A REMARK ON ECONOMIC SURVIVAL GAME

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1. INTRODUCTION

In prof. Miyasawa's paper, "An economic survival game", there seems to be one problem that requires more detailed consideration. More prseicely, the functional equation (2. 1) in theorem 1 holds true only under the conditions that $V(x)$ defined in the above paper is finite for $x>0$, and that optimal strategies which attain this value exist, and among which at least some are stationary in the sense that withdrawal policy is independent of the periods or stages of the game. In this note we shall deal with the problem more rigorously, considering first the game Ω_T where the game should be terminated at the T-th period unless the ruin has not occurred before, and then all that remains should be withdrawn; then considering the limiting case when T tends to infinity.

2. THE EXISTENCE OF THE LIMITING VALUE

We denote the value of the game Ω_T as $V_T(x)$, the meaning of which is similar to that given in (1) and will be seen immediately.

Evidently, $$V_0(x)=\max(x, 0).$$

We define
$$G_T(x)=P\int_{-x}^{\infty} V_T(x+z)\mathrm{dF}(z:x), \qquad \text{for } x>0$$
$$=0 \qquad \text{for } x\leqq 0$$

then it is easily seen that
$$V_T(x)=\sup_{0\leqq y\leqq x}(y+G_{T-1}(x-y))$$
$$=x+\sup_{0\leqq y\leqq x}(G_{T-1}(y)-y) \qquad \text{for } x>0$$
$$=0 \qquad \text{for } x\leqq 0$$

Lemma 1: $V_T(x)$ and $G_T(x)$ are monotone nondecreasing with T.

Proof: We proceed by induction.

Since $G_1(0)=0$, $$V_1(x)\geq x=V_0(x) \qquad \text{for } x>0$$
$$0=V_0(x) \qquad \text{for } x\leqq 0$$

$$G_1(x)=\rho\int V_1(x+z)\,\mathrm{dF}(z\,;\,x)$$
$$\geqq\rho\int V_0(x+z)\,\mathrm{dF}(z:x)=G_0(x).$$

Assume that $V_T(x)\geqq V_{T-1}(x)$ and $G_T(x)\geqq G_{T-1}(x)$, then

$$V_{T+1}(x)=x+\sup(G_T(y)-y)$$
$$\geqq x+\sup(G_{T-1}(y)-y)=V_T(x),$$
$$G_{T+1}(x)=\rho\int V_{T+1}(x+z)\,\mathrm{dF}(z:x)$$
$$\geqq\rho\int V_T(x+z)\,\mathrm{dF}(z:x)=G_T(x)$$

which completes the proof.

Lemma 2: If $\displaystyle\int_{-x}^{\infty}(x+z)\,\mathrm{dF}(z\,;\,x)\leqq\frac{1}{\rho}x+B$ for all $x>0$

for some positive constant B,

$$G_T(x)-G_{T-1}(x)\leqq B\rho^{T+1}$$
$$V_T(x)-V_{T-1}(x)\leqq B\rho^{T}$$

for all $x>0$.

Proof: First it is observed that

$$G_0(x)=\rho\int V_0(x+z)dx(z:x)$$
$$=\rho\int_{-x}^{\infty}(x+z)dF(z:x)\leqq x+B\rho.$$

Hence $\quad V_1(x)-V_0(x)=\sup_y(G_T(y)-y)\leqq B\rho.$

Assume that $V_T(x)-V_{T-1}(x)\leqq B\rho^T$ then

$$G_T(x)-G_{T-1}(x)=\rho\int\{V_T(x+z)-V_{T-1}(x+z)\}\,\mathrm{dF}(z:x)$$
$$\leqq\rho\int B\rho^T\,\mathrm{dF}(z:x)=B\rho^{T+1}$$
$$V_{T+1}-V_T(x)=\sup_y(G_T(y)-y)-\sup_y(G_{T-1}(y)-y)$$
$$\leqq\sup_y(G_T(y)-G_{T-1}(y))\leqq B\rho^{T+1}$$

which completes the proof.

Theorem 1: If the condition of lemma 2 holds, $V_T(x)$ and $G_T(x)$ converge uniformly to limits $V(x)$ and $G(x)$ respectively, as T tends to infinity, and for which it holds that

$$V(x)=x+\sup_{0\leq y\leq x}(G(y)-y) \qquad \text{for } x>0$$
$$=0 \qquad \text{for } x\leqq 0$$
$$G(x)=\rho\int V(x+z)\mathrm{dF}(z:x).$$

Proof: The uniform convergence is garanteed by lemmas 1 and 2, and by the uniformity of convergence relations among G and V are maintained in the limit.

Corollary: If $F(x:z)$ is independent of x and

$$E(Z)=\int z\,\mathrm{dF}(z)$$

exists and is finite, G_T and V_T converge.

Proof: The existence of $E(Z)$ implies

$$\xi\overset{d}{=}\int_0^\infty z\,\mathrm{dF}(z)<\infty$$

and
$$\int_{-x}^\infty(x+z)\mathrm{dF}(z)\leqq x\int_{-\infty}^\infty \mathrm{dF}(z)+\int_0^\infty z\,\mathrm{dF}(z)=x+\xi$$

Lemma 3: If some $C>\frac{1}{\rho}$ and $D>0$,

$$\int_{-x}^\infty(x+z)\ \mathrm{dF}\ (z:x)\geqq Cx-D \qquad \text{for all } x>0$$

then
$$\begin{matrix}V_T(x)\\ G_{T-1}(x)\end{matrix}\geqq(\rho C)^T x-D\rho^T(C^T-1)/(C-1) \qquad \text{for all } x>0.$$

Proof: The proof is done also by induction.
For V_0 the inequailty holds evidently, and assuming that it holds for V_T we have,

$$\begin{aligned}G_T(x)&=\rho\int V_T(x+z)\ \mathrm{dF}\ (z:x)\\ &\geqq\rho\int_{-x}^\infty((\rho C)^T(x+z)-D\rho^T(C^T-1)/(C-1))\mathrm{dF}(z:x)\\ &\geqq\rho((\rho C)^T(Cx-D-D\rho^T(C^T-1)/(C-1))\\ &=(\rho C)^{T+1}x-D\rho^{T+1}(C^{T+1}-1)/(C-1)\\ V_{T+1}(x)&=x+\sup_{0\leq y\leq x}(G_T(y)-y)\\ &\geqq x+\sup_{0\leq y\leq x}(\{(\rho C)^{T+1}y-D\rho^{T+1}(C^{T+1}-1)/(C-1\}-y)\\ &=(\rho C)^{T+1}x-D\rho^{T+1}(C^{T+1}-1)/(C-1).\end{aligned}$$

Theorem 2: If the condition of lemma 3 holds the value $V_T(x)$ diverges to infinity.

When neither of the conditions of lemmas 2 or 3 holds, the convergence or divergence of $V_T(x)$ cannot be ascertained immediately, and it may happen that it depends on the value of x.

Lemma 4: $V_T(x)$ is monotone increasing in x.
The proof of this lemma is obvious.

Theorem 3: If for some $x_0>0$, it is shown that

$$V_T(x_0)\to\infty \qquad \text{as } T\to\infty;$$

then for all x such that $x>x_0$ or

$$\int_{x_0}^{\infty} dF(z : x)>0$$

$$V_T(x)\to\infty \qquad \text{when } T \text{ tends to infinity.}$$

Corollary: If $V_T(x_0)\to\infty$, and there exists a sequnce of x's : $x_1<x_2\cdots<x_n$, such that $x_n\geqq x_0$ and that

$$\int_{x_i}^{\infty} dF(z : x_{i-1})>0 \qquad i=2, 3, \cdots,$$

then $V(x_1)\to\infty$.

3. THE EXISTENCE OF OPTIMAL STRATEGIES IN Ω_T

We shall next consider the problem whether there exists an optimal strategy which attains the value of the game discussed in the previous section. We shall denote a withdrawal strategy at the initial period of Ω_T as $w_T(x)$, then whole of a strategy in Ω_T can be identified with a sequence $\{w_k(x)\}$ $\quad k=1, 2, \cdots T$.

Lemma 4: For all T and for any $\varepsilon>0$, there exists a sequence $\{w_k(x)\}$ which is $\varepsilon-$best, that is, which attains a value not smaller than $V_T(x)-\varepsilon$.

Proof: Let $w_k(x)=w^0$ be such that

$$G_{T-1}(y^0)-y^0\geqq \sup\,(G(y)-y)-\frac{\varepsilon}{2^k} \qquad \text{wherer } y_0=x-w^0$$

then the sequence $\{w_k(x)\}$ satisfies the condition.

Theorem 4: If $V_T(x)$ converges to a function $V(x)$, then there exists a strategy in Ω which attains a value not smaller than $V(x)-\varepsilon$.

We shall hereafter assume that the distribution of z, *i. e.*, $F(z : x)$ is independent of x, and will be denoted by $F(z)$.

Under this assumption it holds that

Lemma 5: $V_T(x)-V_{T-1}(x)$ and $G_T(x)-G_{T-1}(x)$ are monotone non decreasing for $x>0$, for $T=1, 2, \cdots\cdots$

Proof: We shall proceed by induction.
It is evident that

$$V_1(x)-V_0(x)=\sup_{0\leq y\leq x}(G_0(y)-y)$$

is monotone in x.

If $V_T(x)-V_{T-1}(x)$ is monotone non decreasing,

$$G_T(x)-G_{T-1}(x)=\rho\int_{-x}^{\infty}\{V_T(x+z)-V_{T-1}(x+z)\}\mathrm{dF}(z)$$

is also. We shall show that the monotonicity of the left side of the above implies in turn that of

$$V_{T+1}(x)-V_T(x)=\sup_{0\leq y\leq x}(G_T(y)-y)-\sup_{0\leq y\leq x}(G_{T-1}(y)-y).$$

Let $x_1>x_2>0$, then $\sup_{0\leq y\leq x_1}(G_T(y)-y)=\max\{\sup_{0\leq y\leq x_2}(G_T(y)-y), \sup_{x_2\leq y\leq x_1}(G_T(y)-y)\}$ If $\sup_{0\leq y\leq x_1}(G_{T-1}(y)-y)=\sup_{0\leq y\leq x_2}(G_{T-1}(y)-y)$, then

$$\begin{aligned}V_{T+1}(x_1)-V_T(x_1)&\geqq\sup_{0\leq y\leq x_2}(G_T(y)-y)-\sup_{0\leq y\leq x_2}(G_{T-1}(y)-y)\\&=V_{T+1}(x_2)-V_T(x_2)\,;\end{aligned}$$

and if $\sup_{0\leq y\leq x_2}(G_{T-1}(y)-y)>\sup_{0\leq y\leq x_2}(G_{T-1}(y)-y)$

then there exists an x_0 such that

$$x_1>x_0>x_2 \quad \text{and} \quad G_{T-1}(x)-x_0>G_{T-1}(y)-y \qquad \text{for all } 0\leqq y\leqq x_2.$$

Since $G_T(x)-G_{T-1}(x)$ is monotone,

$$\begin{aligned}G_T(x)-x_0&\geqq G_T(y)-y-(G_{T-1}(y)-y)+G_{T-1}(x_0)-x_0\\&>G_T(y)-y \qquad\qquad \text{for all } 0\leqq y\leqq x_2.\end{aligned}$$

Consequently,

$$\begin{aligned}&\sup_{0\leq y\leq x_1}(G_T(y)-y)-\sup_{0\leq y\leq x_1}(G_{T-1}(y)-y)\\&\geq\sup_{x_2\leq y\leq x_1}(G_T(y)-y)-\sup_{x_2\leq y\leq x_1}(G_{T-1}(y)-y)\\&\geq\inf_{x_2\leq y\leq x_1}(G_T(y)-G_{T-1}(y))\\&\geq\sup_{0\leq y\leq x_2}(G_T(y)-G_{T-1}(y))\\&\geq\sup_{0\leq y\leq x_2}(G_T(y)-y)-\sup_{0\leq y\leq x_2}(G_{T-1}(y)-y)\end{aligned}$$

form which it follows that $V_{T+1}(x_1)-V_T(x_1)\geq V_{T+1}(x_2)-V_T(x_2)$.

Generally there does not necessarily exist an optimal strategy in Ω_T, but if $G_T(x)$ is right continuous, there is an optimal strategy, for then $G_T(x)-x$ will have no negative jumps since $G_T(x)$ is monotone non decreasing, and $\sup\limits_{0\leq y\leq x}(G_T(y)-y)$ is attained by some value of y.

However, optimal strategy may not exist even in the simplest case Ω_1. It is easy to see that this actually happens when $G_0(y)-y$ is monotone decreasing for $0<y\leqq x$ and $\lim\limits_{y\to 0} G_0(y)\neq 0$. This circumstance is due to the assumption that as long as we have however small amount of money we can continue the game; but it is not perfectly realistic, and the assumption that we can continue the game unless our money gets *below* some fixed amount K is easier to handle and may seem more realistic. We assume K is equal to zero, for it causes no essential loss of generality. (Though some complexity may occur about the "clearance".) In this case the definitions of $G_T(x)$ and $V_T(x)$ are slightly altered so that

$$G_T(x)=\rho\int_{-x}^{\infty} V_T(x+z)\,\mathrm{dF}(z) \qquad \text{for } x\geq 0.$$

and

$$V_T(x)=x+\sup_{0\leq y\leq x}(G_{T-1}(y)-y) \qquad \text{for } x\geq 0.$$

Lemma 6: $G_T(x)$ and $V_T(x)$ defined as above are right continuous for $T=0, 1, 2, \cdots\cdots$

Proof: $G_0(x)=\rho\int_{-x}^{\infty}(x+z)dF(z)=\rho x(1-F(-x-0))+\rho\int_{-x}^{\infty} z\,dF(z)$ is certainly right continuous.

It is easy to see that the right continuity of $G_{T-1}(x)$ implies that of $V_T(x)$; and since $V_T(x+\Delta x+z)$ is monotone in Δx, $\lim\limits_{\Delta x\to +0} V_T(x+\Delta x+z)=V_T(x)$ implies $\lim\limits_{\Delta x\to +0} G_T(x+\Delta x)=G_T(x)$

Thus it is shown that $\sup\,(G_T(y)-y)$ is attained in the interval $[0, x]$, and there there exists avalue $y_T{}^0$ such that $y_T{}^0=\min\{y^*: G_T(y^*)-y^*=\sup\,(G_T(y)-y)\}$

Lemma 7: $y_T{}^0$ for $T=1, 2, \cdots\cdots$ are monotone non decreasing.

Proof:

$$\begin{aligned}&G_T(y_T{}^0)-G_{T-1}(y_T{}^0)\\ &=\max_y(G_T(y)-y)-(G_{T-1}(y_T{}^0)-y_T{}^0)\\ &\geqq G_T(y)-y_{T-1}{}^0-\max_y(G_{T-1}(y)-y)\\ &=G_T(y_{T-1}{}^0)-G_{T-1}(y_{T-1}{}^0)\end{aligned}$$

which, combined with the monotonicity of G_T-G_{T-1}, establishes the lemma.

Theorem 5: The functions $V(x)$ and $V(x)$ are right continuous, and if we put $y^0=\lim_{T\to\infty} y_T{}^0$, it holds that

$$G(y^0)-y^0=\max_{0\leq y\leq x}(G(y)-y).$$

Proof: The uniformity of convergence of $G_T(x)$ and $V_T(x)$ assures the theorem.

4. THE EXISTENCE OF OPTIMAL STRATEGY IN Ω

We shall now prove the following

Theorem 6: If $G_T(x)$ and $V_T(x)$ converge uniformly to $G(x)$ and $V(x)$ respectively, and let $\tilde{w}=\{w^0(x)\}$ be a stationary strategy such that $w^0(x)=x-y^0$, and $G(y^0)-y^0=\max(G(y)-y)$, and y^0 is bounded. Then $\tilde{w}$ is an optimal strategy for Ω_T.

Proof: We shall show that

$$V(x,\tilde{w})=\lim_T V_T(x,\tilde{w})=\lim_T V_T(x)=V(x).$$

Let ε be any positive constant, then for sufficiently large T

$$\begin{aligned}&\sup_{0\leq y\leq x}(G_T(y)-y)-(G_T(y^0)-y^0)\\ &\qquad=\sup_{0\leq y\leq x}(G_T(y)-y)-\sup\,(G(y)-y)+G(y^0)-G_T(y^0)\\ &\qquad\leqq \sup\,(G_T(y)-G(y))+G(y^0)-G_T(y^0)<\varepsilon.\end{aligned}$$

Put $M_T=\sup_x (V_T(x)-V_T(x,\tilde{w})$, since

$$\begin{aligned}V_T(x)-V_T(x,\tilde{w})&=\sup_{0\leq y\leq x}(G_{T-1}(y)-y)\\ &\quad-(\rho\int V_{T-1}(y_0+z:\tilde{w})\,\mathrm{dF}(z:x)-y^0)\\ &=\sup_{0\leq y\leq x}(G_{T-1}(y)-y)-(G_{T-1}(y^0)-y^0)\\ &\quad+(\rho\int\{V_{T-1}(y^0+z)-V_{T-1}(y^0+z\,;\tilde{w})\}\mathrm{dF}(z:x),\end{aligned}$$

it is shown that $M_T=C_T+\rho M_{T-1}$, where C_T is a positive constant and $C_T\to 0$ as T.

It is easily seen that $V_0(x,w)=x-y^0$, and since $V_0(x)-V_0(x,w)=y^0$ is bounded, M_0 is finite. And in view of the lemma below, it is shown that $M_T\to 0$, which completes the proof.

Lemma 8: If for two sequences of positive constants $\{M_T\}$ and $\{C_T\}$

$$M_T \leqq C_T + \rho M_{T-1} \qquad T=1, 2, \cdots$$
$$0<\rho<1$$

then $\lim C_T=0$ implies $\lim M_T=0$.

Proof: Let $N_T=M_T/\rho^T$ then $N_T-N_{T-1} \leqq C_T/\rho^T$, from which it it seen that
$$M_T \leqq \rho^T N^T \leqq \sum_{k=0}^{T} \rho^{T-k} C_k + \rho^T M_0.$$

Assume that for $k>K$, $C_k<\varepsilon$, then for $T>K$,

$$M_T \leqq \rho^T \sum_{k=1}^{K} C_k/\rho^k + \sum_{K+1}^{T} \rho^{T-k}\varepsilon + \rho_T M_0$$
$$\leqq \rho^T \sum_{k=1}^{K} C_k/\rho^k + \varepsilon/(1-\rho) + \rho^T M_0.$$

Since ε can be taken arbitrarily small, the lemma is proved.

Corollary: If $F(z:x)$ is indepent of x, theorem 6 holds.

Proof: What remains to be proved is the boundedness of y^0.

In a similar way like in lemma 2 and theorem 1, it is easily proved that $G(x) \leqq x+C$, from which it follows that $y^0 \leqq C/(1-\rho)$, since for $y>C/(1-\rho) G(y)-y$ becomes negative.

REFERENCE

(1) K. Miyasawa; An economic survival game. *this issue.*