NOTE ON A RELATION BETWEEN THE MARKOV CHAIN AND THE BIRTH AND DEATH PROCESS

KANEHISA UDAGAWA AND NORIKO ITO

Faculty of Engineering, Nagoya University (Received June 9, 1961)

§ 1. INTRODUCTION

The problem of finding the probability distribution in the stationary state in the birth and death process has been solved mainly by solving the difference equations which are derived from the differential equations on $P_n(t) = P\{X_t = E_n\}$ [1]. Here, X_t signifies the random variable at time t. This is an attempt to solve the problem by quite a different method.

The birth and death process can be considered as a continuous Markov process with time t as parameter. From definition of the process it is impossible that two or more transitions occur at a time. Then, at least in the practical problems of operations research, we can assume that time interval can be taken sufficiently short to neglect the probability of two or more transitions in a interval, i. e., we fix the time interval in a finite length, h, satisfying the above condition. Throughout the paper no further considerations are given to the validity of the existence of such a short time interval.

If we observe the state every h unit time, we shall be able to know the state after $nh = t$ unit time by clarifying the state of the n-th transition. On the other hand, if any state continues for τ unit time, it is observed n times $((\tau/h) - 1 < n < (\tau/h) + 1)$. Hence, if any state is observed n times, it will continue for $(n-1)h < \tau < (n+1)h$ unit time. It is harmless to regard it as $\tau = nh$. Consequently we can take the birth and death process as a Markov process with the discrete parameters which has a sufficiently short time interval h. We think that the result of the trial which is made every h unit time (h is sufficiently short interval) is a Markov chain [2], and we can apply the theorem of Markov chain. The transition probabilities p_{ij}'s from E_i to E_j are properly given as follows.

$$p_{i,\,i+1} = \lambda_i h \quad (i \geq 0), \qquad p_{i,\,i-1} = \mu_i h \quad (i \geq 1),$$
$$p_{i,\,i} = 1 - \lambda_i h - \mu_i h \quad (i \geq 1), \qquad p_{0,\,0} = 1 - \lambda_0 h,$$
$$p_{i,\,j} = 0 \quad (|j - i| \geq 2).$$

27

Let $p_{ij}^{(n)}$ be the probability that X_t starts from E_i and that reaches to E_j by the n-th transition. If we look for $\lim_{n\to\infty} p_{ij}^{(n)}$, we can obtain the probability that X_t is in state E_j after a sufficiently long time. Moreover, this probability does not concern with the initial state. Consequently it gives the absolute probability of E in the stationary state.

§ 2. THE CASE WHERE THE NUMBER OF VALUE X_t CAN TAKE IS FINITE

Let X_t be the random variable at time t, and E_0, E_1, $\cdots$, E_r be the value which X_t can take. In that case, as the transition is possible only from E_r to E_{r-1}, the transition probabilities are

$p_{i, i+1}=\lambda_i h$ $(0\leq i\leq r-1)$, $p_{i, i-1}=\mu_i h$ $(1\leq i\leq r)$,
$p_{i, i}=1-\lambda_i h-\mu_i h$ $(1\leq i\leq r-1)$, $p_{0, 0}=1-\lambda_0 h$, $p_{rr}=1-\mu_r h$,
$p_{i, j}=0$ $(|j-i|\geq 2)$.

Using these values we look for the recurrence probability of E_0, $f_0^{(n)}$ $(n=1, 2, 3, \cdots)$ (the probability that the first return to E_0 occurs at time n). For convenience' sake, put

$$p_{i, i+1}\cdot p_{i+1, i}=\alpha_i \qquad p_{i, i}=\beta_i.$$

Apparently

$$f_0^{(1)}=1-\lambda_0 h. \tag{1}$$

For $n\geq 2$, in one case, in first 1 unit time, transition $E_0\to E_1$ occurs, but in the next $(n-2)$ unit time no transition occurs. In the last 1 unit time $E_1\to E_0$ occurs, and the state returns to E_0. In the other case, let E_l be the farthest state from E_0 in n unit time $(2\leq l\leq r)$, and let s_1, s_2, $\cdots$, s_{l-1}; m_1, m_2, $\cdots$, m_l satisfy the equation

$$n=2+2\sum_{i=1}^{l-1} s_i+\sum_{i=1}^{l} m_i \quad (s_i\geq 1, \ m_i\geq 0). \tag{2}$$

Then, in the first 1 unit time transition $E_0\to E_1$ occurs ; in the next $(n-2)$ unit time, transitions $E_1\to E_1$; $E_2\to E_2$; $\cdots$; $E_l\to E_l$; $E_1\to E_2$, $E_2\to E_1$; $E_2\to E_3$; $E_3\to E_2$; $\cdots$; $E_{l-1}\to E_l$, $E_l\to E_{l-1}$ occur m_1, m_2, $\cdots$, m_l ; s_1, s_2, $\cdots$, s_{l-1} times respectively ; and in the last 1 unit time transition $E_1\to E_0$ occurs and the state returns to E_0. First, the probability of the former is

$$\alpha_0\beta_1^{n-2}.$$

Taking the order of transition into consideration, the probability of the latter is

$$A_l \equiv \binom{s_1+s_2-1}{s_1-1}\binom{s_2+s_3-1}{s_2-1}\cdots\cdots\binom{s_{l-1}+s_l-1}{s_{l-1}-1}\binom{s_1+m_1}{m_1}\binom{s_1+s_2+m_2-1}{m_2}$$

$$\cdots\cdots\binom{s_{l-2}+s_{l-1}+m_{l-1}-1}{m_{l-1}}\binom{s_{l-1}+m_l-1}{m_l}$$

$$\cdot \alpha_0\alpha_1{}^{s_1}\alpha_2{}^{s_1}\cdots\cdots\alpha_{l-1}{}^{s_{l-1}}\beta_1{}^{m_1}\beta_2{}^{m_1}\cdots\cdots\beta_l{}^{m_l}. \tag{3}$$

Hence,

$$f_0{}^{(n)}=1-\lambda_0 h+\alpha_0\beta_1{}^{n-2}+\sum_{l=2}^{r}\left[\sum_{\substack{n=2+2\sum_{i=1}^{l-1}s_i+\sum_{i=1}^{l}m_i}}A_l\right]. \tag{4}$$

Here, $\displaystyle\sum_{n=2+2\sum s_i+\sum m_i}$ means the sum of A_l about $s_1, s_2, \cdots, s_{l-1}$; $m_1, m_2,$

$\cdots, m_l$ which satisfy (3).

We introduce the lemma for computing $\displaystyle\sum_{n=1}^{\infty}f_0{}^{(n)}$, and $\displaystyle\sum_{n=1}^{\infty}nf_0{}^{(n)}$.

Lemma

If

$$u_0=1,$$
$$u_j=\lambda_1\lambda_2\cdots\lambda_j+\mu_1\lambda_2\cdots\lambda_j+\cdots+\mu_1\mu_2\cdots\mu_j \quad (j\geqq 1),$$
$$v_j=\lambda_0\lambda_1\cdots\lambda_j\mu_1\mu_2\cdots\mu_{j+1} \quad (j\geqq 0),$$

then

(i) $\displaystyle\frac{\lambda_0\mu_1}{\lambda_1+\mu_1}\sum_{s_k=1}^{\infty}\left[\frac{\lambda_k\mu_{k+1}}{(\lambda_k+\mu_k)(\lambda_{k+1}+\mu_{k+1})}\right]^{s_k}$

$\displaystyle\times\prod_{i=1}^{k-1}\left\{\sum_{s_i=1}^{\infty}\binom{s^i+s_{i+1}-1}{s_i-1}\left[\frac{\lambda_i\mu_{i+1}}{(\lambda_i+\mu_i)(\lambda_{i+1}+\mu_{i+1})}\right]^{s_i}\right\}=\frac{v_k}{u_k u_{k+1}},$ (5)

(ii) $\displaystyle\frac{\lambda_0\mu_1}{\lambda_1+\mu_1}\sum_{s_k=1}^{\infty}\left[\frac{\lambda_k}{\lambda_k+\mu_k}\right]^{s_k}$

$\displaystyle\times\prod_{i=1}^{k=1}\left\{\sum_{s_i=1}^{\infty}\binom{s_i+s_{i+1}-1}{s_i-1}\left[\frac{\lambda_i\mu_{i+1}}{(\lambda_i+\mu_i)(\lambda_{i+1}+\mu_{i+1})}\right]^{s_i}\right\}=\frac{\lambda_0\lambda_1\cdots\lambda_k}{u_k},$ (6)

(iii) $\displaystyle\frac{\lambda_0\mu_1}{\lambda_1+\mu_1}\prod_{i=1}^{k}\left\{\sum_{s_i=1}^{\infty}\binom{s_i+s_{i+1}-1}{s_i-1}\left[\frac{\lambda_i\mu_{i+1}}{(\lambda_i+\mu_i)(\lambda_{i+1}+\mu_{i+1})}\right]^{s_i}\right\}$

$\displaystyle=\frac{v_k}{u_k u_{k+1}}\left[\frac{(\lambda_{k+1}+\mu_{k+1})u_k}{u_{k+1}}\right]^{s_{k+1}+1}. \tag{7}$

Using these lemmas,

$$\sum_{n=1}^{\infty} f_0{}^{(n)} = 1 - \lambda_0 h + \sum_{m_1=0}^{\infty} \alpha_0 \beta_1{}^{m_1} +$$

$$\times \sum_{l=2}^{r} \left[\sum_{s_{l-1}=1}^{\infty} \cdots \cdots \sum_{s_1=1}^{\infty} \cdot \sum_{m_1=0}^{\infty} \cdots \cdots \sum_{m_l=0}^{\infty} A_l \right] = 1, \qquad (8)$$

$$\sum_{n=1}^{\infty} n f_0{}^{(n)} = 1 - \lambda_0 h + \sum_{m_1=0}^{\infty} \alpha_0 (2+m_1) \beta_1{}^{m_1} + \sum_{l=2}^{r} \left[\sum_{s_{l-1}=1}^{\infty} \cdots \cdots \sum_{s_1=1}^{\infty} \cdot \sum_{m_1=0}^{\infty} \right.$$

$$\left. \cdots \cdots \sum_{m_l=0}^{\infty} \left(2 + 2 \sum_{i=1}^{l-1} s_i + \sum_{i=1}^{l} m_i \right) A_l \right] = 1 + \sum_{i=0}^{r-1} \frac{\lambda_0 \lambda_1 \cdots \lambda_i}{\mu_1 \mu_2 \cdots \mu_{i+1}}. \qquad (9)$$

These relations hold unconditionally. The proves of $(5) \sim (9)$ are given in the appendix.

Generally, in order to look for the mean recurrence time of E_k, $\sum_{n=1}^{\infty} n f_k{}^{(n)}$, we distinguish three cases: where the first transition is $E_k \to E_{k-1}$, $E_k \to E_k$, and $E_k \to E_{k+1}$. Then,

$$\sum_{n=1}^{\infty} n f_k{}^{(n)} = \sum_{n=1}^{\infty} n P \{ (\text{the first transition is } E_k \to E_{k-1}) \cap (E_{k-1} \to E_k$$

occurs for the first time at the n-th transition)$\} + \sum_{n=1}^{\infty} n P \{ (\text{the first transition is } E_k \to E_{k-1}) \cap (E_{k-1} \to E_k \text{ or } E_{k+1} \to E_k \text{ occurs for the first time at the } n\text{-th transition}) \} + \sum_{n=1}^{\infty} n P \{ (\text{the first transition is } E_k \to E_{k+1}) \cap (E_{k+1} \to E_k$

occurs for the first time at the n-th transition). $\qquad (10)$

Here, (9) is applied to the first and the third term, the number of their states being $k+1$ and $r-k+1$ respectively. But in the first term, the replacement of parameter, $\lambda_i \to \mu_{k-i}$, $\mu_i \to \lambda_{k-i}$ is required, in the third term $\lambda_i \to \lambda_{i+k}$, $\mu_i \to \mu_{i+k}$ is required. Moreover as the first transition is $E_k \to E_{k-1}$ or $E_k \to E_{k+1}$, we must deduce $1 - \mu_k h$ or $1 - \lambda_k h$ respectively from the value which is obtained by applying (9). The second term is apparently $1 - \lambda_k h - \mu_k h$. Hence,

$$\sum_{n=1}^{\infty} n f_n{}^{(k)} = \left\{ 1 + \frac{\mu_k}{\lambda_{k-1}} + \frac{\mu_k \mu_{k-1}}{\lambda_{k-1} \lambda_{k-2}} + \cdots + \frac{\mu_k \mu_{k-1} \cdots \mu_1}{\lambda_{k-1} \lambda_{k-2} \cdots \lambda_0} - (1 - \mu_k h) \right\}$$

$$+ \{ 1 - \lambda_k h - \mu_k h \}$$

$$+ \left\{ 1 + \frac{\lambda_k}{\mu_{k+1}} + \frac{\lambda_k \lambda_{k+1}}{\mu_{k+1} \mu_{k+2}} + \cdots + \frac{\lambda_k \lambda_{k+1} \cdots \lambda_{r-1}}{\mu_{k+1} \mu_{k+2} \cdots \mu_r} - (1 - \lambda_k h) \right\}$$

$$=1+ \sum_{i=1}^{k} \frac{\mu_k \cdots \mu_{k-i+1}}{\lambda_{k-1} \cdots \lambda_{k-i}} + \sum_{i=0}^{r-k-1} \frac{\lambda_k \cdots \lambda_{k+i}}{\mu_{k+1} \cdots \mu_{k+i+1}}. \tag{11}$$

This is the finite value unconditionally. Consequently, we can apply the theorem of Markov chain[2], and obtain the next theorem.

Theorem 1 In the birth and death process such as X_t can take the value $E_0, E_1, \cdots, E_r$, put

$$P\{X_t=E_k\}=P_k(t),$$

then, we obtain

$$p_0=\lim_{t \to \infty} P_0(t) =\left(1+ \sum_{i=0}^{r-1} \frac{\lambda_0 \lambda_1 \cdots \lambda_i}{\mu_1 \mu_2 \cdots \mu_{i+1}} \right)^{-1},$$

$$\tag{12}$$

$$p_k=\lim_{t \to \infty} P_k(t) =\left(1+ \sum_{i=1}^{k} \frac{\mu_k \cdots \mu_{k-i+1}}{\lambda_{k-1} \cdots \lambda_{k-i}} + \sum_{i=0}^{r-k-1} \frac{\lambda_k \cdots \lambda_{k+i}}{\mu_{k+1} \cdots \mu_{k+i+1}} \right)^{-1} (k \geqq 1)$$

This probability does not change even if we know the initial value.

For example, we consider the next problem. This is the problem whose solution has already obtained by W. Feller[3].

We consider automatic machines which normally require no human care. However at any time a machine may break down and call for a human service. If at time t the machine is in working state, the probability that it will call for a service before time $t+h$ is $\lambda h+o(h)$. Conversely, if at time t the machine is being served, the probability that the serving time terminates before $t+h$ and the machine reverts to the working state is $\mu h+o(h)$. We suppose that m machines with the same parameters λ and μ, and working independently are served by a single repairman. A machine which breaks down is served immediately unless the repairman is serving another machine, in which case a waiting line is formed. We say that the system is in state E_k if k machines are not working. Let $P_k(t)$ be the probability that k machines break down at time t, and we try to look for

$$p_k=\lim_{t \to \infty} P_k(t).$$

In this case the parameters are

$$\lambda_0=m\lambda, \qquad \mu_0=0,$$
$$\lambda_k=(m-k)\lambda, \qquad \mu_k=\mu \quad (1 \leqq k \leqq m).$$

If we substitute these values in (12), we obtain

$$p_k = \left(1 + \frac{\mu}{(m-k+1)\lambda} + \frac{\mu^2}{(m-k+1)(m-k+2)\lambda^2} + \cdots \right.$$

$$+ \frac{\mu_k}{(m-k+1)(m-k+2)\cdots(m-1)m\lambda^k} + \frac{(m-k)\lambda}{\mu}$$

$$\left. + \frac{(m-k)(m-k-1)\lambda^2}{\mu^2} + \cdots + \frac{(m-k)(m-k-1)\cdots 2\cdot 1\lambda^{m-k}}{\mu^{m-k}} \right)^{-1}$$

$$= \frac{k! \binom{m}{k}\left(\frac{\lambda}{\mu}\right)^k}{\sum\limits_{i=0}^{m} i! \binom{m}{i}\left(\frac{\lambda}{\mu}\right)^i} \quad (k=0, 1, \cdots, m). \tag{13}$$

Consequently,

$$p_m = \frac{m!\left(\frac{\lambda}{\mu}\right)^m}{\sum\limits_{i=0}^{m} i!\binom{m}{i}\left(\frac{\lambda}{\mu}\right)^i} = \frac{1}{\sum\limits_{i=0}^{m}\frac{1}{i!}\left(\frac{\mu}{\lambda}\right)^i},$$

$$\tag{14}$$

$$p_{m-k} = \frac{(m-k)!\binom{m}{m-k}\left(\frac{\lambda}{\mu}\right)^{m-k}}{\sum\limits_{i=0}^{m} i!\binom{m}{i}\left(\frac{\lambda}{\mu}\right)^i} = \frac{1}{k!}\left(\frac{\mu}{\lambda}\right)^k p_m \quad (0 \le k \le m-1).$$

This accords with the result of W. Feller[4].

§ 3. THE CASE WHERE THE NUMBER OF VALUE X_t CAN TAKE IS INFINITE

Transition probabilities are given as follows.

$$p_{i,\,i+1} = \lambda_i h \quad (i \ge 0), \qquad p_{i,\,i-1} = \mu_i h \quad (i \ge 1),$$
$$p_{i,\,i} = 1 - \lambda_i h - \mu_i h \quad (i \ge 1), \qquad p_{0,\,0} = 1 - \lambda_0 h, \tag{15}$$
$$p_{i,\,j} = 0 \quad (|j-i| \ge 2).$$

Using these values, we look for the recurrence probability $f_0^{(n)}$. Apparently we obtain

$$f_0^{(1)} = 1 - \lambda_0 h. \tag{16}$$

As to $n \ge 2$, in the same way that we derive (13),

$$f_0^{(n)} = \alpha_0 \beta_1^{n-2} + \sum_{l=2}^{\infty}\left[\sum_{\substack{n=2+2\sum\limits_{i=1}^{l-1} s_i + \sum\limits_{i=1}^{l} m_i}} A_l \right]. \tag{17}$$

hence,

$$\sum_{n=1}^{\infty} f_0^{(n)} = 1 - \lambda_0 h + \sum_{m_1=0}^{\infty} \alpha_0 \beta_1^{m_1} + \sum_{l=2}^{\infty} \left[\sum_{s_{l-1}=1}^{\infty} \cdots \cdots \sum_{s_1=1}^{\infty} \cdot \sum_{m_1=0}^{\infty} \cdots \cdots \sum_{m_l=0}^{\infty} A_l \right]$$

$$= 1 - \lambda_0 h + h \sum_{i=0}^{\infty} \frac{v_i}{u_i u_{i+1}}.$$

however, as the following relation is proved by mathematical induction,

$$\sum_{i=0}^{k-1} \frac{v_i}{u_i u_{i+1}} = \frac{\lambda_0 \sum_{i=1}^{k} \frac{\mu_1 \mu_2 \cdots \mu_i}{\lambda_1 \lambda_2 \cdots \lambda_i}}{1 + \sum_{i=1}^{k} \frac{\mu_1 \mu_2 \cdots \mu_i}{\lambda_1 \lambda_2 \cdots \lambda_i}}$$

we obtain

$$\sum_{n=1}^{\infty} f_0^{(n)} = 1 - \lambda_0 h + \frac{\lambda_0 h \sum_{i=1}^{\infty} \frac{\mu_1 \mu_2 \cdots \mu_i}{\lambda_1 \lambda_2 \cdots \lambda_i}}{1 + \sum_{i=1}^{\infty} \frac{\mu_1 \mu_2 \cdots \mu_i}{\lambda_1 \lambda_2 \cdots \lambda_i}}. \tag{18}$$

Hence, according to convergence or divergence of the series $\sum_{i=1}^{\infty} \frac{\mu_1 \mu_2 \cdots \mu_i}{\lambda_1 \lambda_2 \cdots \lambda_i}$, it is introduced that $\sum_{n=1}^{\infty} f_0^{(n)} < 1$ or $\sum_{n=1}^{\infty} f_0^{(n)} = 1$ respectively. If $\lim_{n \to \infty} \frac{\mu_n}{\lambda_n} < 1$, $\sum_{i=1}^{\infty} \frac{\mu_1 \mu_2 \cdots \mu_i}{\lambda_1 \lambda_2 \cdots \lambda_i}$ converges, and if $\lim_{n \to \infty} \frac{\mu_n}{\lambda_n} > 1$, even if $\lim_{n \to \infty} \frac{\mu_n}{\lambda_n} = \infty$, it diverges. Moreover if $\lim_{n \to \infty} \frac{\mu_n}{\lambda_n} = 1$, as $\lim_{n \to \infty} \frac{\mu_1 \mu_2 \cdots \mu_n}{\lambda_1 \lambda_2 \cdots \lambda_n} \neq 0$ it diverges. If the limit of $\frac{\mu_n}{\lambda_n}$ does not exist, we cannot decide whether $\sum_{i=1}^{\infty} \frac{\mu_1 \mu_2 \cdots \mu_i}{\lambda_1 \lambda_2 \cdots \lambda_i}$ converges or diverges. Hence, from (18) we obtain

$$\sum_{n=0}^{\infty} f_0^{(n)} \begin{cases} = 1 & \left(\text{if } \lim_{n \to \infty} \frac{\mu_n}{\lambda_n} \geqq 1 \right), \\ < 1 & \left(\text{if } \lim_{n \to \infty} \frac{\mu_n}{\lambda_n} < 1 \right). \end{cases} \tag{19}$$

Next, the mean recurrence time of E_0, $\sum_{n=1}^{\infty} n f_0^{(n)}$ is obtained in the similar way as (9) was introduced.

$$\sum_{n=1}^{\infty} n f_0^{(n)} = 1 - \lambda_0 h + \sum_{m_1=0}^{\infty} \alpha_0 (2 + m_1) \beta_1^{m_1}$$

$$+ \sum_{l=2}^{\infty} \left[\sum_{s_{l-1}=1}^{\infty} \cdots \cdots \sum_{s_0=1}^{\infty} \cdot \sum_{m_l=0}^{\infty} \cdots \cdots \sum_{m_l=0}^{\infty} \left(2+2 \sum_{i=1}^{l-1} s_i + \sum_{i=1}^{l} m_i \right) A_l \right]$$

$$=1+ \sum_{i=0}^{\infty} \frac{\lambda_0 \lambda_1 \cdots \lambda_i}{\mu_1 \mu_2 \cdots \mu_i +1}. \qquad (20)$$

If $\lim\limits_{n\to\infty} \dfrac{\lambda_{n-1}}{\mu_n} < 1$ it converges, and if $\lim\limits_{n\to\infty} \dfrac{\lambda_{n-1}}{\mu_n} \geqq 1$ it diverges.

Let us seek the mean recurrence time of arbitrary state E_k, $\sum\limits_{n=1}^{\infty} n f_k^{(n)}$. Similarly to finite states, we distinguish three cases: where the first transition is $E_k \to E_{k-1}$, $E_k \to E_k$, and $E_k \to E_{k+1}$. By changing the parameter, we can apply (9) or (20). Then we obtain

$$\sum_{n=1}^{\infty} n f_k^{(n)} = 1 + \sum_{i=0}^{k-1} \frac{\mu_k \cdots \mu_{k-i}}{\lambda_{k-1} \cdots \lambda_{k-i-1}} + \sum_{i=0}^{\infty} \frac{\lambda_k \cdots \lambda_{k+i}}{\mu_{k+1} \cdots \mu_{k+i+1}}.$$

Whether the term of infinite series converges or not depends upon whether (20) converges or not.

Applying the theorem of Markov chain[2] to the above, we obtain the next theorem.

Theorem 2 In the birth and death process such as X_t can take the value E_0, E_1, E_2, $\cdots$, let $P_k(t)$ be
$$P\{X_t = E_k\} = P_k(t).$$

If $\lim\limits_{n\to\infty} \dfrac{\mu_n}{\lambda_n} \geqq 1$ and $\lim\limits_{n\to\infty} \dfrac{\lambda_n}{\mu_{n-1}} > 1$, we obtain

$$p_0 = \lim_{t\to\infty} P_0(t) = \left(1 + \sum_{i=0}^{\infty} \frac{\lambda_0 \lambda_1 \cdots \lambda_i}{\mu_1 \mu_2 \cdots \mu_{i+1}} \right)^{-1}$$

$$p_k = \lim_{t\to\infty} P_k(t)$$

$$= \left(1 + \sum_{i=1}^{k} \frac{\mu_k \cdots \mu_{k-i+1}}{\lambda_{k-1} \cdots \lambda_{k-i}} + \sum_{i=0}^{\infty} \frac{\lambda_k \cdots \lambda_{k+i}}{\mu_{k+1} \cdots \mu_{k+i+1}} \right)^{-1} \quad (k \geqq 1). \qquad (21)$$

Otherwise we obtain

$$p_k = \lim_{k\to\infty} P_k(t) = 0. \qquad (22)$$

When $\lim\limits_{n\to\infty} \dfrac{\mu_n}{\lambda_n} < 1$ or $\lim\limits_{n\to\infty} \dfrac{\mu_n}{\lambda_{n-1}} \leqq 1$, the probability that the length of queue is finite is 1.

For example, we consider the next problem. This is the problem whose solusion has already been obtained by W. Feller[5].

Suppose that there are a trunks and that the probability of a conversation ending during the interval $(t, t+h)$ is $\mu h + o(h)$, and that two or more conversations ending is $o(h)$. The probability of an incoming calling during the interval $(t, t+h)$ is $\lambda h + o(h)$ and that of two or more incoming calling is $o(h)$. If all trunks are busy, each new call joins a waiting line and waits until a trunk is freed. Let us look for the probability $p_k = \lim_{n \to \infty} P_k(t)$ that k calls are in cenverastion or in waiting line after a long time. In this case

$$\lambda_k = \lambda, \qquad \mu_k = \begin{cases} k\mu & (1 \leqq k \leqq a), \\ a\mu & (k \geqq a). \end{cases}$$

First of all, from

$$\lim_{n \to \infty} \frac{\mu_n}{\lambda_n} = \lim_{n \to \infty} \frac{\mu_n}{\lambda_{n-1}} = \frac{a\mu}{\lambda},$$

if $(a\mu/\lambda) \leqq 1$ then we obtain $p_k = 0$ ($k = 0, 1, 2, \cdots$). This means that if $(a\mu/\lambda) \leqq 1$ the queue is infinitely long. If $(a\mu/\lambda) > 1$, p_k is decided as probability distribution. From (21) we obtain

$$
\begin{aligned}
p_0 &= \left[1 + \frac{\lambda}{\mu} + \frac{1}{2!} \left(\frac{\lambda}{\mu} \right)^2 + \cdots + \frac{1}{a!} \left(\frac{\lambda}{\mu} \right)^a + \frac{1}{a!\,a} \left(\frac{\lambda}{\mu} \right)^{a+1} \right. \\
&\qquad \left. + \cdots + \frac{1}{a!\,a^{n-a}} \left(\frac{\lambda}{\mu} \right)^n + \cdots \right]^{-1} \\
&= \left[1 + \frac{\lambda}{\mu} + \frac{1}{2!} \left(\frac{\lambda}{\mu} \right)^2 + \cdots + \frac{1}{a!} \left(\frac{\lambda}{\mu} \right)^a + \frac{1}{a!} \left(\frac{\lambda}{\mu} \right)^a \frac{a\mu}{a\mu - \lambda} \right]^{-1},
\end{aligned}
$$

$$
\begin{aligned}
p_k &= \left[1 + k\frac{\mu}{\lambda} + k(k-1) \left(\frac{\mu}{\lambda} \right)^2 + \cdots + k! \left(\frac{\mu}{\lambda} \right)^k + \frac{1}{k+1} \frac{\lambda}{\mu} \right. \\
&\qquad + \frac{1}{(k+1)(k+2)} \left(\frac{\lambda}{\mu} \right)^2 + \cdots + \frac{1}{(k+1)(k+2)\cdots(a-1)a} \left(\frac{\lambda}{\mu} \right)^{a-k} \\
&\qquad \left. + \frac{1}{(k+1)(k+2)\cdots(a-1)a^2} \left(\frac{\lambda}{\mu} \right)^{a-k+1} + \cdots \right]^{-1} \\
&= \frac{1}{k!} \left(\frac{\lambda}{\mu} \right)^k p_0 \quad (k \leqq a),
\end{aligned}
$$

$$
\begin{aligned}
p_k &= \left[a\frac{\mu}{\lambda} + a^2 \left(\frac{\mu}{\lambda} \right)^2 + \cdots + a^{k-a} \left(\frac{\mu}{\lambda} \right)^{k-a} + a^{k-a} a \left(\frac{\mu}{\lambda} \right)^{k-a+1} \right. \\
&\qquad \left. + a^{k-a} a(a+1) \left(\frac{\mu}{\lambda} \right)^{k-a+2} + \cdots + a^{k-a} a! \left(\frac{\mu}{\lambda} \right)^k \right.
\end{aligned}
$$

$$+1+\frac{1}{a}\frac{\lambda}{\mu}+\frac{1}{a^2}\left(\frac{\lambda}{\mu}\right)^2+\cdots\Bigg]^{-1}$$
$$=\frac{1}{a!\,a^{k-a}}\left(\frac{\lambda}{\mu}\right)^k p_0 \quad (k\leqq a).$$

Namely if $(a\mu/\lambda)>1$, we obtain

$$p_k=\begin{cases}\dfrac{1}{k!}\left(\dfrac{\lambda}{\mu}\right)^k p_0 & (k\leqq a),\\[2mm] \dfrac{1}{a!\,a^{k-a}}\left(\dfrac{\lambda}{\mu}\right)^k p_0 & (k\geqq a).\end{cases}\tag{24}$$

This accords with the result of W. Feller[6].

§ 4.   CONCLUSION

Although the birth and death process is a Markov process with continuous time t as parameter, we can take it as the Markov process with the discrete parameters which has a sufficiently short time interval h. Hence we looked for the probability distribution in the stationary state using the theorem of Markov chain. If we suppose, in other words, that the parameters of input and output in state E_n are λ_n and μ_n respectively, the probability p_k that X_t is in state E_k after a long time is represented as follows.

When the number of states is finite $(r+1)$

$$p_0=\left(1+\sum_{i=0}^{r-1}\frac{\lambda_0\lambda_1\cdots\lambda_i}{\mu_1\mu_2\cdots\mu_{i+1}}\right)^{-1},$$
$$p_k=\left(1+\sum_{i=1}^{k}\frac{\mu_k\cdots\mu_{k-i+1}}{\lambda_{k-1}\cdots\lambda_{k-i}}+\sum_{i=0}^{r-k+1}\frac{\lambda_k\cdots\lambda_{k+i}}{\mu_{k+1}\cdots\mu_{k+i+1}}\right)^{-1}\quad (k\geqq 1).$$

When the number of states is infinite, if $\lim\limits_{n\to\infty}\dfrac{\mu_n}{\lambda_n}\geqq 1$ and $\lim\limits_{n\to\infty}\dfrac{\mu_n}{\lambda_{n-1}}>1$,

$$p_0=\left(1+\sum_{i=0}^{\infty}\frac{\lambda_0\lambda_1\cdots\lambda_i}{\mu_1\mu_2\cdots\mu_{i+1}}\right)^{-1},$$
$$p_k=\left(1+\sum_{i=1}^{k}\frac{\mu_k\cdots\mu_{k-i+1}}{\lambda_k\cdots\lambda_{k-i}}+\sum_{i=0}^{\infty}\frac{\lambda_k\cdots\lambda_{k+i}}{\mu_{k+1}\cdots\mu_{k+i+1}}\right)^{-1}\quad (k\geqq 1).$$

if $\lim\limits_{n\to\infty}\dfrac{\mu_n}{\lambda_n}<1$ or $\lim\limits_{n\to\infty}\dfrac{\mu_n}{\lambda_{n-1}}\leqq 1$,

$$p_k=0\quad (k\geqq 0).$$

They were introduced by quite a different method from the usual one that solves the difference equation derived from the differential equation. We made sure that in each case the result accords with that obtained as to some simple problems by the usual method.

APPENDIX : THE PROOVES OF (5) ~ (9).

The proof of (5) :

$$\frac{\lambda_0\mu_1}{\lambda_1+\mu_1}\sum_{s_k=1}^{\infty}\left[\frac{\lambda_k\mu_{k+1}}{(\lambda_k+\mu_k)(\lambda_{k+1}+\mu_{k+1})}\right]^{s_k}\prod_{i=1}^{k-1}\left\{\sum_{s_i=1}^{\infty}\binom{s_i+s_{i+1}-1}{s_i-1}\left[\frac{\lambda_i\mu_{i+1}}{(\lambda_i+\mu_i)(\lambda_{i+1}+\mu_{i+1})}\right]^{s_i}\right\}$$

$$=\frac{\lambda_0\mu_1}{\lambda_1+\mu_1}\sum_{s_k=1}^{\infty}\left[\frac{\lambda_k\mu_{k+1}}{(\lambda_k+\mu_k)(\lambda_{k+1}+\mu_{k+1})}\right]^{s_k+1}$$

$$\times\prod_{i=2}^{k-1}\left\{\sum_{s_i=0}^{\infty}\binom{s_i+s_{i+1}+1}{s_i}\left[\frac{\lambda_i\mu_{i+1}}{(\lambda_i+\mu_i)(\lambda_{i+1}+\mu_{i+1})}\right]^{s_i+1}\right\}\times\frac{\dfrac{\lambda_1\mu_2}{(\lambda_1+\mu_1)(\lambda_2+\mu_2)}}{\left[1-\dfrac{\lambda_1\mu_2}{(\lambda_1+\mu_1)(\lambda_2+\mu_2)}\right]^{s_k+2}}$$

$$=\frac{v_1}{u_1u_2}\sum_{s_k=0}^{\infty}\left[\frac{\lambda_k\mu_{k+1}}{(\lambda_k+\mu_k)(\lambda_{k+1}+\mu_{k+1})}\right]^{s_k+1}\prod_{i=3}^{k-1}\left\{\sum_{s_i=0}^{\infty}\binom{s_i+s_{i+1}+1}{s_i}\left[\frac{\lambda_i\mu_{i+1}}{(\lambda_i+\mu_i)(\lambda_{i+1}+\mu_{i+1})}\right]^{s_i+1}\right\}$$

$$\times\sum_{s_2=0}^{\infty}\binom{s_2+s_3+1}{s_2}\left[\frac{(\lambda_1+\mu_1)\lambda_2\mu_3}{(\lambda_3+\mu_3)u_2}\right]^{s_2+1}$$

$$=\frac{v_1}{u_1u_2}\sum_{s_k=0}^{\infty}\left[\frac{\lambda_k\mu_{k+1}}{(\lambda_k+\mu_k)(\lambda_{k+1}+\mu_{k+1})}\right]^{s_k+1}$$

$$\times\prod_{i=4}^{k-1}\left\{\sum_{s_i=0}^{\infty}\binom{s_i+s_{i+1}+1}{s_i}\left[\frac{\lambda_i\mu_{i+1}}{(\lambda_i+\mu_i)(\lambda_{i+1}+\mu_{i+1})}\right]^{s_i+1}\right\}$$

$$\times\sum_{s_3=0}^{\infty}\binom{s_3+s_4+1}{s_3}\left[\frac{\lambda_3\mu_4u_2}{(\lambda_4+\mu_4)u_3}\right]^{s_3+1}$$

$$=\cdots\cdots$$

$$=\frac{v_{k-2}}{u_{k-2}u_{k-2}}\sum_{s_k=0}^{\infty}\left[\frac{\lambda_k\mu_{k+1}}{(\lambda_k+\mu_k)(\lambda_{k+1}+\mu_{k+1})}\right]^{s_k+1}\sum_{s_{k-1}=0}^{\infty}\binom{s_{k-1}+s_k+1}{s_{k-1}}\left[\frac{\lambda_{k-1}\mu_k u_{k-2}}{(\lambda_k+\mu_k)u_{k-1}}\right]^{s_{k-1}+1}$$

$$=\frac{v_{k-1}}{u_{k-1}u_k}\sum_{s_k=0}^{\infty}\left[\frac{\lambda_k\mu_{k+1}}{(\lambda_k+\mu_k)(\lambda_{k+1}+\mu_{k+1})}\right]^{s_k+1}$$

$$=\frac{v_k}{u_ku_{k+1}}.$$

The proof of (6) :

From the 7-th line in the proof of (5),

$$\frac{\lambda_0\mu_1}{\lambda_1+\mu_1}\sum_{s_k=1}^{\infty}\left[\frac{\lambda_k}{\lambda_k+\mu_k}\right]^{s_k}\prod_{i=1}^{k-1}\left\{\sum_{s_{i-1}=1}^{\infty}\binom{s_i+s_{i+1}-1}{s_i-1}\left[\frac{\lambda_i\mu_{i+1}}{(\lambda_i+\mu_i)(\lambda_{i+1}+\mu_{i+1})}\right]^{s_i}\right\}$$

$$=\frac{v_{k-1}}{u_{k-1}u_k}\sum_{s_k=0}^{\infty}\left[\frac{\lambda_k}{\lambda_k+\mu_k}\right]^{s_k+l}$$

$$=\frac{\lambda_0\lambda_1\cdots\lambda_k}{u_k}.$$

The proof of (7):

From the 6-th line in the proof of (5),

$$\frac{\lambda_0\mu_1}{\lambda_1+\mu_1}\prod_{i=1}^{k}\left\{\sum_{s_i=1}^{\infty}\binom{s_i+s_{i+1}-1}{s_{i-1}}\left[\frac{\lambda_i\mu_{i+1}}{(\lambda_i+\mu_i)(\lambda_{i+1}+\mu_{i+1})}\right]^{s_i}\right\}$$

$$=\frac{v_{k-1}}{u_{k-1}u_k}\sum_{s_k=0}^{\infty}\binom{s_k+s_{k+1}+1}{s_k}\left[\frac{\lambda_k\mu_{k+1}u_{k-1}}{(\lambda_{k+1}+\mu_{k+1})u_k}\right]^{s_k+1}$$

$$=\frac{v_k}{u_ku_{k+1}}\left[\frac{(\lambda_{k+1}+\mu_{k+1})u_k}{u_{k+1}}\right]^{s_{k+1}+l}.$$

The proof of (8):

$$\sum_{n=1}^{\infty}f_0^{(n)}=1-\lambda_0h+\sum_{m_1=0}^{\infty}\alpha_0\beta_1^{m_1}+\sum_{l=2}^{r}\left[\sum_{s_{l-1}=1}^{\infty}\cdots\cdots\sum_{s_1=1}^{\infty}\cdot\sum_{m_1=0}^{\infty}\cdots\cdots\sum_{m_l=0}^{\infty}A_l\right]$$

$$=1-\lambda_0k+\frac{\lambda_0\mu_1h}{\lambda_1+\mu_1}+\sum_{l=2}^{r-1}\left[\frac{\lambda_0\mu_1h}{\lambda_1+\mu_1}\sum_{s_{l-1}=1}^{\infty}\left[\frac{\lambda_{l-1}\mu_l}{(\lambda_{l-1}+\mu_{l-1})(\lambda_l+\mu_l)}\right]^{s_{l-1}}\right.$$

$$\times\prod_{i=1}^{l-2}\left\{\sum_{s_i=1}^{\infty}\binom{s_i+s_{i+1}-1}{s_i-1}\left[\frac{\lambda_i\mu_{i+1}}{(\lambda_i+\mu_i)(\lambda_{i+1}+\mu_{i+1})}\right]^{s_i}\right\}\right]$$

$$+\sum_{s_{r-1}=1}^{\infty}\left[\frac{\lambda_{r-1}}{\lambda_{r-1}+\mu_{r-1}}\right]^{s_{r-1}}\prod_{i=1}^{r-2}\left\{\sum_{s_i=1}^{\infty}\binom{s_i+s_{i+1}-1}{s_i-1}\left[\frac{\lambda_i\mu_{i+1}}{(\lambda_i+\mu_i)(\lambda_{i+1}+\mu_{i+1})}\right]^{s_i}\right\}.$$

Using the lemma (i) in the 4-th term and (ii) in the 5-th term,

$$\sum_{n=1}^{\infty}f_0^{(n)}=1-\lambda_0h+\frac{\lambda_0\mu_1h}{\lambda_1+\mu_1}+\left(\frac{v_1}{u_1u_2}+\frac{v_2}{u_2u_3}+\cdots+\frac{v_{r-2}}{u_{r-2}u_{r-1}}+\frac{\lambda_0\lambda_1\cdots\lambda_{r-1}}{u_{r-1}}\right)h.$$

Then we use the relation $\dfrac{\lambda_0\lambda_1\cdots\lambda_{i+1}}{u_{i+1}}+\dfrac{v_i}{u_iu_{i+1}}=\dfrac{\lambda_0\lambda_1\cdots\lambda_i}{u_i}$

$$\sum_{n=1}^{\infty}f_0^{(n)}=1-\lambda_0h+\frac{\lambda_0\mu_1h}{\lambda_1+\mu_1}+\frac{\lambda_0\lambda_1\mu}{\lambda_0+\mu_1}=1.$$

Consequently,

$$\sum_{n=1}^{\infty}f_0^{(n)}=1.$$

The proof of (9):

$$\sum_{n=1}^{\infty} n f_0^{(n)} = 1 - \lambda_0 h + \sum_{m_1=0}^{\infty} \alpha_0 (2 + m_1) \beta_1^{m_1}$$

$$+ \sum_{l=2}^{r} \left[\sum_{s_{l-1}=1}^{\infty} \cdots \cdots \sum_{s_1=1}^{\infty} \cdot \sum_{m_1=0}^{\infty} \cdots \cdots \sum_{m_l=0}^{\infty} \left(2 + 2 \sum_{i=1}^{l-1} s_i + \sum_{i=1}^{l} m_i \right) A_l \right]$$

$$= 1 - \lambda_0 h + \frac{v_0 h}{u_1}$$

$$+ \sum_{l=2}^{r-1} \left[\sum_{s_{l-1}=1}^{\infty} \cdots \cdots \sum_{s_1=1}^{\infty} \cdot \sum_{m_1=0}^{\infty} \cdots \cdots \sum_{m_l=0}^{\infty} A_l \right.$$

$$+ \sum_{s_{l-1}=1}^{\infty} \cdots \cdots \sum_{s_1=1}^{\infty} \cdot \sum_{m_1=0}^{\infty} \cdots \cdots \sum_{m_l=0}^{\infty} (s_{l-1} + m_l) A_l$$

$$+ \sum_{s_{l-1}=1}^{\infty} \cdots \cdots \sum_{s_1=1}^{\infty} \cdot \sum_{m_1=0}^{\infty} \cdots \cdots \sum_{m_l=0}^{\infty} (s_{l-2} + s_{l-1} + m_{l-1}) A_l + \cdots \cdot$$

$$+ \sum_{s_{l-1}=1}^{\infty} \cdots \cdots \sum_{s_1=1}^{\infty} \cdot \sum_{m_1=0}^{\infty} \cdots \cdots \sum_{m_l=0}^{\infty} (s_1 + s_2 + m_2) A_l$$

$$+ \sum_{s_{l-1}=1}^{\infty} \cdots \cdots \sum_{s_l=1}^{\infty} \cdot \sum_{m_1=0}^{\infty} \cdots \cdots \sum_{m_l=0}^{\infty} (1 + s_1 + m_2) A_l$$

$$+ \sum_{s_{r-1}=1}^{\infty} \cdots \cdots \sum_{s_1=1}^{\infty} \cdot \sum_{m_1=0}^{\infty} \cdots \cdots \sum_{m_r=0}^{\infty} A_r$$

$$+ \sum_{s_{r-1}=1}^{\infty} \cdots \cdots \sum_{s_1=1}^{\infty} \cdot \sum_{m_1=0}^{\infty} \cdots \cdots \sum_{m_r=0}^{\infty} (s_{r-1} + m_r) A_r$$

$$+ \sum_{s_{r-1}=1}^{\infty} \cdots \cdots \sum_{s_1=1}^{\infty} \cdot \sum_{m_1=0}^{\infty} \cdots \cdots \sum_{m_l=0}^{\infty} (s_{r-2} + s_{r-1} + m_{r-1}) A_r + \cdots \cdot$$

$$+ \sum_{s_{r-1}=1}^{\infty} \cdots \cdots \sum_{s_1=1}^{\infty} \cdot \sum_{m_1=0}^{\infty} \cdots \cdots \sum_{m_r=0}^{\infty} (s_1 + s_2 + m_2) A_r$$

$$+ \sum_{s_{r-1}=1}^{\infty} \cdots \cdots \sum_{s_1=1}^{\infty} \cdot \sum_{m_1=0}^{\infty} \cdots \cdots \sum_{m_r=0}^{\infty} (1 + s_1 + m_1) A_r$$

$$= 1 - \lambda_0 h + \frac{v_0 h}{u_1} + \sum_{l=2}^{r} \left\{ \sum_{s_{l-1}=1}^{\infty} \cdots \cdots \sum_{s_1=1}^{\infty} \cdot \sum_{m_1=0}^{\infty} \cdots \cdots \sum_{m_l=0}^{\infty} \right\} A_l$$

$$+ \sum_{t=1}^{r} \left[\sum_{s_{l-1}=1}^{\infty} \cdots \cdots \sum_{s_1=1}^{\infty} \cdot \sum_{m_1=0}^{\infty} \cdots \cdots \sum_{m_l=0}^{\infty} (s_{t-1} + m_t) A_t \right.$$

$$+ \sum_{l=t+1}^{r} \left\{ \sum_{s_{l-1}=1}^{\infty} \cdots \cdots \sum_{s_1=1}^{\infty} \cdot \sum_{m_1=0}^{\infty} \cdots \cdots \sum_{m_l=0}^{\infty} (s_{t-1} + s_t + m_t) A_l \right\} \right]. \tag{25}$$

For $2 \leqq l \leqq r - 1$, using the lemma (i),

$$\sum_{s_{l-1}=1}^{\infty} \cdots \sum_{s_1=1}^{\infty} \cdot \sum_{m_1=0}^{\infty} \cdots \sum_{m_l=0}^{\infty} A_l$$

$$= \frac{\lambda_0 \mu_1 h}{\lambda_1 + \mu_1} \sum_{s_{l-1}=1}^{\infty} \left[\frac{\lambda_{l-1} \mu_l}{(\lambda_{l-1} + \mu_{l-1})(\lambda_l + \mu_l)} \right]^{s_{l-1}}$$

$$\times \prod_{i=1}^{l-2} \left\{ \sum_{s_i=1}^{\infty} \binom{s_i + s_{i+1} - 1}{s_i - 1} \left[\frac{\lambda_i \mu_{i+1}}{(\lambda_i + \mu_i)(\lambda_{i+1} + \mu_{i+1})} \right]^{s_i} \right\}$$

$$= \frac{v_{l-1} h}{u_{l-1} u_l}. \tag{26}$$

Using the lemma (iii),

$$\sum_{s_{l-1}=1}^{\infty} \cdots \sum_{s_1=1}^{\infty} \cdot \sum_{m_1=0}^{\infty} \cdots \sum_{m_l=0}^{\infty} (s_{l-1} + m_l) A_l$$

$$= \sum_{s_{l-1}=1}^{\infty} \left[\frac{\lambda_{l-1} \mu_l h}{\lambda_{l-1} + \mu_{l-1}} \right]^{s_{l-1}} \sum_{m_l=0}^{\infty} (s_{l-1} + m_l + 1) \binom{s_{l-1} + m_l}{m_l} \beta_l^{m_l}$$

$$\times \prod_{i=1}^{l-2} \left\{ \sum_{s_i=1}^{\infty} \binom{s_i + s_{i+1} - 1}{s_i - 1} \left[\frac{\lambda_i \mu_{i+1}}{(\lambda_i + \mu_i)(\lambda_{i+1} + \mu_{i+1})} \right]^{s_i} \right\}$$

$$= \frac{v_{l-2}}{u_{l-2} u_{l-1} (\lambda_l + \mu_l)} \sum_{s_{l-1}=0}^{\infty} (s_{l-1} + 1) \left[\frac{\lambda_{l-1} \mu_l u_{l-2}}{(\lambda_l + \mu_l) u_{l-1}} \right]^{s_{l-1}+1}$$

$$= \frac{v_{l-2}}{u_{l-2} u_{l-1} (\lambda_l + \mu_l)} \frac{\dfrac{\lambda_{l-1} \mu_l u_{l-2}}{(\lambda_l + \mu_l) u_{l-1}}}{\left[1 - \dfrac{\lambda_{l-1} \mu_l u_{l-2}}{(\lambda_l + \mu_l) u_{l-1}} \right]^2}$$

$$= \frac{v_{l-1}}{u_l^2}. \tag{27}$$

Generally for $2 \leq t \leq r-1$, using the lemma (i) $\sim$ (iii),

$$\sum_{s_{l-1}=1}^{\infty} \cdots \sum_{s_1=1}^{\infty} \cdot \sum_{m_1=0}^{\infty} \cdots \sum_{m_l=0}^{\infty} (s_{t-1} + s_t + m_t) A_l$$

$$= \prod_{s_{l-1}=1}^{\infty} \left[\frac{\lambda_{l-1} \mu_l}{(\lambda_{l-1} + \mu_{l-1})(\lambda_l + \mu_l)} \right]^{s_{l-1}} \prod_{i=t+1}^{l-2} \left\{ \sum_{s_i=1}^{\infty} \binom{s_i + s_{i+1} - 1}{s_i - 1} \left[\frac{\lambda_i \mu_{i+1}}{(\lambda_i + \mu_i)(\lambda_{i+1} + \mu_{i+1})} \right]^{s_i} \right\}$$

$$\times \sum_{s_t=1}^{\infty} \binom{s_t + s_{t+1} - 1}{s_t - 1} \left[\frac{\lambda_t \mu_{t+1} h}{\lambda_{t+1} + \mu_{t+1}} \right]^{s_t} \sum_{s_{t-1}=1}^{\infty} \binom{s_{t-1} + s_t - 1}{s_{t-1} - 1} \left(\frac{\lambda_{t-1} \mu_t h}{\lambda_{t-1} + \mu_{t-1}} \right)^{s_{t-1}}$$

$$\times \sum_{m_t=0}^{\infty} (s_{t-1} + s_t + m_t) \binom{s_{t-1} + s_t + m_t - 1}{m_t} \beta_t^{m_t} \frac{\lambda_0 \mu_1 h}{\lambda_1 + \mu_1}$$

$$\times \prod_{i=1}^{t-2} \left\{ \sum_{s_i=1}^{\infty} \binom{s_i + s_{i+1} - 1}{s_i - 1} \left[\frac{\lambda_i \mu_{i+1}}{(\lambda_i + \mu_i)(\lambda_{i+1} + \mu_{i+1})} \right]^{s_i} \right\}$$

$$
= \sum_{s_{l-1}=0}^{\infty} \left[\frac{\lambda_{l-1}\mu_l}{(\lambda_{l-1}+\mu_{l-1})\,(\lambda_l+\mu_l)} \right]^{s_{l-1}+1}
$$

$$
\times \prod_{i=t+1}^{l-2} \left\{ \sum_{s_i=0}^{\infty} \binom{s_i+s_{i+1}-1}{s_i-1} \left[\frac{\lambda_i\mu_{i+1}}{(\lambda_i+\mu_i)\,(\lambda_{i+1}+\mu_{i+1})} \right]^{s_{i+1}} \right\}
$$

$$
\times \sum_{s_t=0}^{\infty} \binom{s_t+s_{t+1}+1}{s_t} \left[\frac{\lambda_t\mu_{t+1}h}{\lambda_{t+1}+\mu_{t+1}} \right]^{s_{t+1}}
$$

$$
\times \sum_{s_{t-1}=0}^{\infty} (s_{t-1}+s_{t+2}) \binom{s_t+s_{t+1}+1}{s_t} \left(\frac{\lambda_{t-1}\mu_t h}{\lambda_{t-1}+\mu_{t-1}} \right)^{s_{t+1}} \left(\frac{1}{(\lambda_t+\mu_t)\,h} \right)^{s_{t-1}+s_{t}+3} \frac{v_{t-2}h}{u_{t-2}u_{t-1}}
$$

$$
\times \left[\frac{(\lambda_{t-1}+\mu_{t-1})\,u_{t-2}}{u_{t-1}} \right]^{s_{t-1}+1}
$$

$$
= \frac{v_{t-2}}{u_{t-2}u_{t-1}} \frac{1}{\lambda_t+\mu_t} \sum_{s_{l-1}=0}^{\infty} \left[\frac{\lambda_{l-1}\mu_l}{(\lambda_{l-1}+\mu_{l-1})\,(\lambda_l+\mu_l)} \right]^{s_{l-1}+1}
$$

$$
\times \prod_{i=t+1}^{l-2} \left\{ \sum_{s_i=0}^{\infty} \binom{s_i+s_{i+1}+1}{s_i} \left[\frac{\lambda_i\mu_{i+1}}{(\lambda_i+\mu_i)\,(\lambda_{i+1}+\mu_{i+1})} \right]^{s_{i+1}} \right\}
$$

$$
\times \sum_{s_t=0}^{\infty} (s_t+2) \binom{s_t+s_{t+1}+1}{s_t} \left[\frac{\lambda_t\mu_{t+1}}{(\lambda_t+\mu_t)\,(\lambda_{t+1}+\mu_{t+1})} \right]^{s_{t+1}}
$$

$$
\times \sum_{s_{t-1}=0}^{\infty} \binom{s_{t-1}+s_t+2}{s_{t-1}} \left[\frac{\lambda_{t-1}\mu_t}{(\lambda_{t-1}+\mu_{t-1})\,(\lambda_t+\mu_t)} \right]^{s_{t-1}+1} \left[\frac{(\lambda_{t-1}+\mu_{t-1})\,u_{t-2}}{u_{t-1}} \right]^{s_{t-1}+1}
$$

$$
= \frac{v_{t-2}}{u_{t-2}u_{t-1}(\lambda_t+\mu_t)} \sum_{s_{l-1}=0}^{\infty} \left[\frac{\lambda_{l-1}\mu_l}{(\lambda_{l-1}+\mu_{l-1})\,(\lambda_l+\mu_l)} \right]^{s_{l-1}+1}
$$

$$
\times \prod_{i=t+1}^{l-2} \left\{ \sum_{s_i=0}^{\infty} \binom{s_i+s_{i+1}+1}{s_i} \left[\frac{\lambda_i\mu_{i+1}}{(\lambda_i+\mu_i)\,(\lambda_{i+1}+\mu_{i+1})} \right]^{s_{i+1}} \right\}
$$

$$
\times \sum_{s_t=0}^{\infty} (s_t+2) \binom{s_t+s_{t+1}+1}{s_t} \left[\frac{\lambda_t\mu_{t+1}u_{t-1}}{(\lambda_{t+1}+\mu_{t+1})\,u_t} \right]^{s_{t+1}} \frac{\lambda_{t-1}\mu_t\,(\lambda_t+\mu_t)\,u_{t-2}u_{t-1}}{u_t^2}
$$

$$
= \frac{v_{t-1}}{u_t^2} \sum_{s_{l-1}=0}^{\infty} \left[\frac{\lambda_{l-1}\mu_l}{(\lambda_{l-1}+\mu_{l-1})\,(\lambda_l+\mu_l)} \right]^{s_{l-1}+1}
$$

$$
\times \prod_{i=t+2}^{l-2} \left\{ \sum_{s_i=0}^{\infty} \binom{s_i+s_{i+1}+1}{s_i} \left[\frac{\lambda_i\mu_{i+1}}{(\lambda_i+\mu_i)\,(\lambda_{i+1}+\mu_{i+1})} \right]^{s_{i+1}} \right\}
$$

$$
\times \sum_{s_{t+1}=0}^{\infty} (s_{t+1}+2) \binom{s_{t+1}+s_{t+2}+1}{s_{t+1}} \left[\frac{\lambda_{t+1}\mu_{t+2}}{(\lambda_{t+1}+\mu_{t+1})\,(\lambda_{t+2}+\mu_{t+2})} \right]^{s_{t+1}+1}
$$

$$
\left[\frac{(\lambda_{t+1}+\mu_{t+1})\,u_t}{u_{t+1}} \right]^{s_{t+1}+1} \left[\frac{\lambda_t\mu_{t+1}u_{t-1}}{u_{t+1}} \right]^{2} + \frac{2v_{t-1}}{u_t^2} \sum_{s_{t-1}=0}^{\infty} \left[\frac{\lambda_{l-1}\mu_l}{(\lambda_{l-1}+\mu_{l-1})\,(\lambda_l+\mu_l)} \right]^{s_{l-1}+1}
$$

$$
\times \prod_{i=t+1}^{l-2} \left\{ \sum_{s_i=0}^{\infty} \binom{s_i+s_{i+1}+1}{s_i} \left[\frac{\lambda_i\mu_{i+1}}{(\lambda_i+\mu_i)\,(\lambda_{i+1}+\mu_{i+1})} \right]^{s_{i+1}} \right\}
$$

$$\times \sum_{s_t=0}^{\infty} \binom{s_t+s_{t+1}+1}{s_t}\left[\frac{\lambda_t \mu_{t+1} u_{t-1}}{(\lambda_{t+1}+\mu_{t+1})u_t}\right]^{s_t+1}$$

$$=\frac{v_t \lambda_t \mu_{t+1} u_{t-1}^2}{u_t^2 u_{t+1}^2}\sum_{s_{t-1}=0}^{\infty}\left[\frac{\lambda_{l-1}\mu_l}{(\lambda_{l-1}+\mu_{l-1})(\lambda_l+\mu_l)}\right]^{s_{l-1}+1}$$

$$\times\prod_{i=t+2}^{l-2}\left\{\sum_{s_i=0}^{\infty}\binom{s_i+s_{i+1}+1}{s_i}\left[\frac{\lambda_i \mu_{i+1}}{(\lambda_i+\mu_i)(\lambda_{i+1}+\mu_{i+1})}\right]^{s_i+1}\right\}$$

$$\times\sum_{s_{t+1}=0}^{\infty}(s_{t+1}+2)\binom{s_{t+1}+s_{t+2}+1}{s_{t+1}}\left[\frac{\lambda_{t+1}\mu_{t+2}u_t}{(\lambda_{t+2}+\mu_{t+2})u_{t+1}}\right]^{s_{t+1}+1}+\frac{2v_{l-1}u_{t-1}}{u_t u_{l-1}u_l}$$

$$=\frac{v_t \lambda_t \mu_{t+1}u_{t-1}^2}{u_t^2 u_{t+1}^2}\sum_{s_{t-1}=0}^{\infty}\left[\frac{\lambda_{l-1}\mu_l}{(\lambda_{l-1}+\mu_{l-1})(\lambda_l+\mu_l)}\right]^{s_{l-1}+1}$$

$$\times\prod_{i=t+3}^{l-2}\left\{\sum_{s_i=0}^{\infty}\binom{s_i+s_{i+1}+1}{s_i}\left[\frac{\lambda_i\mu_{i+1}}{(\lambda_i+\mu_i)(\lambda_{i+1}+\mu_{i+1})}\right]^{s_i+1}\right\}$$

$$\times\sum_{s_{t+2}=0}^{\infty}(u_{t+2}+2)\binom{s_{t+2}+s_{t+3}+1}{s_{t+2}}\left[\frac{\lambda_{t+2}\mu_{t+3}}{(\lambda_{t+2}+\mu_{t+2})(\lambda_{t+3}+\mu_{t+3})}\right]^{s_{t+2}+1}$$

$$\times\left[\frac{(\lambda_{t+2}+\mu_{t+2})u_{t+1}}{u_{t+2}}\right]^{s_{t+2}+1}\left(\frac{\lambda_{t+1}\mu_{t+2}u_t}{u_{t+2}}\right)^2+\frac{2v_t\lambda_t\mu_{t+1}u_{t-1}^2}{u_t^2 u_{t+1}^2}$$

$$\times\sum_{s_{t-1}=0}^{\infty}\left[\frac{\lambda_{l-1}\mu_l}{(\lambda_{l-1}+\mu_{l-1})(\lambda_l+\mu_l)}\right]^{s_{l-1}+1}$$

$$\times\prod_{i=t+2}^{l-2}\left\{\sum_{s_i=0}^{\infty}\binom{s_i+s_{i+1}+1}{s_i}\left[\frac{\lambda_i\mu_{i+1}}{(\lambda_i+\mu_i)(\lambda_{i+1}+\mu_{i+1})}\right]^{s_i+1}\right\}$$

$$\times\sum_{s_{t+1}=0}^{\infty}\binom{s_{t+1}+s_{t-2}+1}{s_{t+1}}\left[\frac{\lambda_{t+1}\mu_{t+2}u_t}{(\lambda_{t+2}+\mu_{t+2})u_{t+1}}\right]^{s_{t+1}+1}+\frac{2v_{l-1}u_{t-1}}{u_t u_{l-1}u_l}$$

$$=\frac{v_{t+1}\lambda_t\lambda_{t+1}\mu_{t+1}\mu_{t+2}u_{t-1}^2}{u_{t+1}^2 u_{t+2}^2}\sum_{s_{t-1}=0}^{\infty}\left[\frac{\lambda_{l-1}\mu_l}{(\lambda_{l-1}+\mu_{l-1})(\lambda_l+\mu_l)}\right]^{s_{l-1}+1}$$

$$\times\prod_{i=t+3}^{l-2}\left\{\sum_{s_i=0}^{\infty}\binom{s_i+s_{i+1}+1}{s_i}\left[\frac{\lambda_i\mu_{i+1}}{(\lambda_i+\mu_i)(\lambda_{i+1}+\mu_{i+1})}\right]^{s_i+1}\right\}$$

$$\times\left\{\sum_{s_{t+2}=0}^{\infty}(s_{t+2}+2)\binom{s_{t+2}+s_{t+3}+1}{s_{t+2}}\left[\frac{\lambda_{t+2}+\mu_{t+3}u_{t+1}}{(\lambda_{t+3}+\mu_{t+3})u_{t+2}}\right]^{s_{t+2}+1}\right\}$$

$$+\frac{2v_{l-1}\lambda_t\mu_{t+1}u_{t-1}^2}{u_t u_{t+1}u_{l-1}u_l}+\frac{2v_{l-1}u_{t-1}^2}{u_{t-1}u_t u_{l-1}u_l}$$

$$=\cdots\cdots\cdots\cdots$$

$$=\frac{v_{l-2}\lambda_t\lambda_{t+1}\cdots\cdots\lambda_{l-2}\mu_{t+1}\mu_{t+2}\cdots\cdots\mu_{l-1}u_{t-1}^2}{u_{l-2}^2 u_{l-1}^2}\sum_{s_{l-1}=0}^{\infty}(s_{l-1}+2)\left[\frac{\lambda_{l-1}\mu_l u_{l-2}}{(\lambda_l+\mu_l)u_{l-1}}\right]^{s_{l-1}+1}$$

$$+\frac{2v_{l-1}\lambda_l\lambda_{l+1}\cdots\lambda_{l-3}\mu_{l+1}\mu_{l+2}\cdots\mu_{l-2}u_{l-1}^2}{u_{l-3}u_{l-2}u_{l-1}u_l}+\cdots+\frac{2v_{l-1}\lambda_l\mu_{l+1}u_{l-1}^2}{u_lu_{l+1}u_{l-1}u_l}+\frac{2v_{l-1}u_{l-1}^2}{u_{l-1}u_lu_{l-1}u_l}$$

$$=\frac{v_{l-1}u_{l-1}^2}{u_{l-1}u_l}\left[\frac{\lambda_l\cdots\lambda_{l-2}\mu_{l+1}\cdots\mu_{l-1}(\lambda_l+\mu_l)}{u_{l-2}u_l}+\frac{\lambda_l\cdots\lambda_{l-2}\mu_{l+1}\cdots\mu_{l-1}}{u_{l-2}u_{l-1}}+2\right.$$

$$\left.\times\sum_{i=t-1}^{l-3}\frac{\lambda_t\cdots\lambda_i\mu_{i+1}\cdots\mu_{i+1}}{u_iu_{i+1}}\right].\tag{28}$$

Next, for $l=r$,

$$\sum_{s_{r-1}=1}^{\infty}\cdots\cdots\sum_{s_1=1}^{\infty}\cdot\sum_{m_1=0}^{\infty}\cdots\cdots\sum_{m_r=0}^{\infty}A_r=\frac{\lambda_0\lambda_1\cdots\lambda_{r-1}}{u_{r-1}},\tag{29}$$

$$\sum_{s_{r-1}=1}^{\infty}\cdots\cdots\sum_{s_1=1}^{\infty}\cdot\sum_{m_1=0}^{\infty}\cdots\cdots\sum_{m_r=0}^{\infty}(s_{r-1}+s_r+m_r)A_r$$

$$=\frac{v_{r-2}\lambda_t\lambda_{t+1}\cdots\lambda_{r-2}\mu_{t+1}\mu_{t+2}\cdots\mu_{r-1}u_{t-1}^2}{u_{r-2}^2u_{r-1}^2}\sum_{s_{r-1}=0}^{\infty}(s_{r-1}+2)\left(\frac{\lambda_{r-1}u_{r-2}}{u_{r-1}}\right)^{s_{r-1}+1}$$

$$+\frac{2\lambda_0\lambda_1\cdots\lambda_{r-1}u_{t-1}^2}{u_{r-1}}\sum_{i=t-1}^{r-3}\frac{\lambda_t\cdots\lambda_i\mu_{i+1}\cdots\mu_{i+1}}{p_ip_{i+1}}$$

$$=\frac{\lambda_0\lambda_1\cdots\lambda_{r-1}u_{t-1}^2}{u_{r-1}}\left[\frac{\lambda_t\cdots\lambda_{r-2}}{\mu_1\cdots\mu_lu_{r-2}}+\frac{\lambda_t\cdots\lambda_{r-2}\mu_{t+1}\cdots\mu_{r-1}}{u_{r-2}u_{r-1}}+2\right.$$

$$\left.\times\sum_{i=t-1}^{r-3}\frac{\lambda_t\cdots\lambda_i\mu_{i+1}\cdots\mu_{i+1}}{u_iu_{i+1}}\right],\tag{30}$$

$$\sum_{s_{r-1}=1}^{\infty}\cdots\cdots\sum_{s_1=1}^{\infty}\cdot\sum_{m_1=0}^{\infty}\cdots\cdots\sum_{m_r=0}^{\infty}(s_{r-1}+m_r)A_r$$

$$=\frac{v_{r-2}}{u_{r-2}u_{r-1}\mu_r}\sum_{s_{r-1}=0}^{\infty}(s_{r-1}+1)\left[\frac{\lambda_{r-1}u_{r-2}}{u_{r-1}}\right]^{s_{r-1}+1}$$

$$=\frac{\lambda_0\lambda_1\cdots\lambda_{r-1}}{\mu_1\mu_2\cdots\mu_r}.\tag{31}$$

Using (26) and (29),

$$\frac{v_0h}{u_1}+\sum_{l=2}^{r}\left\{\sum_{s_{l-1}=1}^{\infty}\cdots\cdots\sum_{s_1=1}^{\infty}\cdot\sum_{m_1=0}^{\infty}\cdots\cdots\sum_{m_l=0}^{\infty}\right\}$$

$$A_l=h\left\{\frac{v_0}{u_1}+\frac{v_1}{u_1u_2}+\frac{v_2}{u_2u_3}+\cdots+\frac{v_{r-2}}{u_{r-2}u_{r-1}}+\frac{\lambda_0\lambda_1\cdots\lambda_{r-1}}{u_{r-1}}\right\}=\lambda_0h.$$

Using (27), (28), (30), and (31),

$$\sum_{s_{l-1}=1}^{\infty}\cdots\cdots\sum_{s_1=1}^{\infty}\cdot\sum_{m_1=0}^{\infty}\cdots\cdots\sum_{m_l=0}^{\infty}(s_{l-1}+m_l)A_l+\sum_{l=t+1}^{r}$$

$$\times\left\{\sum_{s_{t-1}=1}^{\infty}\cdots\cdots\sum_{s_1=1}^{\infty}\cdot\sum_{m_1=0}^{\infty}\cdots\cdots\sum_{m_l=0}^{\infty}(s_{t-1}+s_t+m_t)A_l\right\}$$

$$=\frac{v_{t-1}}{u_t^2}+\sum_{l=t+1}^{r-1}\frac{v_{k-1}u_{t-1}^2}{u_{k-1}u_k}$$

$$\times\left[2\left(\frac{1}{u_{t-1}u_t}+\frac{\lambda_t\mu_{t+1}}{u_tu_{t+1}}+\frac{\lambda_t\lambda_{t+1}\mu_{t+1}\mu_{t+2}}{u_{t+1}u_{t+2}}+\cdots\cdots+\frac{\lambda_t\lambda_{t+1}\cdots\cdots\lambda_{l-3}\mu_{t+1}\mu_{t+2}\cdots\cdots\mu_{l-2}}{u_{l-3}u_{l-2}}\right)\right.$$

$$\left.+\frac{\lambda_t\lambda_{t+1}\cdots\cdots\lambda_{l-2}\mu_{t+1}\mu_{t+2}\cdots\cdots\mu_{l-1}(\lambda_l+\mu_l)}{u_{l-2}u_l}+\frac{\lambda_t\lambda_{t+1}\cdots\cdots\lambda_{l-2}\mu_{t+1}\mu_{t+2}\cdots\cdots\mu_{l-1}}{u_{l-2}u_{l-1}}\right]$$

$$+\frac{\lambda_0\lambda_1\cdots\cdots\lambda_{r-1}u_{t-1}^2}{u_{r-1}}$$

$$\times\left[2\left(\frac{1}{u_{t-1}u_t}+\frac{\lambda_t\mu_{t+1}}{u_tu_{t+1}}+\frac{\lambda_t\lambda_{t+1}\mu_{t+1}\mu_{t+2}}{u_{t+1}u_{t+2}}+\cdots\cdots+\frac{\lambda_t\lambda_{t+1}\cdots\cdots\lambda_{r-3}\mu_{t+1}\mu_{t+2}\cdots\cdots\mu_{r-2}}{u_{r-3}u_{r-2}}\right)\right.$$

$$\left.+\frac{\lambda_t\lambda_{t+1}\cdots\cdots\lambda_{r-2}}{\mu_1\mu_2\cdots\cdots\mu_tu_{r-2}}+\frac{\lambda_t\lambda_{t+1}\cdots\cdots\lambda_{r-2}\mu_{t+1}\mu_{t+2}\cdots\cdots\mu_{r-1}}{u_{r-2}u_{r-1}}\right]$$

$$=\frac{2\lambda_0\cdots\cdots\lambda_{t+1}u_{t-1}^2}{u_{t-1}u_tu_{t+1}}+\frac{2\lambda_t\mu_{t+1}\lambda_0\cdots\cdots\lambda_{t+2}u_{t-1}^2}{u_tu_{t+1}u_{t+2}}+\cdots\cdots$$

$$+\frac{2\lambda_t\cdots\cdots\lambda_{r-4}\mu_{t+1}\cdots\cdots\mu_{r-3}\lambda_0\cdots\cdots\lambda_{r-2}u_{t-1}^2}{u_{r-4}u_{r-3}u_{r-2}}$$

$$+\frac{2\lambda_t\cdots\cdots\lambda_{r-3}\mu_{t+1}\cdots\cdots\mu_{r-2}\lambda_0\cdots\cdots\lambda_{r-2}u_{t-1}^2}{u_{r-3}u_{r-2}u_{r-1}}$$

$$+\frac{v_{t-1}}{u_t^2}+\frac{v_tu_{t-1}^2}{u_{t-1}u_t^2u_{t+1}}+\frac{v_{t+1}u_{t-1}^2\lambda_t\mu_{t+1}}{u_tu_{t+1}^2u_{t+2}}+\cdots\cdots+\frac{v_{r-2}u_{t-1}^2\lambda_t\cdots\cdots\lambda_{r-3}\mu_{t+1}\cdots\cdots\mu_{r-2}}{u_{r-3}u_{r-2}^2u_{r-1}}$$

$$+\frac{\lambda_0\lambda_1\cdots\cdots\lambda_{r-1}\lambda_t\cdots\cdots\lambda_{r-2}\mu_{t+1}\cdots\cdots\mu_{r-1}u_{t-1}^2}{u_{r-2}u_{r-1}^2}$$

$$+\frac{v_t(\lambda_{t+1}+\mu_{t+1})u_{t-1}^2}{u_{t-1}u_tu_{t+1}^2}+\frac{v_{t+1}\lambda_t\mu_{t+1}(\lambda_{t+2}+\mu_{t+2})u_{t-1}^2}{u_tu_{t+1}u_{t+2}^2}+\cdots\cdots$$

$$+\frac{v_{r-2}\lambda_t\cdots\cdots\lambda_{r-3}\mu_{t+1}\cdots\cdots\mu_{r-2}(\lambda_{r-1}+\mu_{r-1})u_{t-1}^2}{u_{r-3}u_{r-2}u_{r-1}^2}$$

$$+\frac{\lambda_0\lambda_1\cdots\cdots\lambda_{r-1}\lambda_t\cdots\cdots\lambda_{r-2}u_{t-1}^2}{\mu_1\mu_2\cdots\cdots\mu_tu_{r-2}u_{r-1}}$$

$$=\frac{\lambda_0\lambda_1\cdots\cdots\lambda_{t-1}}{\mu_1\mu_2\cdots\cdots\mu_t}.$$

Then we obtain

$$\sum_{n=1}^{\infty} nf_0{}^{(n)} = 1 + \sum_{i=0}^{r-1} \frac{\lambda_0 \lambda_1 \cdots \lambda_i}{\mu_1 \mu_2 \cdots \mu_{i+1}}.$$

REFERENCES :

(1) W. Feller ; An Introduction to Probability Theory and its Applications (1958) Wiley

                                                   p 413,

(2) ·································· p 356,

(3) ·································· p 416,

(4) ·································· p 417, (7. 16), (7. 16 a),

(5) ·································· p 415,

(6) ·································· p 415, (7. 9), (7. 10).