

ON THE ECONOMICAL ASSIGNMENT OF  
COMPONENT TOLERANCES

NORIHIRO YAMAKAWA

*Kasado Works, Hitachi Ltd.*

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1. INTRODUCTION

The problem considered is that of how to select component tolerances so as to minimize production costs—assuming a situation in which the product is assembled from a number of component parts. Several approaches have been made to this problem in the recent years (e. g. Taguchi [1] and Evans [2]). In this paper, the author, taking several assumptions for granted, proposes a graphical solution of this problem by means of an iterative scheme. The three assumptions taken here are as follows:

- 1) All component parts are under statistical control;
- 2) The statistical behavior of the component response can be estimated as a function of the component's production cost; and
- 3) The production cost of the assembled product can be estimated as the sum of the production costs of component parts.

2. MATHEMATICAL FORMULATION OF THE PROBLEM

Let us suppose that the principal response  $y$  of the assembled product can be represented as a function of the responses  $x_i$ 's ( $i=1, 2, \dots, n$ ) of its component parts. Let us also assume that  $x_i$  can be measured by the deviation from the center of tolerance of  $x_i$ .

$$y=y(x_1, x_2, \dots, x_n) \quad (1)$$

Also,  $y$  has an allowable tolerance  $T_y$ , and therefore a set of component tolerances ( $T_1, T_2, \dots, T_n$ ) are assigned for the component responses ( $x_1, x_2, \dots, x_n$ ) respectively. If  $T_i = 2t_i\sigma_i$ , and if the responses  $x_i$ 's are distributed independently, then Formula (1) can be linearized, and using the ordinary linear propagation of error formula, the relation between the tolerance of  $y$  and  $T_i$ 's can be expressed in the following formula

$$T_y = \sqrt{a_1^2 T_1^2 + a_2^2 T_2^2 + \dots + a_n^2 T_n^2} \quad (2)$$

where

$$a_i^2 \equiv \left( \frac{\partial y}{\partial x_i} \right) x_i = m_i, \quad (3)$$

and  $m_i$  and  $\sigma_i^2$  are, respectively, the mean and variance of the  $i$ -th component response before inspection.

Under these conditions, we are able to assign a set of tolerances of the component responses. However, it will be understood from Formula (2) that many sets of  $T_i$ 's may exist for any given  $T_y$ . Our problem is to choose, from among the many possible sets of  $T_i$ 's satisfying Formula (2), the most economical set of tolerances (hereafter called the optimal solution).

In this type of problem, it is necessary that

$$y = y(x_1, x_2, \dots, x_n) \quad (4-1)$$

and that

$$C = \varphi(T_1, T_2, \dots, T_n, T_y, \xi_1, \dots, \xi_n), \quad (4-2)$$

where  $C$  is the total cost of production and  $\{\xi_i\}$  represents a set of parameters such as  $m_i$  and  $\sigma_i^2$  in the statistical distribution of the preinspection response of the  $i$ -th component part.

In actual application, it can be assumed that

$$C = \sum_{i=1}^n \varphi_i(T_i, \xi_i) + \varphi_y(T_y, T_1, \dots, T_n, \xi_1, \dots, \xi_n), \quad (5)$$

where  $\varphi_i$  represents the production cost of the  $i$ -th component part as a function of  $T_i$  and  $\xi_i$  and  $\varphi_y$  represents the loss for which the principal response  $y$  will incur at the outside of its allowable tolerance.

Let us assume that

$$\varphi_y = P_y C_y,$$

where  $P_y$  represents the fraction defective that the principal response  $y$  will fall outside of its allowable tolerance, and  $C_y$  is the loss of a defective

assembly. If  $C_y$  can be assumed to be a constant, and if it can be assumed that  $y$  is distributed according to a normal distribution, then the mean  $m_y$  and the variance  $\sigma_y^2$  can be calculated from the following transformations,

$$k \equiv (y - m_y) / \sigma_y \quad (6-1)$$

and

$$T_y = (k_2 - k_1) \sigma_y \quad (k_2 > k_1), \quad (6-2)$$

where  $k$  is the normalized variate of  $y$ , and  $k_1$  and  $k_2$  represent the lower and upper limits, respectively, of the normalized tolerance of  $y$ .

The  $m_y$  and  $\sigma_y^2$  are also determined by the following relations

$$\sigma_y^2 = \sum_{i=1}^n a_i \sigma_i'^2 \quad (7-1)$$

and

$$m_y = y(m_1', m_2', \dots, m_n'), \quad (7-2)$$

where  $m_i'$  and  $\sigma_i'^2$  represent, respectively, the mean and variance of the  $x_i$  of acceptable products.

We can say, therefore, that our problem consists with to select the component tolerance so as to minimize production cost

$$\varphi = \sum_{i=1}^n \varphi_i(T_i, \xi_i) \quad (8)$$

subject to restraints on Formula (7), where

$$\xi_i = (m_i, \sigma_i^2) \quad (9-1)$$

$$m_i' = \int_{X_{1i}}^{X_{2i}} x_i f_i(x_i) dx_i \quad (9-2)$$

and

$$\sigma_i'^2 = \int_{X_{1i}}^{X_{2i}} (x_i - m_i')^2 f_i(x_i) dx_i, \quad (9-3)$$

$f_i(x_i)$  is the pdf of the  $i$ -th final acceptable component response, and  $X_{1i}$  and  $X_{2i}$  are the lower and upper allowable limits of the  $i$ -th component response.

In practice, Formula (8) is represented by the following three typical situations:

$$\varphi_i = \varphi_i(T_i) \quad (10-1)$$

$$\varphi_i = \varphi_i(T_i, \sigma_i^2) \quad (10-2)$$

and

$$\varphi_i = \varphi_i(T_i, \sigma_i^2, m_i). \quad (10-3)$$

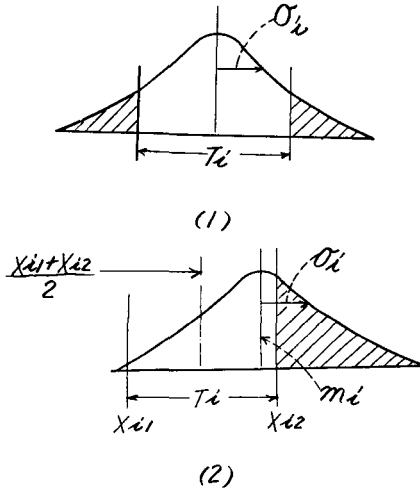


Fig. 1. Representation of Two Models

- (1) Corresponds to the Case of Formula (10-2)
- (2) Corresponds to the Case of Formula (10-3)

Before calculations, let us assume that the preinspection component responses  $x_i$ 's are normally distributed independently of each other.

We can say that the  $i$ -th component's production cost is

$$\varphi_i = (C_i p_i + H_i), \quad (11)$$

where  $C_i$  is the additional costs required for the repairing of a defective component part, and  $p_i$  is the fraction defective of the  $i$ -th components, and  $H_i$  is the production cost as a function of a certain variance changing according to the processing method.

Of the two restraints:

$$\sigma_y^2 = \sum_{i=1}^n a_i^2 \sigma_i'^2$$

and

$$m_y = y(m_1', m_2', \dots, m_n'),$$

the latter can be eliminated, since if the first term  $C_i p_i$  of Formula (11) is minimized, this will be equivalent to making  $m_i$ 's to be zero.

Consequently, the optimal tolerance  $T_i = 2t_i \sigma_i$  can be obtained from

The first one of these (10-1) has been solved by G. Taguchi under the following assumption

$$\varphi_i = \beta_i (T_i)^{-a_i}$$

The other two situations are visualized in Fig. 1 (1) and (2), respectively. The author reports below on the results of his research with regard to (10-2).

### 3. ASSIGNMENT OF THE OPTIMAL TOLERANCES

Let us now solve the problem given above, assuming that the production cost can be grasped as a function of the tolerance of component response  $T_i$  and the variance of component response  $\sigma_i'^2$ , as represented by Formula (10-2). Before

the solution of the following simultaneous equations

$$\left. \begin{aligned} \frac{\partial}{\partial t_i}(\varphi + \lambda \sigma_y^2) &= 0 \\ \frac{\partial}{\partial \sigma_i^2}(\varphi + \lambda \sigma_y^2) &= 0 \end{aligned} \right\}, \quad (12)$$

where  $\lambda$  is the Lagrange multiplier.

Now, if  $f(t)$  is taken to be the pdf of the component response before inspection, then

$$\sigma_i'^2 = \sigma_i^2 \int_{-t_i}^{t_i} t^2 f_i(t) dt \equiv \sigma_i^2 k(t_i) \quad (13)$$

and

$$p_i = 1 - \int_{-t_i}^{t_i} f(t) dt = p(t_i) \quad (14)$$

Furthermore we can put

$$H_i = \beta_i (\sigma_i^2)^{-\alpha_i} \quad (15)$$

where  $\alpha_i$  and  $\beta_i$  represent parameters with respect to several controlling conditions for the  $i$ -th component part. This results in

$$\begin{aligned} \frac{\partial t_i}{\partial p_i} &\neq 0 & (t_i \neq \infty), \\ \frac{\partial \sigma_i}{\partial H_i} &\neq 0. \end{aligned}$$

Then, from Formula (12) we obtain

$$\left. \begin{aligned} \frac{\partial}{\partial p_i}(\varphi + \lambda \sigma_y^2) &= 0 \\ \frac{\partial}{\partial H_i}(\varphi + \lambda \sigma_y^2) &= 0 \end{aligned} \right\}. \quad (16)$$

We then obtain

$$\left. \begin{aligned} C_i + \lambda \frac{\partial}{\partial p_i} \left[ \sum_i a_i^2 \sigma_i^2 k(t_i) \right] &= 0 \\ 1 + \lambda \frac{\partial}{\partial H_i} \left[ \sum_i a_i^2 \sigma_i^2 k(t_i) \right] &= 0 \end{aligned} \right\}. \quad (17)$$

Here, denoting as

$$\left. \begin{aligned} \phi_i &= \sigma_i^2 \frac{\partial k(t_i)}{\partial p_i} \\ \phi_i &= (\sigma_i)^{\alpha_i+1} k(t_i) / \alpha_i \end{aligned} \right\}, \quad (18)$$

we finally obtain

$$\left. \begin{aligned} \lambda\phi_i &= C_i/a_i^2 \\ \lambda\phi_i &= \beta_i/a_i^2 \end{aligned} \right\} \quad (19)$$

Consequently, we are able to determine the optimal set of tolerances of component responses  $(t_1, t_2, \dots, t_n; \sigma_1, \sigma_2, \dots, \sigma_n)$ . The  $i$ -th tolerance  $T_i$  can be calculated from  $T_i = 2t_i\sigma_i$ . The solution of Formula (19) is obtained from the intersection of curve  $\phi_i$  and the other curve  $\phi_i$  on the graph representing the optimal relations between  $k(t_i)$  and  $\sigma_i^2$ .

#### 4. ITERATIVE SCHEME

##### 4. 1. Calculation of $\phi_i$ and $\phi_i$

In order to establish an iterative scheme, it is necessary to calculate the values of  $\phi_i$  and  $\phi_i$  as the functions of  $t_i$ ,  $\sigma_i^2$ ,  $\alpha_i$  and  $\beta_i$ . First, we must consider the pdf of the final acceptable component response  $f_i(t)$ .

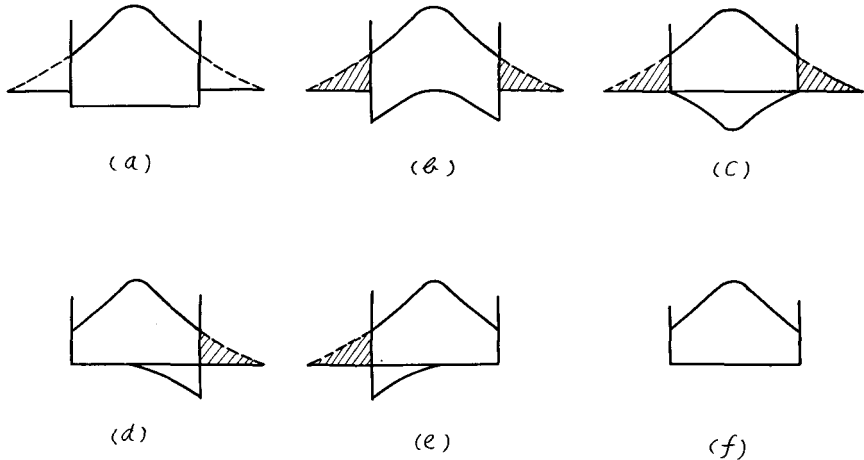


Fig. 2 Extreme Six Cases for  $f_i(t)$

Several possible  $f_i(t)$  are represented graphically in Fig. 2. Generally, these pdf can be formulated in the following formula:

$$f_i(t) = f(t) + g(t), \quad (-t_i \leq t \leq t_i), \quad (20)$$

where  $f(t)$  is the pdf of the acceptable component response after inspection, which is equal to the pdf of the component response before inspection within the tolerance ( $T_i$ ), and  $g(t)$  is the pdf of the acceptable component response after repairing the component parts rejected at the first inspection.

When repair is not possible, that is, in Fig. 2 (e), the pdf  $f_i(t)$  of the acceptable component response will be

$$f_i(t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} / \int_{-t_i}^{t_i} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt, \quad (-t_i \leq t \leq t_i). \quad (21)$$

On the other hand, if the defective components are repairable, then the final acceptable pdf of the repaired component response  $g(t)$  may be assumed such as Fig. 2 (a), (b) and (c), in which (a), (b) and (c) each represent cases where  $g(t)$  can be assumed as a uniform distribution function, as a parabolic type distribution function, and as a normal distribution function, respectively. Each of them can be expressed in the following terms

$$(a) \quad g(t) = F(t_i)/t_i \quad (-t_i \leq t \leq t_i), \quad (22-1)$$

$$(b) \quad g(t) = 3F(t_i)t^2/t_i^3 \quad (-t_i \leq t \leq t_i), \quad (22-2)$$

and

$$(c) \quad g(t) = 2F(t_i)f(t) \quad (-t_i \leq t \leq t_i), \quad (22-3)$$

where

$$F(t_i) = \int_{t_i}^{\infty} f(t) dt.$$

These two extreme cases (a) and (b) are calculated in this paper. If the  $k(t_i)$ 's in (a) and (b) are represented as  $k_a(t_i)$  and  $k_b(t_i)$ , respectively, then

$$k_a(t_i) = 1 - 2t_i f(t_i) + F(t_i) \left( \frac{2}{3} t_i^2 - 2 \right) \quad (23-1)$$

and

$$k_b(t_i) = 1 - 2t_i f(t_i) + F(t_i) \left( \frac{6}{5} t_i^2 - 2 \right). \quad (23-2)$$

Similarly, if the  $\partial k/\partial p_i$  in the case of (a) and (b) are represented as  $\partial k_a/\partial p_i$  and  $\partial k_b/\partial p_i$ , respectively, then we obtain the following

$$\frac{\partial k_a}{\partial p_i} = -\frac{2}{3} t_i \left[ \frac{F(t_i)}{f(t_i)} + t_i \right] \quad (24-1)$$

and

$$\frac{\partial k_b}{\partial p_i} = -\frac{2}{5} t_i \left[ \frac{F(t_i)}{f(t_i)} + t_i \right]. \quad (24-2)$$

#### 4. 2. Practical Procedure of the Iterative Scheme

On the basis of the foregoing reasoning, the author proposes the following iterative scheme to determine the optimal assignment of the

set of tolerance of the component responses.

(1) Basic Data to be Prepared

- (a) The tolerance to be given for the response of the given assembly ;

$$T_y$$

- (b) The allowable ratio for the principal response falling outside the assembly tolerance  $T_y$  ;  $P_y$

- (c) The relation between the principal response  $y$  and the component responses  $(x_1, x_2, \dots, x_n)$  ;

$$y = y(x_1, x_2, \dots, x_n)$$

- (d) The additional cost required for repairing one defective component part :  $C_i$

$C_i$  is assumed to vary according to the component response number ( $i$ ) but to be a constant not effected by  $t_i$  and  $\sigma_i$ .

- (e) The two parameters  $\alpha_i$  and  $\beta_i$  in the relation between the production cost and the variance of the component response ;

$$H_i = \beta_i / (\sigma_i^2)^{\alpha_i}$$

It may also be possible that there are parts for which any other production method would be unconsiderable, for instance, when there is only one suitable type of tool. In these cases, the data here is not required, and it is sought from the constant  $\sigma_i^2$  as a function of  $\phi_i$  alone.

(2) Computation Procedure

- (a) Compute the  $a_i^2$  from

$$\left( \frac{\partial y}{\partial x_i} \right)_{x_i = m_i}^2 = a_i^2.$$

- (b) Using the given  $T_y$  and  $P_y$  and the normal distribution table, compute the  $k$  of

$$\frac{1}{\sqrt{2\pi}} \int_k^\infty e^{-\frac{t^2}{2}} dt = P_y / 2$$

and compute  $\sigma_y^2$  from

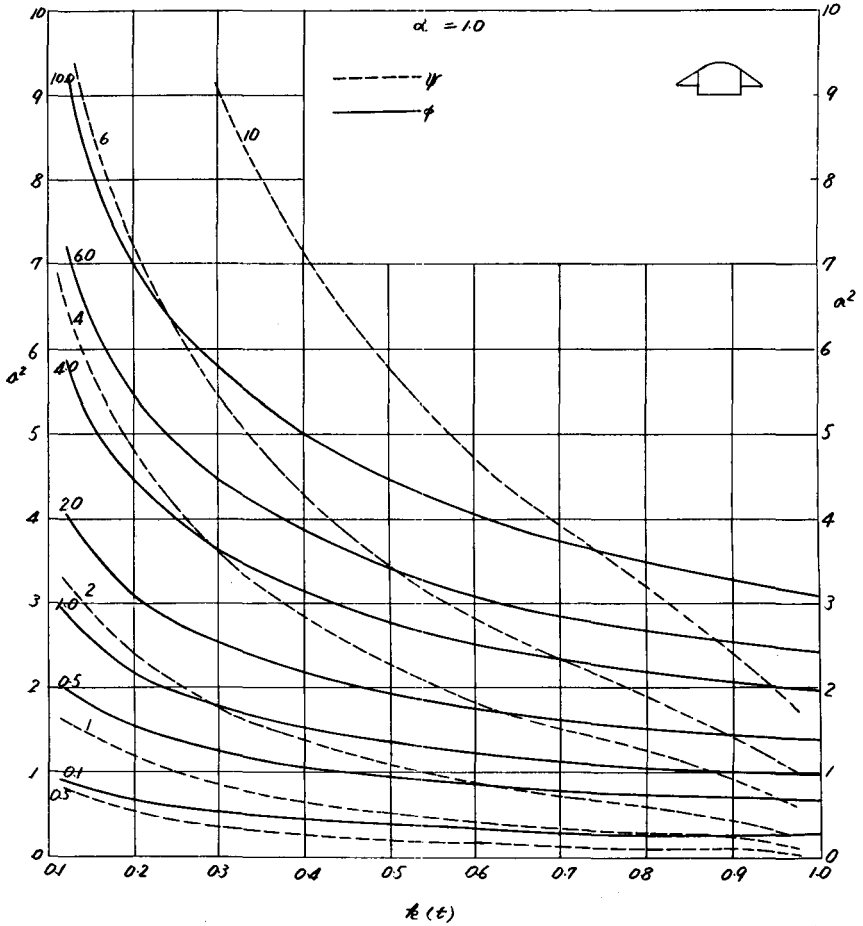
$$\frac{T_y}{2} = k \sigma_y.$$

- (c) Compute  $\phi_i^{(1)} = C_i / a_i^2$  and  $\phi_i^{(1)} = \beta_i / a_i^2$ .

- (d) Select the computation graph (refer to Fig. 3) for the  $\alpha_i$  of the given component part.

- (e) Find the value of  $\sigma_i^{2(1)}$  and  $k^{(1)}(t_i)$  from the intersection of the curves  $\phi_i^{(1)}$  and  $\phi_i^{(1)}$  in Figure (d) calculated at step (c).




 Fig. 3 Computational Graph for  $\alpha=1.0$ 

(f) Compute  $A_1$  from

$$A_1 = \sum \alpha_i^2 \sigma_i^{2(1)} k^{(1)}(t_i) / \sigma_y^2.$$

(g) Compute  $\phi_i^{(2)} = \phi_i^{(1)} A_1$  and  $\phi_i^{(2)} = \phi_i^{(1)} / A_1$ . Find the value of  $\sigma_i^{2(2)}$  and  $k^{(2)}(t_i)$  from the intersection of the curves in the same way as in (e) above.

(h) Compute  $A_2$  from

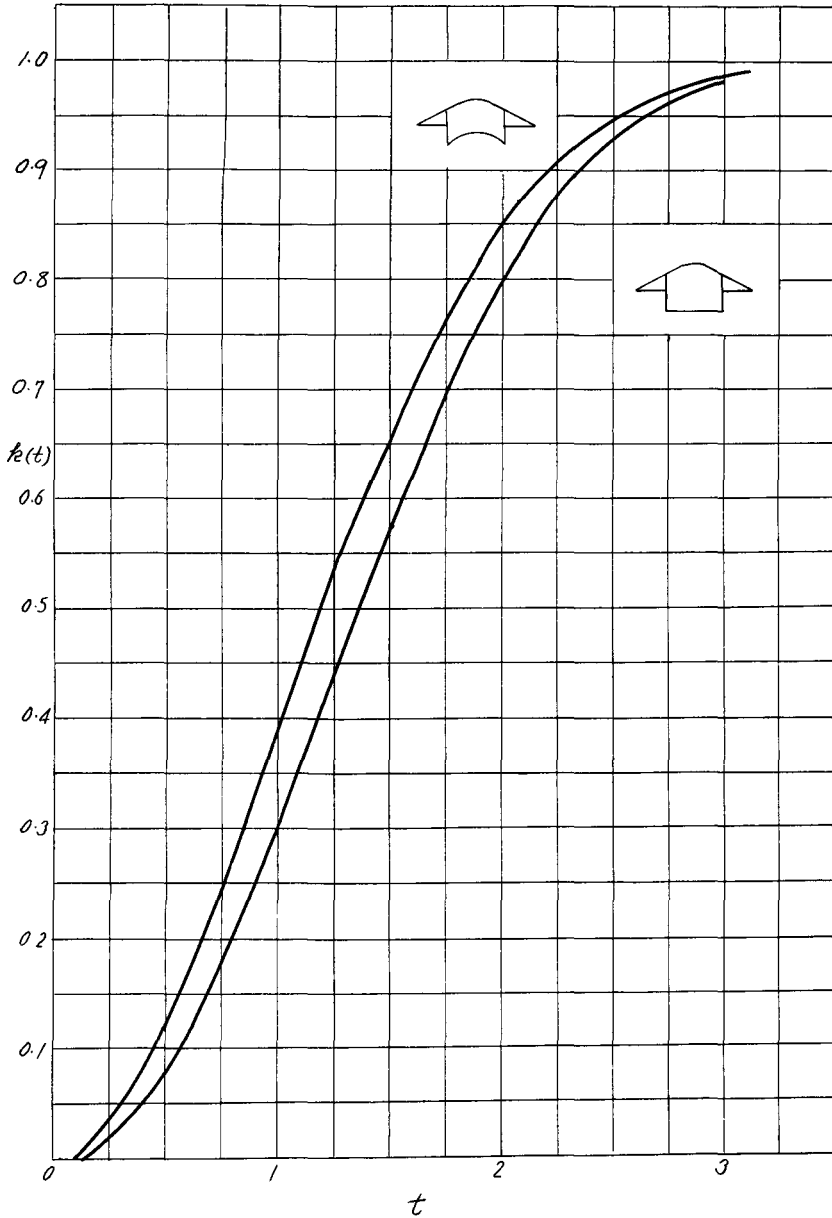


Fig. 4 Relation between  $t$  and  $k(t)$  for given two  $f_i(t_i)$ 's

$$A_2 = \sum_{i=1}^n a_i^2 \sigma_i^{(2)} k^{(2)}(t_i) / \sigma_y^2.$$

(i) Compute

$$\phi_i^{(3)} = \phi_i^{(2)} / A_2 = \phi_i^{(1)} / A_1 \cdot A_2$$

and

$$\phi_i^{(3)} = \phi_i^{(2)} / A_2 = \phi_i^{(1)} / A_1 \cdot A_1.$$

(j) Repeat the above procedure until  $A_j$  approximate to unity.

(k) Substituting  $k(t_i)$  for  $t_i$  in Fig. 4, determine the optimal tolerance of the  $i$ -th component response from

$$T_i = 2t_i^{(j)} \sigma_i^{(j)}.$$

#### 4. 3. Comparison with Other Methods

As mentioned above, other methods have been proposed with regard to the tolerance assignment problem, such as:

- (1) The Taguchi method;
- (2) The Evans method.

Of these, the first corresponds to the situation in which  $p_i$  is equal to the constant in Formula (11) in this paper. In the second method, Evans attempts to find the cost function graphically, rather than analytically. Consequently, it is not useful for assemblies composed of many different component parts, but is effective only for many identical component parts, that is when

$$\sigma_y^2 = \sum a_i^2 \sigma_i'^2 = n \sigma_i'^2.$$

The cost function also becomes

$$\varphi = n \varphi_i(T_i, m_i, \sigma_i).$$

#### 5. CONCLUSION

Through the above theoretical study of the tolerance assignment problem, we can reach to the following conclusions:

- (1) A new method for the numerical calculation of tolerance assignments may be proposed.
- (2) Our method include previous two methods as special cases.

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