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THE ORDER OF n ITEMS PROCESSED ON m MACHINES. (II)

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1. PROBLEM AND SOLUTION

In the previous paper[1] we considered the problem of deciding the order in which n items should be processed by m machines in order to minimize the time required to complete all the operations, by using the functional-equation approach formulated by R. Bellman [2]. In this paper we shall consider the same problem and present a new formulation and derive new results which cover the previous results.

Let m machines be named by $M_1, M_2, \cdots\cdots, M_m$, and let $m_{k,i}$ be the time required to process the ith item on the machine M_k where the processing requires that the machines be used by the same numerical order for any item.

When an optimal scheduling procedure is employed and after the processing of some definite sequence S of items, the machine M_k is committed t_{k-1} hours ahead for the machine M_{k-1}, $k=2, 3, \cdots\cdots, m$, we see that the last machine M_m is committed $t=\sum_{k=1}^{m-1} t_k$ hours ahead for the first machine M_1.

If ith item is processed first after the sequence S of items, then by defining

$f_1(i, t)$ = the time consumed in processing the ith item,

we have (see Fig. 1)

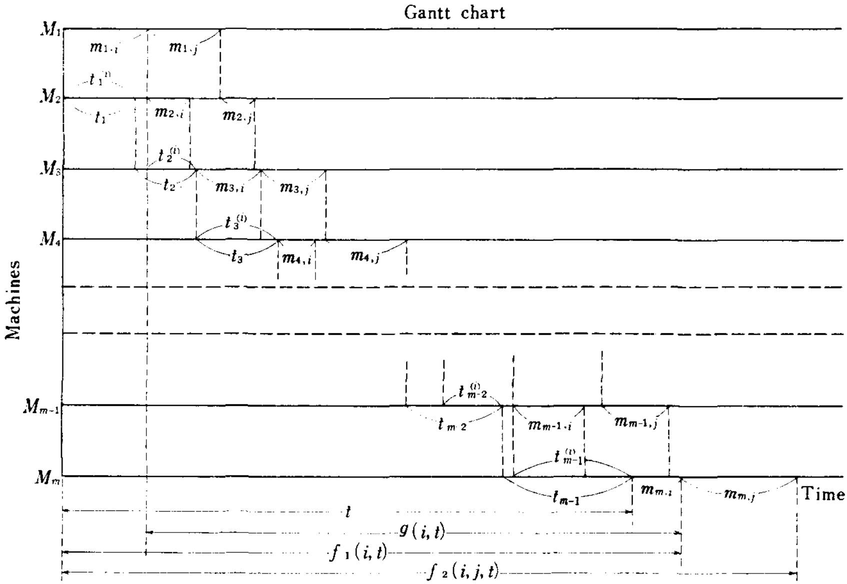

Fig. 1

$$f_1(i,\ t)=m_{1,\,i}+g(i,\ t), \tag{1}$$

where
$$g(i,\ t)=\sum_{k=2}^{m}\{m_{k,\,i}+\max\,(t^{(i)}{}_{k-1}-m_{k-1,\,i},\ 0)\} \tag{2}$$

$$t^{(i)}{}_1=t_1,\ \ t^{(i)}{}_k=t_k-\max\,(m_{k-1,\,i}-t^{(i)}{}_{k-1},\ 0). \tag{3}$$
$$(k=2,\ 3,\ \cdots,\ m-1)$$

If we choose the j th item to follow, then by defining

$f_2(i,\ j,\ t)=$the time consumed in processing both the i th and j th items in this order after the sequence S of items,

we have

$$f_2(i,\ j,\ t)=m_{1,\,i}+f_1(j,\ g(i,\ t)). \tag{4}$$

On the other hand, if we interchange the orders of the i th and the j th item, we obtain similarly

$$f_2(j,\ i,\ t)=m_{1,\,j}+f_1(i,\ g(j,\ t)). \tag{5}$$

So that, in the case when $f_2(i,\ j,\ t)<f_2(j,\ i,\ t)$ after the i th and j th items of the above both cases, if new f-term which follows from $f_2(i,\ j,\ t)$ is smaller than the corresponding f-term for the $f_2(j,\ i,\ t)$ for any of

the following items, then the order of operations which minimizes new f-term is optimal. That is to say, if this condition follows we choose the order of the items which yields minimum of $f_2(i, j, t)$ and $f_2(j, i, t)$.

Hence we obtain the next theorem.

Therem. 1. An optimal ordering is determined by the following rule : When the above mentioned condition follows, item i precedes item j if

$$f_2(i, j, t) > f_2(j, i, t). \tag{6}$$

If there is equality, either ordering is optimal, provided that it is consistent with all the definite preferences.

2. THE VALUE OF $f_1(i, t)$ AND $f_2(i, j, t)$

From (3), we have

$$\begin{aligned} t^{(i)}{}_k = \sum_{e=1}^{k} t_e - \max\Big[&\sum_{e=1}^{k-1} t_e,\ \sum_{e=1}^{k-2} t_e + m_{k-1,\,i},\ \sum_{e=1}^{k-3} t_e \\ &+ \sum_{e=k-2}^{k-1} m_{e,\,i},\ \cdots\cdots,\ \sum_{e=1}^{2} t_e + \sum_{e=3}^{k-1} m_{e,\,i},\ t_1 \\ &+ \sum_{e=2}^{k-1} m_{e,\,i},\ \sum_{e=1}^{k-1} m_{e,\,i}\Big], \quad (\mathrm{k}=1,\ 2,\ \cdots\cdots,\ m-1) \end{aligned} \tag{7}$$

hence we obtain from (1) and (2).

$$\begin{aligned} f_1(i,\ t) = \sum_{k=1}^{m} m_{k,\,i} - \sum_{k=1}^{m-1} m_{k,\,i} - \sum_{k=1}^{m-1} \max\Big[&\sum_{e=1}^{k-1} t_e, \\ &\sum_{e=1}^{k-2} t_e + m_{k-1,\,i},\ \cdots\cdots,\ \sum_{e=1}^{2} t_e + \sum_{e=3}^{k-1} m_{e,\,i},\ t_1 \\ &+ \sum_{e=2}^{k-1} m_{e,\,i},\ \sum_{e=1}^{k-1} m_{e,i}\Big] + \sum_{k=1}^{m-1} \max\Big[\sum_{e=1}^{k} t_e,\ \sum_{e=1}^{k-1} t_e \\ &+ m_{k,\,i},\ \sum_{e=1}^{k-2} t_e + \sum_{e=k-1}^{k} m_{e,\,i},\ \cdots\cdots,\ t_1 \\ &+ \sum_{e=2}^{k} m_{e,\,i},\ \sum_{e=1}^{k} m_{e,\,i}\Big] \\ = m_{m,\,i} + \max\Big[&\sum_{e=1}^{m-1} t_e,\ \sum_{e=1}^{m-2} t_e + m_{m-1,\,i},\ \sum_{e=1}^{m-3} t_e \\ &+ \sum_{e=m-2}^{m-1} m_{e,\,i},\ \cdots\cdots,\ \sum_{e=1}^{2} t_e + \sum_{e=3}^{m-1} m_{e,\,i},\ t_1 \\ &+ \sum_{e=2}^{m-1} m_{e,\,i},\ \sum_{e=1}^{m-1} m_{e,\,i}\Big] \end{aligned} \tag{8}$$

Next, for $g(i, t)$, as the term corresponding to t_e is $m_{e+1, i}+\max(t^{(i)}{}_e-m_{e, i}, 0)$, we have from (8), being

$$\sum_{e=1}^{k}\{m_{e+1, i}+\max(t^{(i)}{}_e-m_{e, i}, 0)\}+\sum_{e=k+1}^{m-1}m_{e, j}$$

$$=m_{k+1, i}-m_{1, i}+\max\Big[\sum_{p=1}^{k}t_p, \sum_{p=1}^{k-1}t_p+m_{k, i}, \cdots\cdots, t_1+\sum_{p=2}^{k}m_{p, i}, \sum_{p=1}^{k}m_{p, i}\Big], \quad (k=0, 1, \cdots\cdots, m-1);$$

$$f_1(j, g(i, t))=m_{m, j}-m_{m, i}-m_{1, i}+\max\Big[\sum_{p=1}^{m-1}t_p, \sum_{p=1}^{m-2}t_p+m_{m-1, i}, \cdots\cdots, t_1+\sum_{p=2}^{m-1}m_{p, i}, \sum_{p=1}^{m-1}m_{p, i}, m_{m-1, i}+m_{m-1, j}-m_{m, i}+\max\Big[\sum_{p=1}^{m-2}t_p, \sum_{p=1}^{m-3}t_p+m_{m-2, i}, \cdots\cdots, t_1+\sum_{p=2}^{m-2}m_{p, i}, \sum_{p=1}^{m-2}m_{p, i}\Big], \cdots\cdots, m_{2,i}+\sum_{e=2}^{m-1}m_{e, j}-m_{m, i}+\max[t_1, m_{1, i}], m_{1, i}+\sum_{e=1}^{m-1}m_{e, j}-m_{m, i}\Big] \tag{9}$$

So that, from (4) we have

$$f_2(i, j, t)=m_{m, j}+m_{m, i}+\max\Big[\sum_{p=1}^{m-1}t_p, \sum_{p=1}^{m-2}t_p+m_{m-1, i}, \cdots\cdots, t_1+\sum_{p=2}^{m-1}m_{p, i}, \sum_{p=1}^{m-1}m_{p, i}, m_{m-1, i}+m_{m-1, j}-m_{m, i}+\max\Big[\sum_{p=1}^{m-2}t_p, \sum_{p=1}^{m-3}t_p+m_{m-2, i}, \cdots\cdots, t_1+\sum_{p=2}^{m-2}m_{p, i}, \sum_{p=1}^{m-2}m_{p, i}\Big], m_{m-2, i}+\sum_{e=m-2}^{m-1}m_{e, j}-m_{m, i}+\max\Big[\sum_{p=1}^{m-3}t_p, \sum_{p=1}^{m-4}t_p+m_{m-3, i}, \cdots\cdots, t_1+\sum_{p=2}^{m-3}m_{p, i}, \sum_{p=1}^{m-3}m_{p, i}\Big], \cdots\cdots, m_{2, i}+\sum_{e=2}^{m-1}m_{e, j}-m_{m, i}+\max[t_1, m_{1, i}], m_{1, i}+\sum_{e=1}^{m-1}m_{e, j}-m_{m, i}\Big] \tag{10}$$

Similarly we obtain the value of $f_2(j, i, t)$ by exchanging j for i and i for j.

3. SPECIALIZATION: THE CASE WHEN $\min_i m_{m,i} \geqq \max_i m_{m-1,i}$.

In this case, as we have

$$t_{m-1} \geqq m_{m,e} \geqq \max_i m_{m-1,i}$$

the condition of the theorem 1 holds.

Since we have

$$\sum_{p=1}^{m-1} t_p \geqq \sum_{p=1}^{m-2} t_p + m_{m-1,i}; \quad \sum_{p=1}^{m-1} t_p \geqq \sum_{p=1}^{m-2} t_p + m_{m-1,j}$$

we obtain from (6) and (10)

$$C_{ij} < C_{ji} \tag{11}$$

where

$$\begin{aligned} C_{ij} = \max \Big[& \sum_{p=1}^{m-1} t_p, \ \sum_{p=1}^{m-3} t_p + \sum_{p=m-2}^{m-1} m_{p,i}, \ \sum_{p=1}^{m-4} t_p \\ & + \sum_{p=m-3}^{m-1} m_{p,i}, \ \cdots\cdots, \ t_1 + \sum_{p=2}^{m-1} m_{p,i}, \ \sum_{p=1}^{m-1} m_{p,i}, \ m_{m-1,i} \\ & + m_{m-1,j} - m_{m,i} + \max \Big[\sum_{p=1}^{m-2} t_p, \ \sum_{p=1}^{m-3} t_p \\ & + m_{m-2,i}, \ \cdots\cdots, \ t_1 + \sum_{p=2}^{m-2} m_{p,i}, \ \sum_{p=1}^{m-2} m_{p,i} \Big], \ m_{m-2,i} \\ & + \sum_{e=m-2}^{m-1} m_{e,j} - m_{m,i} + \Big[\sum_{p=1}^{m-3} t_p, \ \sum_{p=1}^{m-4} t_p + m_{m-3,i}, \ \cdots\cdots, \ t_1 \\ & + \sum_{p=2}^{m-3} m_{p,i}, \ \sum_{p=1}^{m-3} m_{p,i} \Big], \ \cdots\cdots, \ m_{2,i} + \sum_{e=2}^{m-1} m_{e,j} - m_{m,i} \\ & + \max [t_1, \ m_{1,i}], \ m_{1,i} + \sum_{e=1}^{m-1} m_{e,j} - m_{m,i} \Big] \end{aligned}$$

and C_{ji} means a formula obtained by exchanging j for i and i for j in C_{ij}.

Let us express C'_{ij}, C'_{ji} as the formulas obtained by dropping the first term $\sum_{p=1}^{m-1} t_p$ in C_{ij}, C_{ji}, respectively. Then, if

$$C'_{ij} > C'_{ji} \tag{11'}$$

holds, the left hand of (11) is not larger than the right hand of (11). So that we can use (11′) as a criterion; that is to say, if (11′) holds item i precedes item j.

Next, for deciding the first item, we use the formula introduced by putting all the $t_p(p=1, 2, \cdots\cdots, m-2)$ as zero in (11′). By a simple computation we obtain

$$D_{ij}<D_{ji} \tag{12}$$

where

$$D_{ij}=\min\Big[\sum_{p=1}^{m-1} m_{p,\,j},\ \sum_{p=1}^{m-3} m_{p,j}+\sum_{p=m-1}^{m} m_{p,\,i},\ \sum_{p=1}^{m-4} m_{p,\,j}+\sum_{p=m-2}^{m} m_{p,\,i},\ \cdots\cdots,\ m_{1,\,j}+\sum_{p=3}^{m} m_{p,\,i},\ \sum_{p=2}^{m} m_{p,\,i}\Big],$$

and D_{ji} means a formula obtained by exchanging j for i and i for j in D_{ij}.

Consequently we obtained the next theorem:

Theorem. 2.

When $$\min_i m_{m,\,i}\geqq\max_i m_{m-1,\,i}$$

holds, an optimal ordering is determined dy the following rule:

(I) The first item is determined by (12).

(II) Next, by using the definite t_p obtained from the operation of the first item i, we determine the second item by (11′) and we continue this procedure.

4. SPECIALIZATION: THE CASE WHEN $\min_i m_{k,\,i}\geqq\max_i m_{k+1,\,i}$ $(k=1, 2, \cdots\cdots, h-1)$ AND $\min_i m_{k+1,\,i}\geqq\max_i m_{k,\,i}$ $(k=h+1, h+2, \cdots\cdots, m-1)$ HOLD, WHERE h IS A CONSTANT $(1\leqq h\leqq m-1)$

In this case, as we have

$$t_k=m_{k+1,\,e}\leqq\min_i m_{k,\,i} \quad (k=1, 2, \cdots\cdots, h-1) \tag{13}$$

$$t_k\geqq m_{k+1,\,e}\geqq\max_i \mathrm{m}_{k,\,i} \quad (k=h+1, h+2, \cdots\cdots, m-1) \tag{14}$$

the condition of the theorem 1 holds and in (8), from (13) we obtain

$$\sum_{e=1}^{m-1} m_{e,\,i}\geqq\sum_{e=1}^{h-1} t_e+\sum_{e=h}^{m-1} m_{e,\,i};\ \sum_{e=1}^{h-2} t_e+\sum_{e=h-1}^{m-1} m_{e,\,i};\ \cdots\cdots;\ t_1+\sum_{e=2}^{m-1} m_{e,\,i}$$

and from (14) we obtain

$$\sum_{e=1}^{m-1} t_e\geqq\sum_{e=1}^{m-2} t_e+m_{m-1,\,i};\ \sum_{e=1}^{m-3} t_e+\sum_{e=m-2}^{m-1} m_{e,\,i};\ \cdots\cdots;\ \sum_{e=1}^{h} t_e+\sum_{e=h+1}^{m-1} m_{e,\,i}.$$

So that (8) reduces to (15)

$$f_1(i,\ t)=m_{m,\,i}+\max\left[\sum_{e=1}^{m-1} t_e,\ \sum_{e=1}^{m-1} m_{e,\,i}\right]. \tag{15}$$

Similarly (9) reduces to (16)

$$\begin{aligned} f_1(j,\ g(i,\ t)) &= m_{m,\,j}+m_{m,\,i}-m_{1,\,i} \\ &+\max\left[\sum_{p=1}^{m-1} t_p,\ \sum_{p=1}^{m-1} m_{p,\,i},\ m_{1,\,i}-m_{m,\,i}+\sum_{p=1}^{m-1} m_{p,\,j}\right] \end{aligned} \tag{16}$$

Hence, from $f_2(i,\ j,\ t)<f_2(j,\ i,\ t)$ we have

$$\begin{aligned} &\max\left[\sum_{p=1}^{m-1} t_p,\ \sum_{p=1}^{m-1} m_{p,\,i},\ m_{1,\,i}-m_{m,\,i}+\sum_{p=1}^{m-1} m_{p,\,j}\right] \\ &<\max\left[\sum_{p=1}^{m-1} t_p,\ \sum_{p=1}^{m-1} m_{p,\,j},\ m_{1,\,j}-m_{m,\,j}+\sum_{p=1}^{m-1} m_{p,\,i}\right] \end{aligned} \tag{17}$$

So that, if

$$\begin{aligned} &\max\left[\sum_{p=1}^{m-1} m_{p,\,i},\ m_{1,\,i}-m_{m,\,i}+\sum_{p=1}^{m-1} m_{p,\,j}\right] \\ &<\max\left[\sum_{p=1}^{m-1} m_{p,\,j},\ m_{1,\,j}-m_{m,\,j}+\sum_{p=1}^{m-1} m_{p,\,i}\right] \end{aligned} \tag{18}$$

then the left hand of (17) is not larger than the right hand of (17).

From (18) we easily obtain

$$\min\left[\sum_{p=1}^{m-1} m_{p,\,j},\ \sum_{p=2}^{m} m_{p,\,i}\right]>\min\left[\sum_{p=1}^{m-1} m_{p,\,i},\ \sum_{p=2}^{m} m_{p,\,j}\right]$$

Consequently we obtain the next theorem.

Theorem. 3. When, for a certain constant h $(1\leqq h\leqq m-1)$

$$\min_i m_{k,\,i}\geqq\max_i m_{k+1,\,i}\quad (k=1,\ 2,\ \cdots\cdots,\ h-1)$$

and

$$\min_i m_{k+1,\,i}\geqq\max_i m_{k,\,i}\quad (k=h+1,\ h+2,\ \cdots\cdots,\ m-1)$$

hold, an optimal ordering is determined by the following rule: Item i precedes item j if

$$\min\left[\sum_{p=1}^{m-1} m_{p,\,i},\ \sum_{p=2}^{m} m_{p,\,j}\right]<\min\left[\sum_{p=1}^{m-1} m_{p,j},\ \sum_{p=2}^{m} m_{p,\,i}\right] \tag{19}$$

If there is equality, either ordering is optimal.

From the theorem 3 for $h=m-1$ and $h=1$ respectively, we obtain the next corollary.

Corollary. When either

(a) $\min_i m_{k,\,i}\geqq\max_i m_{k-1,\,i}\quad (k=1,\ 2,\ \cdots\cdots,\ m-2)$

or (b) $\min_i m_{k+1,\,i}\geqq\max_i m_{k,\,i}\quad (k=2,\ 3,\ \cdots\cdots,\ m-1)$

holds, an optimal ordering is determined by the following rule: Item i precedes item j if (19) holds. If there is equality, either ordering is optimal.

Thus theorem 3 and its corollary generalize the corllary in the former paper.

Especially for the case $m=3$, this corollary coincides with the Johnson's criterion [3].

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