

THE ORDER OF n ITEMS PROCESSED ON m MACHINES.**ICHIRO NABESHIMA**

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1. PROBLEM

In this paper we consider the problem of deciding the order in which n items should be processed by m machines in order to minimize the time required to complete all the operations.

Let m machines be named by M_1, M_2, \dots, M_m , and let $m_{k,i}$ be the time required to process the i th item on the machine M_k where the processing requires that the machines be used by the same numerical order for any item.

2. SOLUTION

In the following, we use the functional-equation approach. [1]

Let us define

$f[i, j, \dots; t] =$ the time consumed in processing the remained items after the processing of some definite sequence S of items, when the machine M_k is committed t_{k-1} hours ahead for the machine M_{k-1} , $k=2, 3, \dots, m$; and an optimal scheduling procedure is employed.

In this case, the last machine M_m is committed $t = \sum_{k=1}^{m-1} t_k$ hours ahead for the first machine M_1 .

Then, after the sequence S of items, if i th item is processed first, we have (see Fig. 1)

$$f[i, j, \dots; t] = m_{1,i} + f[0, j, \dots; g(i, t)]$$

$$g(i, t) = \sum_{k=2}^m \{m_{k,i} + \max(t_k^{(i)} - m_{k-1,i}, 0)\}$$

where

$$t_1^{(i)} = t_1,$$

$$t_k^{(i)} = t_k - \max(m_{k-1,i} - t_{k-1}^{(i)}, 0)$$

$$k=2, 3, \dots, m-1.$$

If we choose the j th item to follow, we obtain, by putting

$$A_k^{(i)} = m_{k+1,i} - m_{k,j} + \max(t_k^{(i)} - m_{k,j}, 0),$$

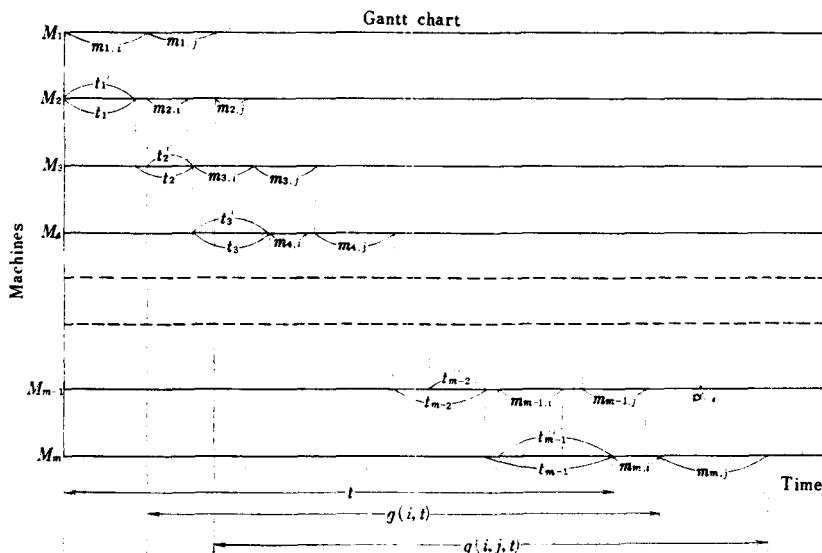


Fig. 1.

$$\begin{aligned}
 k &= 1, 2, \dots, m-1; \\
 f[i, j, \dots, t] &= m_{1,t} + m_{1,j} + f[0, 0, \dots; g(i, j, t)] \\
 g(i, j, t) &= m_{2,j} + \max(A_1^{(t)}, 0) \\
 &\quad + m_{3,j} + \max[A_2^{(t)} - \max(-A_1^{(t)}, 0), 0] \\
 &\quad + m_{4,j} + \max[A_3^{(t)} - \max[\max(-A_1^{(t)}, 0) - A_2^{(t)}, 0], 0] \\
 &\quad + m_{5,j} + \max[A_4^{(t)} - \max[\max[\max(-A_1^{(t)}, 0) - A_2^{(t)}, 0] - A_3^{(t)}, 0], 0] \\
 &\quad + \dots \\
 &\quad + m_{m,j} + \max[A_{m-1}^{(t)} - \max[\max[\dots \max[\max[\max(-A_1^{(t)}, 0) \\
 &\quad - A_2^{(t)}, 0] - A_3^{(t)}, 0] \dots] - A_{m-3}^{(t)}, 0] - A_{m-2}^{(t)}, 0], 0] \\
 &= \sum_{k=2}^m (m_{k,j} + \max[A_{k-1}^{(t)} - \max[\max[\dots \max[\max[\max(-A_1^{(t)}, 0) \\
 &\quad - A_2^{(t)}, 0] - A_3^{(t)}, 0] \dots] - A_{k-2}^{(t)}, 0], 0])
 \end{aligned} \tag{1}$$

On the other hand, if we interchange the orders of the i th item and the j th item, we obtain similarly

$$f[i, j, \dots, t] = m_{1,j} + m_{1,t} + f[0, 0, \dots; g(j, i, t)]$$

$$=g(j, i, t) \sum_{k=2}^m \{m_{k,i} + \max[A_{k-1}^{(j)} - \max[\max[\dots \max[\max[\max[\\(-A_1^{(j)}, 0) - A_2^{(j)}, 0] - A_3^{(j)}, 0] \dots] - A_{k-2}^{(j)}, 0], 0]\}] \quad (2)$$

where

$$\begin{aligned} t_1^{(j)} &= t_1 \\ t_k^{(j)} &= t_k - \max(m_{k-1,j} - t_{k-1}^{(j)}, 0), \\ k &= 1, 2, \dots, m-1. \end{aligned}$$

and

$$A_k^{(j)} = m_{k+1,j} + m_{k,i} + \max(t_k^{(j)} - m_{k,j}, 0). \\ k=1, 2, \dots, m-1.$$

So that, in the case when $g(i, j, t) < g(j, i, t)$, after the i th and j th item of the above both cases, if new t -term g which follows from $g(i, j, t)$ is smaller than the corresponding t -term for the $g(j, i, t)$ for any of the following items, than the order of operations which minimizes new t -term is optimal. That is to say, if this condition follows we choose the order of the items which yields minimum of $g(i, j, t)$ and $g(j, i, t)$.

Hence, we obtain the next theorem.

Theorem. An optimal ordering is determined by the following rule: When the above mentioned condition follows, item i precedes item j if

$$g(i, j, t) < g(j, i, t)$$

If there is equality, either ordering is optimal, provided that it is consistent with all the definite preferences.

3. SPECIALIZATION

In the case when $m=2$, the condition of the theorem holds and we have the same result as both S. Johnson and R. Bellman had shown. [1], [2].

For the case when $m \geq 3$, for example, if

$$\min_i m_{k,i} \geq \max_i m_{k+1,i}, \\ k=1, 2, \dots, m-2;$$

then, as we have

$$t_k^{(i)}, t_k^{(j)} \leq t_k = m_{k+1,i}, \\ k=1, 2, \dots, m-2;$$

the condition of the theorem holds. (see Fig. 2)

In this special case, we obtain, in (1)

$$A_k^{(i)} = m_{k+1,i} - m_{k,j}, \quad k=1, 2, \dots, m-2;$$

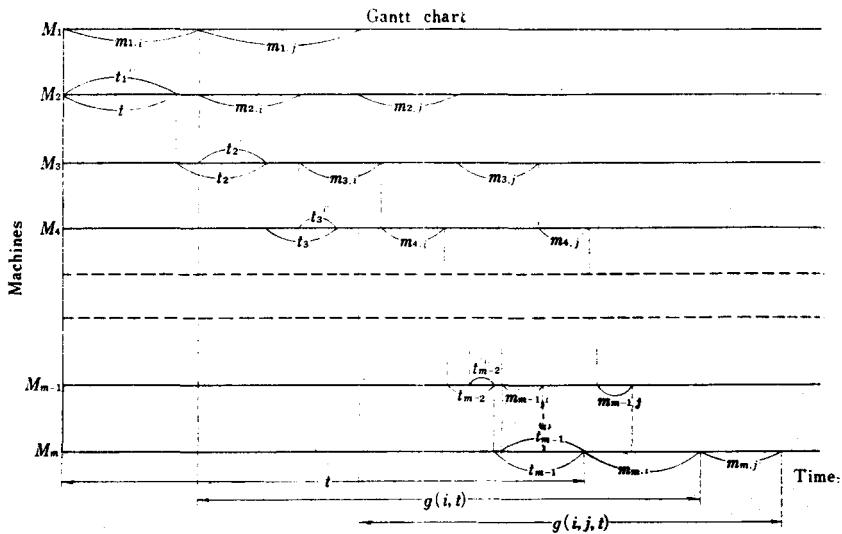


Fig. 2.

and than

$$\begin{aligned} \max(-A_1^{(t)}, 0) &= m_{1,j} - m_{2,i}, \\ \max[\max(-A_1^{(t)}, 0) - A_2^{(t)}, 0] &= \sum_{l=1}^2 m_{l,j} - \sum_{l=2}^3 m_{l,i}, \\ \max[\max[\max(-A_1^{(t)}, 0) - A_2^{(t)}, 0] - A_3^{(t)}, 0] &= \sum_{l=1}^3 m_{l,j} - \sum_{l=2}^4 m_{l,i}, \\ &\dots \end{aligned}$$

Consequently, from (1) we have

$$\begin{aligned} g(i, j, t) &= \sum_{k=2}^m \left\{ m_{k,j} + \max \left[(m_{k,i} - m_{k-1,j}) \right. \right. \\ &\quad \left. \left. - \left(\sum_{l=1}^{k-2} m_{l,j} - \sum_{l=2}^{k-1} m_{l,i} \right), 0 \right] \right\} + m_{m,j} \\ &\quad + \max \left[A_{m-1}^{(t)} - \left(\sum_{l=2}^{m-2} m_{l,j} - \sum_{l=2}^{m-1} m_{l,i} \right), 0 \right] \end{aligned} \quad (3)$$

where

$$\begin{aligned} A_{m-1}^{(t)} &= m_{m,i} - m_{m-1,j} + \max(t_{m-1}^{(t)} - m_{m-1,i}, 0) \\ &= m_{m,i} - m_{m-1,j} - m_{m-1,i} + \max(t_{m-1}^{(t)}, m_{m-1,i}) \end{aligned}$$

$$\begin{aligned}
t_{m-1}^{(i)} &= t_{m-1} - \max(m_{m-2,i} - t_{m-2}^{(i)}, 0) \\
&= -m_{m-2,i} + t_{m-1} + \min(t_{m-2}^{(i)}, m_{m-2,i}) \\
&= -m_{m-2,i} + t_{m-1} + t_{m-2}^{(i)} \\
&= -m_{m-2,i} + t_{m-1} + t_{m-2} - \max(m_{m-3,i} - t_{m-3}^{(i)}, 0) \\
&= -m_{m-2,i} - m_{m-3,i} + t_{m-1} + t_{m-2} + \min(t_{m-3}^{(i)}, m_{m-3,i}) \\
&= -(m_{m-2,i} + m_{m-3,i}) + t_{m-1} + t_{m-2} + t_{m-3}^{(i)} \\
&= \dots \\
&= -\sum_{l=1}^{m-2} m_{l,i} + \sum_{l=1}^{m-1} t_l \\
&= -\sum_{l=1}^{m-2} m_{l,i} + t
\end{aligned}$$

Hence, (3) becomes

$$\begin{aligned}
g(i, j, t) &= \sum_{k=2}^{m-1} \left\{ m_{m,j} + \max \left[\sum_{l=2}^k m_{l,i} - \sum_{l=1}^{k-1} m_{l,j}, 0 \right] \right\} \\
&\quad + m_{m,j} + \max \left[\sum_{l=2}^m m_{l,i} - \sum_{l=1}^{m-1} m_{l,j} - \sum_{l=1}^{m-1} m_{l,i} \right. \\
&\quad \left. + \max \left(t, \sum_{l=1}^{m-1} m_{l,i} \right), 0 \right] \\
&= \sum_{k=2}^m m_{k,j} + \max \left[m_{m,i} - m_{1,i} - \sum_{l=1}^{m-1} m_{l,j} \right. \\
&\quad \left. + \max \left(t, \sum_{l=1}^{m-1} m_{l,i} \right), 0 \right] \\
&= -m_{1,i} - m_{1,j} + m_{m,i} + m_{m,j} \\
&\quad + \max \left[t, \sum_{l=1}^{m-1} m_{l,i}, m_{1,i} - m_{m,i} + \sum_{l=1}^{m-1} m_{l,j} \right]
\end{aligned}$$

Similarly we have

$$\begin{aligned}
g(j, i, t) &= -m_{1,j} - m_{1,i} + m_{m,j} + m_{m,i} \\
&\quad + \max \left[t, \sum_{l=1}^{m-1} m_{l,j}, m_{1,j} - m_{m,j} + \sum_{l=1}^{m-1} m_{l,i} \right]
\end{aligned}$$

Hence, from $g(i, j, t) < g(j, i, t)$ we have

$$\begin{aligned}
&\max \left[t, \sum_{l=1}^{m-1} m_{l,i}, m_{1,i} - m_{m,i} + \sum_{l=1}^{m-1} m_{l,j} \right] \\
&< \max \left[t, \sum_{l=1}^{m-1} m_{l,j}, m_{1,j} - m_{m,j} + \sum_{l=1}^{m-1} m_{l,i} \right]
\end{aligned} \tag{4}$$

So that, if

$$\begin{aligned} & \max \left[\sum_{t=1}^{m-1} m_{l,t}, m_{1,t} - m_{m,t} + \sum_{t=1}^{m-1} m_{l,t} \right] \\ & < \max \left[\sum_{t=1}^{m-1} m_{l,t}, m_{1,t} - m_{m,t} + \sum_{t=1}^{m-1} m_{l,t} \right] \end{aligned} \quad (5)$$

then the left hand of (4) is not larger than the right hand of (4).

From (5) we have

$$\begin{aligned} & \sum_{t=1}^{m-1} m_{l,t} + \sum_{t=1}^{m-1} m_{l,t} + \max \left[-\sum_{t=1}^{m-1} m_{l,t}, -\sum_{t=2}^m m_{l,t} \right] \\ & < \sum_{t=1}^{m-1} m_{l,t} + \sum_{t=1}^{m-1} m_{l,t} + \max \left[-\sum_{t=1}^{m-1} m_{l,t}, -\sum_{t=2}^m m_{l,t} \right] \end{aligned}$$

hence we easily obtain the criterion,

$$\min \left[\sum_{t=1}^{m-1} m_{l,t}, \sum_{t=2}^m m_{l,t} \right] > \min \left[\sum_{t=1}^{m-1} m_{l,t}, \sum_{t=2}^m m_{l,t} \right].$$

Consequently we obtain the next corollary.

Corollary.

$$\text{When } \min_i m_{k,i} \geq \max_i m_{k+1,i} \\ k=1, 2, \dots, m-2.$$

hold, an optimal ordering is determined by the following rule : Item i precedes item j if

$$\min \left[\sum_{t=1}^{m-1} m_{l,t}, \sum_{t=2}^m m_{l,t} \right] < \min \left[\sum_{t=1}^{m-1} m_{l,j}, \sum_{t=2}^m m_{l,i} \right].$$

If there is equality, either ordering is optimal.

For the case $m=3$, this corollary coincides with the Johnson's criterion. [2]

REFERENCES :

1. R. Bellman: Dynamic Programming of Continuous Processes. 1954. Chap. VIII. The Rand Corporation.
2. S. M. Johnson : "Optimal Two-and Three-Stage Production Schedules with Setup Times Included.,, *Nav. Res. Log. Quart.*, 1, No 1, 61~68 (Mar. 1954).