ON THE NUMBER OF SERVED CUSTOMERS IN A BUSY PERIOD

HIDENORI MORIMURA

Tokyo Institute of Technology
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§ 1. INTRODUCTION

In many applications of queuing theory, it is well known that Monte Carlo methods are useful instead of analytical methods. But when we appeal the method, we may hesitate to decide the number of the Monte Carlo trials. Some rough criteria for the decision are to estimate the deviation from the equilibrium values of some characters. Along this direction there are a few of investigations on waiting time in $M/G/1$ [7], and loss probability in $M/M/s$ [1]. Of course, finite time results are also useful. However, these results are too complicate for practical use.

Another rough criterion is the number of customers who have been served in a busy period, which will be readily observed in the process of the Monte Carlo experiments. If we know the distribution of the number of customers in a busy period in the equilibrium state, we may decide roughly whether the state produced by experiment resembles the equilibrium state or not. The investigations on busy period have been done by many authors. (Takács [9], Conolly [2, 3], Pollaczek [7], Prabhu [8], *inter alia*) But it seems that the number of customers has not been directly studied.

In this paper we shall point out a exact form of the expectation of the numbers in the general case with single server and derivate the distribution of the number in the case $M/M/1$. This disribution already was known by Takács, Pollaczek and Conolly. But these final results are different somewhat each other. It seems that Pollaczek's is true, which is consist with our result. Our method can be used in the general case, but unfortunately the concrete calculation in special cases is tedious. This situation resembles the Pollaczeks' method for busy period.

Furthermore, we shall give a numerical table of the expectation of the number of served customers in a busy period in the case $E_l/E_k/1$. It may be used in some Monte Corlo experiments.

§ 2. NOTATIONS AND A LEMMA

In this paper, we shall use the method of the imbedded queuing process of general type due to Kawata [5]. Then, first of all, some notaions and relations needed to do the following discussions will be introduced here.

As in [5], we shall use following notations. Let

$$t_0 < t_1 < t_2 < \cdots\cdots$$

be a sequence of instants when customers successively arrive at the service station, and set

$$t_j - t_{j-1} = X_j, \qquad j=1,\ 2,\ \cdots\cdots$$

which are interarrival times. Furthermore let $Y_j (j=1,\ 2,\ \cdots\cdots)$ be the service time which is required by the j-th customer who has arrived at the epoch t_{j-1}. Throughout the paper, we assume that each of $\{X_j\}$ and $\{Y_j\}$ is a sequence of independent random variables having identical distributions, and X_j and Y_j are also mutually independent. Set $Z_j = Y_j - X_j$ $(j=1,\ 2,\ \cdots\cdots)$ and

$$S_n = \sum_{j=1}^{n} Z_j,$$

$$a_n = P(S_1>0,\ S_2>0,\ \cdots\cdots,\ S_n>0), \qquad (n \geqq 1),$$

$$a_0 = 1,$$

$$b_n = P(S_n > 0), \qquad (n \geqq 1).$$

We assume also $-\infty < EZ_j < 0$.

Next, we shall denote the number of customrs who have been served in j-th busy period by $N_j (j=1,\ 2,\ \cdots\cdots)$. In the other words, we shall define these as follows. If

$$S_1>0,\ S_2>0,\ \cdots\cdots,\ S_{n_1}>0,\ S_{n_1+1} \leqq 0$$

we shall define as

$$N_1 = n_1 + 1,$$

and if $S_1 \leqq 0$, define as $N_1 = 1$. Furthermore, if

$$S_1'>0,\ S_2'>0,\ \cdots\cdots,\ S_{n_2}'>0,\ S_{n_2+1}' \leqq 0,$$

we shall define as

$$N_2 = n_2 + 1,$$

and if $S_1' \leqq 0$, define as $N_2 = 1$, where $S_j' = \sum_{N_1+1}^{N_1+j} Z_i$. If we repeat this definition successively we can define the sequence of positive and identically distributed random variables $\{N_j\}(j=1, 2, \cdots\cdots)$. It was shown under the assumption $-\infty<EZ_j<0$, that $P(N_j<\infty)=1$ to be hold. (e. g., Kiefer-Wolfowitz [6], in general.)

Thus we can write as

$$q_n=P(N_j=n)=a_{n-1}-a_n, \quad (j,\ n\geqq 1)$$

which means the probability that n customers will be served in a busy period.

Setting

$$A(s)=\sum_{n=0}^{\infty} a_n s^n, \quad |s|<1$$

and

$$B(s)=\sum_{n=1}^{\infty} b_n s^n, \quad |s|<1,$$

we have following lemma which has been shown in [5]. (Lemma 1−3, there)

LEMMA 1. *If* $-\infty<EZ_j<0$, *then* $\sum_{n=0}^{\infty} a_n<\infty$,

and

$$A(s)=\exp\left\{\sum_{n=1}^{\infty}\frac{b_n}{n}s^n\right\}, \quad |s|<1.$$

§ 3. EXPECTATION OF N_j IN GENERAL CASE AND ITS NUMERICAL TABLE IN $E_l/E_k/1$

First of all, we shall describe the following

THEOREM. *If* $-\infty<EZ_j<0$, *then*

(3. 1)
$$\begin{aligned} E(N_j)&=A(1-0)\\ &=\exp\left\{\sum_{1}^{\infty}\frac{b_n}{n}\right\}. \end{aligned}$$

PROOF

It is very easy to prove this theorem. Using the notations and the lemma given in the last section, we have

$$\begin{aligned} E(N_j)&=\sum_{n=1}^{\infty} nq_n\\ &=\sum_{n=1}^{\infty} n(a_{n-1}-a_n) \end{aligned}$$

$$= \sum_{n=0}^{\infty} a_n$$

And, since $\sum_{n=0}^{\infty} a_n < \infty$, from the Abel's teorem

$$E(N_j) = A(1-0)$$

or,

$$= \exp\left\{ \sum_{n=1}^{\infty} \frac{b_n}{n} \right\}.$$

which is (3. 1).

In the following, we shall calculate $A(s)$ in a special case $E_l/E_k/1$, that is, the case of Erlangian input and Erlangian service time.

Let

$$P(X_j < x) = \frac{(l\lambda)^l}{\Gamma(l)} \int_0^x t^{l-1} e^{-l\lambda t} dt \qquad (x>0) \tag{3. 2}$$

and

$$P(Y_j < x) = \frac{(k\mu)^k}{\Gamma(k)} \int_0^x t^{k-1} e^{-k\mu t} dt \qquad (x>0). \tag{3. 3}$$

Obviously, $EX_j = 1/\lambda$ and $EY_j = 1/\mu$. And the characteristic functions of the distribution (3. 2) and (3. 3) are

$$\left(\frac{l\lambda}{l\lambda - it}\right)^l \quad \text{and} \quad \left(\frac{k\mu}{k\mu - it}\right)^k,$$

respectively. Thus the characteristic function of S_n is

$$\left(\frac{k\mu}{k\mu - it}\right)^{kn} \left(\frac{l\lambda}{l\lambda + it}\right)^{ln} \tag{3. 4}$$

which will be decomposed in partial fraction as

$$\sum_{\nu=1}^{kn} \frac{A_\nu}{(k\mu - it)^\nu} + \sum_{\nu=1}^{ln} \frac{B_\nu}{(l\lambda + it)^\nu}. \tag{3. 5}$$

From (3. 4) and (3. 5), we have

$$A_\nu = \frac{(k\mu)^{kn} (l\lambda)^{ln}}{(l\lambda + k\mu)^{ln}} \binom{-ln}{kn-\nu} (-1)^{kn-\nu} \frac{1}{(l\lambda + k\mu)^{kn-\nu}}. \tag{3. 6}$$

If we take the inverse transformation of (3. 5), we have the distribution of S_n. When the inversion has been operated, it is evident that the first term of (3. 5) will give a density function of Γ type which differs from zero for positive argument and equal to zero for non-positive argument. Conversely, the second term of (3. 5) will give a density function of same type which differs from zero for non-positive argument and equals zero for positive argument. Thus for our present purpose to find b_n, only the first term is necessary. Then, putting $\rho=l\lambda/k\mu$, we have

$$b_n=\int_0^\infty dz\left\{\frac{(l\lambda)^{ln}}{(l\lambda+k\mu)^{ln}}e^{-k\mu z}\right.$$

$$\left.\times\sum_{\nu=1}^{kn}\binom{-ln}{kn-\nu}\left(\frac{-k\mu}{l\lambda+k\mu}\right)^{kn-\nu}\frac{(k\mu)^\nu z^{\nu-1}}{\Gamma(\nu)}\right\}$$

$$=\left(\frac{l\lambda}{l\lambda+k\mu}\right)^{ln}\sum_{\nu=1}^{kn}\binom{-ln}{kn-\nu}\left(\frac{-k\mu}{l\lambda+k\mu}\right)^{kn-\nu}$$

$$=\left(\frac{\rho}{1+\rho}\right)^{ln}\sum_{\nu=0}^{kn-1}\binom{-ln}{\nu}\left(\frac{-1}{1+\rho}\right)^{\nu}$$

$$=\left(\frac{\rho}{1+\rho}\right)^{(l+k)n-1}\sum_{\nu=0}^{kn-1}\binom{(l+k)n-1}{\nu}\left(\frac{1}{\rho}\right)^{\nu} \tag{3. 7}$$

$$=\frac{[(l+k)n-1]!}{(kn-1)!(ln-1)!}B_{\frac{\rho}{1+\rho}}(ln,\ kn), \tag{3. 8}$$

where

$$B_x(p,\ q)=\int_0^x t^{p-1}(1-t)^{q-1}dt.$$

Thus, we have

$$\log A(s)=\sum_{n=1}^\infty\frac{b_n}{n}s^n \tag{3. 9}$$

$$=k\sum_{n=1}^\infty\binom{(l+k)n-1}{kn}B_{\frac{\rho}{1+\rho}}(ln,\ kn)s^n$$

and

$$E(N_f)=\exp\left\{k\sum_{\nu=0}^\infty\binom{(l+k)n-1}{kn}B_{\frac{\rho}{1+\rho}}(ln,\ kn)\right\}. \tag{3. 10}$$

In above, we use the following lemma (for the derivation of (3. 7)) which is an interest elementary identity involving binomial coefficients.

LEMMA 2. *If k, l and n are positive integers, then*

$$\left(\frac{\rho}{1+\rho}\right)^{ln}\sum_{\nu=0}^{kn-1}\binom{-ln}{\nu}\left(\frac{-1}{1+\rho}\right)^{\nu}$$

$$=\left(\frac{\rho}{1+\rho}\right)^{ln}\sum_{\nu=0}^{kn-1}\frac{ln(ln+1)\cdots\cdots(ln+\nu-1)}{\nu!}\left(\frac{1}{1+\rho}\right)^{\nu}$$

$$=\left(\frac{\rho}{1+\rho}\right)^{(l+k)n-1}\sum_{\nu=0}^{kn-1}\binom{(l+k)n-1}{\nu}\rho^{-\nu}$$

PROOF Since the first equality is evident, we shall show the second relation. We have

$$\left(\frac{\rho}{1+\rho}\right)^{ln}\sum_{\nu=0}^{kn-1}\frac{ln(ln+1)\cdots\cdots(ln+\nu-1)}{\nu!}\left(\frac{1}{1+\rho}\right)^{\nu}$$

$$=\left(\frac{\rho}{1+\rho}\right)^{(l+k)n-1}\sum_{\nu=0}^{kn-1}\binom{ln+\nu-1}{\nu}(1+\rho)^{kn-1-\nu}\left(\frac{1}{\rho}\right)^{kn-1}$$

$$=\left(\frac{\rho}{1+\rho}\right)^{(l+k)n-1}\sum_{\nu=0}^{kn-1}\binom{ln+\nu-1}{\nu}\sum_{r=0}^{kn-1-\nu}\binom{kn-1-\nu}{r}\left(\frac{1}{\rho}\right)^{kn-1-r},$$

putting $kn-1-r=x$,

$$=\left(\frac{\rho}{1+\rho}\right)^{(l+k)n-1}\sum_{x=0}^{kn-1}\left(\frac{1}{\rho}\right)^{x}\sum_{\nu=0}^{x}\binom{kn-1-\nu}{x-\nu}\binom{ln-1+\nu}{\nu}.$$

Using the formula given by Feller [4] (p. 48, (9. 11)), we have

$$=\left(\frac{\rho}{1+\rho}\right)^{(l+k)n-1}\sum_{x=0}^{kn-1}\left(\frac{1}{\rho}\right)^{x}\binom{kn+ln-1}{x},$$

which is the second relation of this lemma.

By the way, we can calculate the numerical value of $E(N_f)$ based on (3. 10). But it is rather tedious work, then we shall present here a brief table of it for practical uses, while it is not sufficient in some cases.

l	k	λ/μ					
		0.1	0.2	0.3	0.4	0.5	0.6
1		1.111	1.250	1.429	1.667	2.000	2.500
2	1	1.030	1.103	1.216	1.376	1.617	1.967
	2	1.025	1.089	1.190	1.339	1.561	1.898
	3	1.023	1.083	1.180	1.323	1.537	1.866
3	1	1.013	1.063	1.152	1.287	1.493	1.804
	2	1.008	1.045	1.118	1.235	1.415	1.698
	3	1.007	1.039	1.104	1.212	1.381	1.649

When $l=1$ (that is, Poisson arrival) these values will be equal to $1/\left(1-\frac{\lambda}{\mu}\right)$ for all k. In the next section we shall show the fact in a special case $k=1$. However, we can assert the fact for all k as follows.

If we denote by the probability that the system is empty when the n-th customer has just arrived, we can get

(3. 11) $$p_n^{(0)}=q_np_0^{(0)}+q_{n-1}p_1^{(0)}+\cdots\cdots+q_1p_{n-1}^{(0)},$$

and from which

$$\lim_{n\to\infty} p_n^{(0)}=1/\sum_{n=1}^{\infty} np_n.$$

But, it is well known that $\lim_{n\to\infty} p_n^{(0)}=1-\lambda/\mu$ in the case of Poisson arrivals (for instance, [7]), then we can see that $E(N_j)=\sum nq_n=\frac{1}{1-\rho}$.

§ 4. DISTRIBUTION OF N_j IN THE CASE $M/M/1$

In this section, we shall proceed more concrete calculation under the constraint $l=k=1$. From (3. 9)

$$\log A(s)=\int_0^{\frac{\rho}{1+\rho}}\left\{\sum_{n=1}^{\infty}\frac{(2n-1)!}{(n-1)!n!}[(t(1-t))^{n-1}s^n]\right\}dt$$

$$=\int_0^{\frac{\rho}{1+\rho}}\left\{\frac{\left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\cdots\cdots\frac{1}{2}}{n!}2^{2n-1}\right.$$

$$\left.\times[(t(1-t))^{n-1}s^n]\right\}$$

$$=\int_0^{\frac{\rho}{1+\rho}}\left\{\sum_{n=1}^{\infty}\binom{-\frac{1}{2}}{n}[4t(t-1)s]^n\right\}\frac{dt}{2t(t-1)}$$

$$=\int_0^{\frac{\rho}{1+\rho}}\{[1+4t(t-1)s]^{-\frac{1}{2}}-1\}\frac{dt}{2t(t-1)}$$

$$=\int_0^{\frac{\rho}{1+\rho}}\frac{2}{\{1-(1-2t)^2\}}\frac{1-\sqrt{s}\sqrt{(1-2t)^2+(1-s)/s}}{\sqrt{s}\sqrt{(1-2t)^2+(1-s)/s}}dt.$$

Here, putting $1-2t=x$ and $(1-s)/s=a^2$,

$$\log A(s)=\int_1^{\frac{1-\rho}{1+\rho}}\frac{1-\sqrt{s}\sqrt{x^2+a^2}}{\sqrt{s}(x^2-1)\sqrt{x^2+a^2}}dx$$

$$=\int_\alpha^\beta\frac{2}{\sqrt{s}}\frac{1-\sqrt{1-s}-(1+\sqrt{1-s})u^2}{(2au-1+u^2)(2au+1-u^2)}du \tag{4. 1}$$

$$=\left[\frac{1}{2}\log\left|\frac{u^2-2au-1}{u^2+2au-1}\frac{\left(u-\frac{1}{\sqrt{s}}\right)^2-a^2}{\left(u+\frac{1}{\sqrt{s}}\right)^2-a^2}\right|\right]_\alpha^\beta,$$

where we put $\sqrt{x^2+a^2}=a+xu$ and

$$\alpha=\frac{1}{\sqrt{s}}(1-\sqrt{1-s})$$

$$\beta=\frac{1}{\sqrt{s}\,(1-\rho)}(\sqrt{(1+\rho)^2-4\rho s}-(1+\rho)\sqrt{1-s}).$$

Inserting α and β in (4. 1), we have finally

$$A(s)=\frac{2}{\sqrt{(1+\rho)^2-4\rho s}+(1-\rho)} \tag{4. 2}$$

and therefore

$$A(1-0)=\frac{1}{1-\rho}. \tag{4. 3}$$

Thus we have from the theorem,

$$E(N_j)=\frac{1}{1-\rho}. \tag{4. 4}$$

Next, we shall find q_n from (4. 2), expanding $A(s)$ in the power series respect to s as follows:

$$\begin{aligned}A(s)&=\frac{1}{2\rho}\{[(1+\rho)^2-4\rho s]^{\frac{1}{2}}-(1-\rho)\}(1-s)^{-1}\\&=\frac{1}{2\rho}\sum_{\nu=1}^{\infty}\left[\sum_{r=0}^{\nu}\binom{\frac{1}{2}}{r}\frac{(-4\rho)^r}{(1+\rho)^{2r-1}}-(1-\rho)\right]s^\nu\\&=1+\frac{\rho}{1+\rho}s+\sum_{\nu=2}^{\infty}s^\nu\left[\frac{\rho}{1+\rho}-\sum_{r=2}^{\nu}\frac{(2r-3)!!}{(2r)!!}\frac{2\cdot(4\rho)^{r-1}}{(1+\rho)^{2r-1}}\right].\end{aligned}$$

Thus we have

$$a_0=1,\quad a_1=\rho/(1+\rho), \tag{4. 5}$$

$$a_\nu=\frac{\rho}{1+\rho}-\sum_{r=2}^{\nu}\frac{(2r-3)!!}{(2r)!!}\frac{2\cdot(4\rho)^{r-1}}{(1+\rho)^{2r-1}}\qquad(\nu\geqq 2)$$

and therefore,

$$q_1=a_0-a_1=\frac{1}{1+\rho}$$

$$q_n=a_{n-1}a_n=\frac{(2n-3)!!}{(2n)!!}\frac{2(4\rho)^{n-1}}{(1+\rho)^{2n-1}}\qquad(n\geqq 2).$$

Hence, we have

$$q_n=\frac{(2n-2)!}{(n-1)!n!}\frac{\rho^{n-1}}{(1+\rho)^{2n-1}}\qquad(n\geqq 1). \tag{4. 6}$$

which consists with the result by Pollaczek [7], but somewhat differs from those by Takács [9] and Conolly [3].

Of course, it is obvious that $\{q_n\}$ is a probability distribution because $a_n\to 0$ $(n\to\infty)$. But for the sake of checking (4. 6), we shall show that $\sum_{n=1}^{\infty} q_n=1$ by a direct calculation. We shall use to do this the following identity:

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(2n-3)!!}{(2n)!!}x^n=(1+x)^{\frac{1}{2}}-1.$$

Hence,

$$\begin{aligned}\sum_{n=1}^{\infty}q_n&=\frac{1+\rho}{2\rho}\sum_{n=1}^{\infty}\frac{(2n-3)!!}{(2n)!!}\left[\frac{4\rho}{(1+\rho)^2}\right]^n\\&=\frac{1+\rho}{2\rho}\left[1-\left\{1-\frac{4\rho}{(1+\rho)^2}\right\}^{\frac{1}{2}}\right]\\&=\frac{1+\rho}{2\rho}\left(1-\frac{1-\rho}{1+\rho}\right)=1.\end{aligned}$$

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