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THE BUILD-UP TIME OF EQUILIBRIUM WAITING TIME

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§ 1. INTRODUCTION

In many applications of the queuing theory, it will be desired to estimate the time required to approach very close to the asymptotic steady state. If the time is relatively short, we may use some steady-state's result. But there are a few investigations on this subject to date ([1], [5], [7]) except so-called finite time treatments. It seems that the time-dependent studies have been done completely in the case of simple queue from the theoretical view-point, but these results are too complicated for the practical uses. Again, the results due to Pollaczek [7] on the convergence order to the steady state distribution of the waiting time are also not applicable directly in the practical cases.

For the sake of these circumstances, some rough but simple indicators representing the above time will be required for the practical applications of the queuing theory. Recently, Davis [5] proposed a notion of the build-up time of waiting lines to meet the requirement.

In the present paper, we shall introduce an analogous indicator and discuss on it in the case $M/G/1$.

§ 2. THE BUILD-UP TIME OF THE WAITING LINES

For the convenience' sake, we shall quote here the notion and some properties of the build-up time of the waiting lines due to Davis [5] and remark something on these. He dealt with a simple queue system (Poisson arrivals with constant mean rate, λ, exponential service times with common mean b, constant number s of available channels).

(i) If the system is empty at the initial time, the mean number in the queue system $M_1(t)$ at time t is nondecreasing with t and converges to a limit $M_1(\infty)$ as $t\to\infty$. Hence the normalized mean $M_1(t)/M_1(\infty)$ is a distribution-type function.

(ii) (Definition) In these cases where the mean number in the system $M_1(t)$ is a bounded non-decreasing function of t, and where $M_1(0)=0$, the nominal build-up time is

$$T=\left[\int_0^\infty t dM_1(t)\right] \Big/ \left[\int_0^\infty dM_1(t)\right].$$

(iii) If $s=1$ and $b=1$, then $T=1/(1-\lambda)^2$

(iv) He calculated the exact form of T for $s\geqq 1$ and the approximation formulas when $M_1(\infty)\to\infty$. However, these reaults will not be quoted here, since these are not directly concerned with the below.

(v) In the last numerical example of his paper, he conjectures that it takes a time $2T$ to $3T$ before the mean number in the system has reached a value within, say, 10 percent of the asymptotic (steady state's) value.

By the way, it seems that there is a petty mistake of calculation in the derivation of the eq. (29) in [5]. But if we correct it, the final expression of T will be too complicated for the practical uses. Only the case $s=1$, the final expression is simple and valid as quoted in above (iii), because, the eq. (29) may not contribute the quantity T in this case*).

Thus, we shall deal with an analogous indicator in this paper to avoid the complicated finial form of it and to generalize the distribution of the service time.

§ 3. SOME PROPERTIES OF THE MEAN WAITING TIME

In this paper, we shall use the mean virtual waitnig time $E\{W(t)\}$ instead of the mean number in the system $M_1(t)$. Hence, we shall devote this section to discuss some properties of $E\{W(t)\}$.

Throughout this paper, we shall consider the queuing system $M/G/1$, and assume that the system is empty at the initial time. Denote $B(v)$ as the service time distribution with their moments $b_i\equiv\int_0^\infty v^i dB(v)$, if exist. In this paper we shall assume that $b_i (i\leqq 4)$ always exist and $\lambda b_1<1$ to be hold. Furthermore, let $W(t)$ be the waiting time in the queue of a customer if he arrived at time t, which is a notion proposed

*) if $s=1$, the integral in (28) will vanish, since $\sum_0^1 k(1-k)\prod_k=0$.

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by Takács [8] for the first time. The exact definition of $W(t)$ is as follows:

if $t_n<t<t_{n+1}(n=0, 1, 2, \cdots\cdots, t_0=0)$, then

$$W(t)=\begin{cases} W(t_n)-(t-t_n) & \text{if } W(t_n)>t-t_n \\ 0 & \text{if } W(t_n)\leqq t-t_n \end{cases} \tag{3. 1}$$

and if $t=t_n$, then

$$W(t_n)=W(t_n-0)+Y_n \tag{3. 2}$$

where t_n and Y_n denote the arrival moment and the service time of the n-th arrival respectively. Obviously, $W(t_n-0)$ means the waiting time of n-th customer in the literal sense.

Beneš [3] showed that if $w_0=E\{W(0)\}$, $E\{W(t)\}$ will be expressed by the following formula

$$E\{W(t)\}=\int_0^t [P\{W(u)=0|W(0)\}-(1-\lambda b_1)]du+w_0. \tag{3. 3}$$

And the Laplace transform $M^*(\tau)$ or $E\{W(t)\}$ is

$$M^*(\tau)=\frac{w_0}{\tau}+\frac{E\{e^{-\eta W(0)}\}}{\tau\eta}-\frac{1-\lambda b_1}{\tau^2}. \tag{3. 4}$$

This is a result by Beneš [3] and based on the fact that

$$\int_0^\infty e^{-\tau t}P\{W(t)=0|W(0)\}dt=\frac{E\{e^{-\eta W(0)}\}}{\eta}, \tag{3. 5}$$

which was deduced by him and others. In this formula, η is a real unique root such that $0<\eta<1$ of the equation

$$\lambda+\tau=\eta+\lambda B^*(\eta), \tag{3. 6}$$

where $B^*(\eta)$ is the Laplace transform of the service time distribution $B(v)$.

In our case, since $W(0)=0$ is assumd, (3. 4) will be simplified as

$$L(\tau)=\int_0^\infty e^{-\tau t}dE\{W(t)\}=\frac{1}{\eta}-\frac{1-\lambda b_1}{\tau}. \tag{3. 7}$$

Now, we shall note the following properties of $E\{W(t)\}$ and $E\{W(t)^2\}$.

LEMMA. *If $W(0)=0$, that is, the system is empty at the initial time, then, for $t>0$*

(i) *$E\{W(t)\}$ is a strictly increasing function of t,*

(ii) *$E\{[W(t)]^2\}$ will be represented as*

$$E\{[W(t)]^2\}=2(1-\lambda b_1)\int_0^t \left[\frac{\lambda b_2}{2(1-\lambda b_1)}-E\{W(u)\}\right]du, \tag{3. 8}$$

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(iii) $E\{[W(t)]^2\}$ *is a strictly increasing function of* t,

(iv) $E\{[W(t)]^2\}$ *is a concave function of* t.

PROOF.

(i) Since Beneš [4] showed the following expression:

$$P\{W(t)=0|W(0)\} = \frac{1}{t}E[\max(0, t-K(t))] + \frac{1}{t}W(0)P\{K(t)\leqq t\}, \tag{3. 9}$$

we have

$$P\{W(t)=0|W(0)=0\} = E\left[\max\left(0, 1-\frac{K(t)}{t}\right)\right] \geqq E\left(1-\frac{K(t)}{t}\right) = 1-\lambda b_1, \tag{3. 10}$$

where $K(t)$ is the process called by him as cumulative load. If we assume the system $M/G/1$, then $K(t)$ will be represented as compound Poisson processed. Thus the last equality will be deduced directly, since

$$E(K(t)) = \lambda b_1 t. \tag{3. 11}$$

In the above inequality, the equal sign may be deleted because the probability that the system is empty will be positive. Hence, from (3. 3) and (3. 10)

$$E\{W(t)\} = \int_0^t [P\{W(t)=0\ W(0)=0\} - (1-\lambda b_1)]dt \tag{3. 12}$$

is a increasing function of t.

(ii) (3. 8) will be proved more generally in the Appendix I.

(iii), (iv) These are obvious from (i) and (ii).

§ 4. THE BUILD-UP TIME OF WAITING TIME

Since $E\{W(t)\}$ and $E([W(t)]^2\}$ are monotone increasing function, we can define the build-up time concerning with them analogously to § 4. (ii). We shall define as

$$T_{w,1} = \int_0^\infty t dE\{W(t)\}/E\{W(\infty)\} \tag{4. 1}$$

and call it the build-up time of waiting time. Of course, quite analogously, we can consider

$$T_{w,2} = \int_0^\infty t dE\{[W(t)]^2\}/E\{[W(\infty)]^2\} \tag{4. 2}$$

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and so on. However, we shall here restrict ourselves to deal with $T_{w,1}$.

Using (3. 7) we can rewrite as

(4. 3) $$T_{w,1}=-L'(0)/L(0).$$

Now, from (3. 7) we have

(4. 4) $$\frac{dL(\tau)}{d\tau}=-\frac{1}{\eta^2}\frac{d\eta}{d\tau}+\frac{1-\lambda b_1}{\tau^2},$$

and by (3. 6)

$$1=\frac{d\eta}{d\tau}+\lambda\frac{dB^*(\eta)}{d\eta}\frac{d\eta}{d\tau},$$

from which

(4. 5) $$\frac{d\eta}{d\tau}=\frac{1}{1+\lambda B^{*\prime}(\eta)}.$$

On the other hand, since

(4. 6) $$\begin{aligned}B^*(\eta)&=\int_0^\infty e^{-\eta u}dB(u)\\&=\int_0^\infty\left[1-\eta u+\frac{\eta^2u^2}{2}-\frac{\eta^3u^3}{3}+o(\eta^3)\right]dB(u)\\&=1-\eta b_1+\frac{\eta^2}{2}b_2-\frac{\eta^3}{3}b_3+o(\eta^3),\quad(\eta\to 0)\end{aligned}$$

inserting the relation (4. 6) into (3. 6), we have

(4. 7) $$\tau-\eta+\eta\lambda b_1-\frac{\eta^2\lambda}{2}b_2+\frac{\eta^3\lambda}{3}b_3+o(\eta^3)=0,$$

from which we have

(4. 8) $$\eta=o(\tau)$$

or

(4. 9) $$\eta=\frac{\tau}{1-\lambda b_1}\left[1-\frac{\lambda b_2}{2(1-\lambda b_1)^2}\tau+\frac{\lambda b_3(1-\lambda b_1)+3\lambda^2b_2^2}{6(1-\lambda b_1)^4}\tau^2\right]+o(\tau^3).$$

Thus we have

(4. 10) $$\begin{aligned}1+\lambda B^{*\prime}(\eta)&=1+\lambda\{B^{*\prime}(0)+\eta B^{*\prime\prime}(0)+\frac{\eta^2}{2}B^{*\prime\prime\prime}(0)+o(\eta^2)\}\\&=1-\lambda b_1+\lambda b_2\frac{\tau}{1-\lambda b_1}-\frac{\lambda^2b_2^2\tau^2}{2(1-\lambda b_1)^3}-\frac{\lambda b_3}{2}\frac{\tau^2}{(1-\lambda b_1)^2}+o(\tau^2)\\&=(1-\lambda b_1)\left\{1+\frac{\lambda b_2}{(1-\lambda b_1)^2}\tau-\left(\frac{\lambda^2b_2^2}{2(1-\lambda b_1)^4}+\frac{\lambda b_3}{2(1-\lambda b_1)^3}\right)\tau^2+o(\tau^2)\right\}.\end{aligned}$$

Hence, inserting (4. 5) and (4. 10) into (4. 4) we have

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$$(4.11)\quad \frac{dL(\tau)}{d\tau}=\frac{1-\lambda b_1}{\tau^2}-\frac{1}{\eta^2(1+\lambda B^{*\prime}(\eta))}$$

$$=\frac{1-\lambda b_1}{\tau^2}-\frac{1-\lambda b_1}{\tau^2}\left[\left\{1+\frac{\lambda b_2\tau}{(1-\lambda b_1)^2}-\left(\frac{\lambda^2 b_2^2}{2(1-\lambda b_1)^4}\right.\right.\right.$$

$$\left.\left.+\frac{\lambda b_3}{2}\frac{1}{(1-\lambda b_1)^3}\right)\tau^2+o(\tau^2)\right\}\left\{1-\frac{\lambda b_2\tau}{(1-\lambda b_1)^2}\right.$$

$$\left.\left.+\tau^2\left(\frac{\lambda^2 b_2^2}{4(1-\lambda b_1)^4}+\frac{\lambda^2 b_2^2}{(1-\lambda b_1)^4}+\frac{\lambda b_3}{3(1-\lambda b_1)^3}\right)+o(\tau^2)\right\}\right]^{-1}$$

$$=\frac{1-\lambda b_1}{\tau^2}-\frac{1-\lambda b_1}{\tau^2}\frac{1}{\left\{1-\left(\frac{\lambda^2 b_2^2}{4(1-\lambda b_1)^4}+\frac{\lambda b_3}{6(1-\lambda b_1)^3}\right)\tau^2+o(\tau^2)\right\}}$$

and

$$(4.12)\quad -L'(0)=\frac{\lambda^2 b_2^2}{4(1-\lambda b_1)^3}+\frac{\lambda b_2}{6(1-\lambda b_1)^2}.$$

Then we have

$$(4.13)\quad T_{w,1}=\frac{\dfrac{\lambda^2 b_2^2}{4(1-\lambda b_1)^3}+\dfrac{\lambda b_3}{6(1-\lambda b_1)^2}}{\dfrac{\lambda b_2}{2(1-\lambda b_1)}}$$

$$=\frac{\lambda b_2}{2(1-\lambda b_1)^2}-\frac{b_3}{3(1-\lambda b_1)b_2}.$$

Example 1. If the service time obeys the negative exponential distribution with parameter μ, the moments b_i will be as follows:

$$b_1=\frac{1}{\mu},\quad b_2=\frac{2}{\mu^2},\quad b_3=\frac{6}{\mu^3}.$$

Then putting $\rho=\lambda b_1$, from (4. 13) we have

$$(4.14)\quad T_{w,1}=\frac{b^2}{(1-\rho)^2}$$

which consists with T in **§ 2.** ((iii)). This fact may be conjecturable by some known results.

In fact in the case of $M/M/1$ the Laplace transform of the generating function of $P_n(t)$ is given by*)

$$(4.15)\quad \Pi^*(z,\tau)=\int_0^\infty e^{-\tau t}dt\sum_{n=0}^\infty P_n(t)z^n$$

*) Bailey [2]

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$$=\frac{z-\mu(1-z)P_0^*(\tau)}{\tau z-(1-z)(\mu-\lambda z)},$$

where $P_n(t)$ means the probability that the number of customers in the system at time t is n under the initial condition $P_0(0)=1$. And, $P_0^*(\tau)$ is the Laplace transform of $P_0(t)$. From (4. 15) we can easily see that

(4. 16) $$\left[\frac{\partial \Pi^*(z,\ \tau)}{\partial z}\right]_{z=1}=\frac{\tau\mu P_0^*(\tau)+(\lambda-\mu)}{\tau^2}.$$

On the other hand, the corresponding Laplace transform of the moment generating function of $W(t)$ is given by*)

(4. 17) $$\varphi^*(s,\ \tau)=\frac{\{1-sP_0^*(\tau)\}(\mu+s)}{(\mu+s)(\tau-s)+\lambda s},$$

from which we have

(4. 18) $$\left[\frac{\partial\varphi^*(s,\ \tau)}{\partial s}\right]_{s=0}=\frac{\tau\mu P_0^*(\tau)+(\lambda-\mu)}{\mu\tau^2}.$$

Combining (4. 16) and (4. 18) we may assert that

(4. 19) $$E(\eta(t))/E(\eta(\infty))=E(W(t))/E(W(\infty)),$$

where $\eta(t)$ is the number of customers in the system at the time t. Thus we can see that $T_{w,1}=T$ in the queuing system $M/M/1$. Moreover, if we remember that the mean queue size in the equilibrium state is $\rho E(\eta(\infty))$, we can see that the two expressions (4. 16) and (4. 18) will mean the relation $E(L)=\lambda E(W)$ as was noted in [6].

Example 2. If the service time is constant $(=b)$ we can see readily that

$$b_1=b,\ b_2=b^2,\ b_3=b^3.$$

Then, we have putting $\rho=\lambda b$,

(4. 20) $$T_{w,1}=\frac{2+\rho}{6(1-\rho)^2}b.$$

since $\rho<1$, $(2+\rho)/6<\frac{1}{2}$ and we can see that the build-up time for $M/M/1$ will be larger than twice of it for $M/D/1$.

§ 5. SOME CONSIDERATIONS

The build-up time T or $T_{w,1}$ is a rough indicator, and it is vague how use the value to estimate the elapsed time before the $E\{W(t)\}$

*) Beneš [3]. This is deduced from (A. 1) in the Appendix of this paper. Note that $B^*(s)=\frac{\mu}{\mu+s}$ in our case.

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approaches within the fraction ε of $E\{W(\infty)\}$. Davis conjectured that the time is $2T$ or $3T$ when $\varepsilon=0.1$(§ 2 (v)) based upon the infinite server case.

We shall consider here the same subject using some properties of $E\{W(t)\}$ and others.

Takács and Pollaczek showed that the distributions of $W(t)$ and $W(t_n-0)$ converges to the same limit distribution and the convergence order of the later is exponential small order with n.

From these facts and properties of $E\{W(t)\}$, we may conjecture that the curve of $E\{W(t)\}/E\{W(\infty)\}$ is resemble to a exponential distribution as illustrated in Fig. 1.

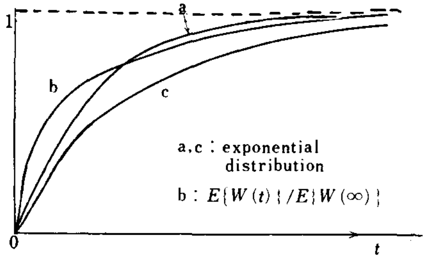

Fig 1

Now an exponential distribution with the mean T_0(a, in Fig, 1) will approach within the fraction ε of $E\{W(\infty)\}$ at $T_0 \log \frac{1}{\varepsilon}$. This value will be 2. 3 T_0 if $\varepsilon=1/10$ and 1. 6 T_0 if $\varepsilon=1/5$ and so on. However, if the "coefficient of variation" of $E\{W(t)\}/E\{W(\infty)\}$ is larger than 1 (b, Fig. 1), which is the coefficient of variation of an exponential distribution, the above constant will must be choosen larger.

In many cases, the "coefficient of variation" of $E\{W(t)\}/E\{W(\infty)\}$ will beyond as we shall illustrate below two examples.

Example 3. The "coefficient of variation" c of $E\{W(t)\}/E\{W(\infty)\}$ may be given by

(5. 1) $$c^2+1=\frac{6\lambda^2b_2{}^2+6\lambda b_2b_3(1-\rho)+b_4(1-\rho)^2}{3\lambda b_2{}^2+2(1-\rho)b_3}$$

which will be deduced in Appendix II.

In the case of $M/M/1$, (5. 1) will be reduced to

(5. 2) $$c^2+1=2+2\rho,$$

from which we have $c=\sqrt{1+2\rho}>1$.

Example 4. In the case of $M/D/1$, (5. 1) will become

(5. 3) $$c^2+1=\frac{6(1+4\rho+\rho^2)}{(2+\rho)^2}=6-\frac{18}{(2+\rho)^2},$$

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from which we can see that $c<1$ for $0<\rho<\frac{3\sqrt{2}-4}{2}$ (about 0.121) and $\sqrt{3}>c>1$ for $\rho>\frac{3\sqrt{2}-4}{2}$.

From the above consideration, we shall support the conjecture (§ 2 (v)) by Davis, in our case too.

Now, perhaps, we shall be perplexed to choose ε. If $\mathrm{Var}\{W(\infty)\}$, is relatively small as compared with $E\{W(\infty)\}$, it will be meaningful that a small value of ε to be taken. However, as we shall show in the following, $\sqrt{\mathrm{Var}\{W(\infty)\}}$ will be considerably large. Thus, in the usual cases, we shall must use the mean waiting time in the steady state as a rough indicator.

Remark 1. Since we have well known formulas:

$$\left\{\begin{aligned} E\{W(\infty)\} &= \frac{\lambda b_2}{2(1-\lambda b_1)} \\ \mathrm{Var}\{W(\infty)\} &= \frac{\lambda b_3}{3(1-\lambda b_1)}+\frac{\lambda_2 b_2{}^2}{4(1-\lambda b_1)^2} \end{aligned}\right. \tag{5. 4}$$

we can calculate the coefficient of variation $C_{W(\infty)}$ of $W(\infty)$ as follows

$$C^2_{W(\infty)}=1+\frac{4b_3(1-\lambda b_1)}{3\lambda b_2{}^2}. \tag{5. 5}$$

While, we assume that $\lambda b_1<1$ and the service time is always possitive, then we can see

$$C_{W(\infty)}>1. \tag{5. 6}$$

APPENDIX

I. (Derivation of (3. 8))

When denote the Laplace transform of $P\{W(t)\leqq x\}$ by $\varphi(s, t)$, the forward Kolmogorov equation implies that

$$\frac{\partial\varphi(s, t)}{\partial t}=\varphi(s, t)[s-\lambda(1-B^*(s))]-sP(0, t), \tag{A. 1}$$

where

$$B^*(s)=\int_0^\infty e^{-st}dB(t)$$

and

$$P(w, t)=P\{W(t)\leqq w\}. \tag{A. 2}$$

This formula was given in [3].

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Differentiating (4. 2) with respect to s, we have

$$\text{(A. 3)} \qquad \frac{\partial^2\varphi(t,\ s)}{\partial s \partial t} = \frac{\partial\varphi(s,\ t)}{\partial s}[s-\lambda(1-B^*(s))] + \varphi(s,\ t)\left[1+\lambda\frac{dB^*(s)}{ds}\right] - P(o,\ t).$$

If let $s\to 0$ here, and integrate it in regard to t, then we have (3. 4) taking the Laplace transform. But, we must differentiate (3. 4) once more in s for the sake of our present purpose. Thus,

$$\text{(A. 4)} \qquad \frac{\partial^3\varphi(s,\ t)}{\partial s^2 \partial t} = \frac{\partial^2\varphi(s,\ t)}{\partial s^2}[s-\lambda(1-B^*(s))] + 2\frac{\partial\varphi(s,\ t)}{\partial s}\left[1+\lambda\frac{dB^*(s)}{ds}\right] + \varphi(s,\ t)\lambda\frac{d^2B^*(s)}{ds^2}.$$

Letting $s\to 0$, we have

$$\text{(A. 5)} \qquad \frac{\partial}{\partial t}E\{[W(t)]^2\} = -2E\{W(t)\}(1-\lambda b_1)+\lambda b_2.$$

Hence,

$$E\{[W(t)]^2\} = w_0^{(2)} - 2(1-\lambda b_1)\int_0^t \left[E\{W(t)\} - \frac{\lambda b_2}{2(1-\lambda b_1)}\right]dt.$$

In our case, we assumed that $W(0)=0$, then (A. 5) will be reduced to (3. 8). Furthermore, we shall note that the Laplace transform of $E\{[W(t)]^2\}$ can be found from (A. 5) directly such that

$$\text{(A. 6)} \qquad V^*(\tau) = \frac{w_0^{(2)}}{\tau} - 2(1-\lambda b_1)\frac{1}{\tau}\left\{M^*(\tau) - \frac{\lambda b_2}{2(1-\lambda b_1)}\frac{1}{\tau}\right\}$$

where

$$w_0^{(2)} = E\{[W(0)]^2\}.$$

In the discussion, we have assumed the differentiability and integrability of $\varphi(s,\ t)$ and $B^*(s)$. But it will be easily shown that these properties are guaranteed under the condition $b_i<\infty (i=1,\ 2,\ 3)$.

II. (Derivation of (5. 1))

Since

$$\text{(A. 7)} \qquad c^2+1 = \frac{L''(0)}{L(0)} \Big/ \left(\frac{L'(0)}{L(0)}\right)^2.$$

We must calculate $L''(0)$ at first. Differentiating (4. 4), we have

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(A. 8)
$$\frac{d^2L(\tau)}{d\tau^2}=-\frac{1}{\eta^2}\frac{d^2\eta}{d\tau^2}+\frac{2}{\eta^3}\left(\frac{d\eta}{d\tau}\right)^2-\frac{2(1-\lambda b_1)}{\tau^3}$$
$$=\frac{\lambda B^{*\prime\prime}(\eta)}{\eta^2(1+\lambda B^{*\prime}(\eta))^3}+\frac{2}{\eta^3(1+\lambda B^{*\prime}(\eta))^2}-\frac{2(1-\lambda b_1)}{\tau^3}.$$

An analogous calculation to (4. 10) and (4. 11) implies

(A. 9)
$$L''(0)=\frac{\lambda b_4}{12(1-\lambda b_1)^3}+\frac{\lambda^2 b_2 b_3}{2(1-\lambda b_1)^4}+\frac{\lambda^3 b_2{}^3}{2(1-\lambda b_1)^5}$$

using a finer formula than (4. 9) of η expanding in terms of τ:

(A. 10)
$$\eta=\frac{\tau}{1-\lambda b_1}\left[1-\frac{\lambda b_2\tau}{2(1-\lambda b_1)}+\frac{\tau}{2(1-\lambda b_1)^3}\left\{\frac{\lambda b_3}{3}+\frac{\lambda^2 b_2{}^2}{1-b_1}\right\}\right.$$
$$\left.-\frac{\tau^3}{4(1-\lambda b_1)^4}\left\{\frac{\lambda b_4}{6}+\frac{5\lambda^2 b_2 b_3}{3(1-\lambda b_1)}+\frac{3\lambda^3 b_2{}^2}{2(1-\lambda b_1)^2}\right\}\right]+o(\tau^3).$$

Inserting (A. 9) and (4. 12) in (A. 7), we have (5. 1).

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