{}_{2n}C_0\frac{a^{2n}}{w^{2n}}\\
&-\left({}_nC_{n-1}\frac{ab^{n-1}}{w^n}+{}_{n+1}C_{n-2}\frac{a^3b^{n-2}}{w^{n+1}}+\cdots\cdots+{}_{n+1}C_{n-1-r}\frac{a^{2r+1}b^{n-1-r}}{w^{n+r}}+\cdots\cdots+{}_{2n}C_0\frac{a^{2n-1}}{w^{2n-1}}\right)
\end{aligned}$$

and the cofficient of z^{2n+1} is given by

$$
{}_{n+1}C_n\frac{ab^n}{w^{n+1}}+{}_{n+2}C_{n-1}\frac{a^3b^{n-1}}{w^{n+2}}+\cdots\cdots+{}_{n+1+r}C_{n-r}\frac{a^{2r+1}b^{n-r}}{w^{n+1+r}}+\cdots\cdots+{}_{2n+1}C_0\frac{a^{2n+1}}{w^{2n+1}}
$$
$$
-\left({}_nC_n\frac{b^n}{w^n}+{}_{n+1}C_{n-2}\frac{a^2b^{n-1}}{w^{n+1}}+\cdots\cdots+{}_{n+r}C_{n-r}\frac{a^{2r}b^{n-r}}{w^{n+r}}+\cdots\cdots+{}_{2n}C_0\frac{a^{2n}}{w^{2n}}\right).
$$

Hence

$$
\begin{aligned}
A_N^{(1)}(z,\ w)=1-z\\
&+z(1-z)\frac{a}{w}\\
&+z^2(1-z)\left\{\frac{b}{w}+{}_2C_0\frac{a^2}{w^2}\right\}\\
&+z^3(1-z)\left\{{}_2C_1\frac{ab}{w^2}+{}_3C_0\frac{a^3}{w^3}\right\}\\
&+z^4(1-z)\left\{{}_2C_2\frac{b^2}{w^2}+{}_3C_1\frac{a^2b}{w_3}+{}_4C_0\frac{a^4}{w^4}\right\}\\
&+\cdots\cdots\\
&+z^{N-1}(1-z)\begin{cases}\left\{{}_nC_{n-1}\dfrac{ab^{n-1}}{w^n}+\cdots\cdots+{}_{n+r}C_{n-1-r}\dfrac{a^{2r+1}b^{n-1-r}}{w^{n+r}}\right.\\ \qquad\left.+\cdots\cdots+{}_{2n-1}C_0\dfrac{a^{2n-1}}{w^{2n-1}}\right\} & (N=2n)\\ \left\{{}_nC_n\dfrac{b^n}{w^n}+\cdots\cdots+{}_{n+r}C_{n-r}\dfrac{a^{2r}b^{n-r}}{w^{n+r}}+\cdots\cdots+{}_{2n}C_0\dfrac{a^{2n}}{w^{2n}}\right\} & (N=2n+1)\end{cases}\\
&+z^N\begin{cases}\left\{{}_nC_n\dfrac{b^n}{w^n}+\cdots\cdots+{}_{n+r}C_{n-r}\dfrac{a^{2r}b^{n-r}}{w^{n+r}}+\cdots\cdots+{}_{2n}C_0\dfrac{a^{2n}}{w^{2n}}\right\} & (N=2n)\\ \left\{{}_{n+1}C_n\dfrac{ab^n}{w^{n+1}}+\cdots\cdots+{}_{n+1-r}C_{n-r}\dfrac{a^{2r+1}b^{n-r}}{w^{n+1+r}}\right.\\ \qquad\left.+\cdots\cdots+{}_{2n+1}C_0\dfrac{a^{2n+1}}{w^{2n+1}}\right\} & (N=2n+1).\end{cases}
\end{aligned}
$$

From the above $A_2(z,\ w)$ is obtained easily. Now we get

$$
\begin{aligned}
&B_1(z)/\{z-w\psi_1[\lambda(1-z)]\}\\
&=-\frac{z^i}{w}\left[a+z\left(\frac{a^2}{w}+b\right)+z^2\left(\frac{ab}{w}+{}_2C_0\frac{a^3}{w^2}+\frac{ab}{w}\right)\right.\\
&\left.+z^3\left({}_2C_1\frac{a^2b}{w^2}+{}_3C_0\frac{a^4}{w^3}+\frac{b^2}{w}+{}_2C_0\frac{a^2b}{w^2}\right)+\cdots\cdots\right].
\end{aligned}
$$

Generally the coefficient of z^{2n} is given by

$$a\left({}_nC_n\frac{b^n}{w^n}+{}_{n+1}C_{n-1}\frac{a^2b^{n-1}}{w^{n+1}}+\cdots\cdots+{}_{n+r}C_{n-r}\frac{a^{2r}b^{n-r}}{w^{n+r}}+\cdots\cdots+{}_{2n}C_0\frac{a^{2n}}{w^{2n}}\right)$$
$$+b\left({}_nC_{n-1}\frac{ab^{n-1}}{w^n}+{}_{n+1}C_{n-2}\frac{a^3b^{n-2}}{w^{n+1}}+\cdots\cdots+{}_{n+r}C_{n-1-r}\frac{a^{2r+1}b^{n-1+r}}{w^{n+r}}\right.$$
$$\left.+\cdots\cdots+{}_{2n-1}C_0\frac{a^{2n-1}}{w^{2n-1}}\right)$$

and the coefficient of z^{2n+1} is given by

$$a\left({}_{n+1}C_n\frac{ab^n}{w^{n+1}}+{}_{n+2}C_{n-1}\frac{a^3b^{n-1}}{w^{n+2}}+\cdots\cdots+{}_{n+1+r}C_{n-r}\frac{a^{2r+1}b^{n-r}}{w^{n+1+r}}\right.$$
$$\left.+\cdots\cdots+{}_{2n+1}C_0\frac{a^{2n+1}}{w^{2n+1}}\right)+b\left({}_nC_n\frac{b_n}{w_n}+{}_{n+1}C_{n-1}\frac{a^2b^{n-1}}{w^{n+1}}\right.$$
$$\left.+\cdots\cdots+{}_{n+r}C_{n-r}\frac{a^{2r}b^{n-r}}{w^{n+r}}+\cdots\cdots+{}_{2n}C_0\frac{a^{2n}}{w^{2n}}\right).$$

From the above $B_N^{(1)}(z, w)$ is calculated. Putting $w=1$ in $A_1(z, w)$ and $A_N^{(1)}(z, w)$, we get $Q_1(z)$ and $Q_N^{(1)}(z)$. $Q_1^*=Q_N^{(1)}(1)$ is given as follows

$$Q_1^*=\begin{cases}{}_nC_nb^n+\cdots\cdots+{}_{n+r}C_{n-r}a^{2r}b^{n-r}+\cdots\cdots+{}_{2n}C_0a^{2n} & (N=2n)\\ {}_{n+1}C_nab^n+\cdots\cdots+{}_{n+1+r}C_{n-r}a^{2r+1}b^{n-r}+\cdots\cdots+{}_{2n+1}C_0a^{2n+1} & (N=2n+1).\end{cases}$$

Putting $a^*=1+s\mu_1+\rho_1$ and $b=-\rho_1$, we have

$$C_1(s, z, w)=\frac{-w}{a^*+bz},$$

and

$$C_1(s, z, w)/\{z-w\phi_1[s+\lambda(1-z)]\}$$
$$=1+z\frac{a^*}{w}$$
$$+z^2\left(\frac{b}{w}+{}_2C_0\frac{a^{*2}}{w^2}\right)$$
$$+z^3\left({}_2C_1\frac{a^*b}{w^2}+{}_3C_0\frac{a^{*3}}{w^3}\right)$$
$$+\cdots\cdots.$$

Generally the coefficient of z^{2n} is given by

$${}_nC_n\frac{b^n}{w^n}+{}_{n+1}C_{n-1}\frac{a^{*2}b^{n-1}}{w^{n+1}}+\cdots\cdots+{}_{n+r}C_{n-r}\frac{a^{*2r}b^{n-r}}{w^{n+r}}+\cdots\cdots+{}_{2n}C_0\frac{a^{*2n}}{w^{2n}}$$

and the coefficient of z^{2n+1} is given by

$${}_{n+1}C_n\frac{a^*b^n}{w^{n+1}}+{}_{n+2}C_{n-1}\frac{a^{*3}b^{n-1}}{w^{n+2}}+\cdots\cdots+{}_{n+1+r}C_{n-r}\frac{a^{*2r+1}b^{n-r}}{w^{n+1+r}}+\cdots\cdots+{}_{2n+1}C_0\frac{a^{*2n+1}}{w^{2n+1}}.$$

Hence $C_N^{(1)}(s, z, w)$ and $D_N^{(1)}(s, z, w)$ are obtained easily.

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