NOTE ON MINIMAX REGRET ORDERING POLICY —STATIC AND DYNAMIC SOLUTIONS AND A COMPARISON TO MAXIMIN POLICY—

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(Received Feb. 1, 1961)

1. INTRODUCTION

There has been an objection to the minimax principle regarding to its too pessimistic character, and several alternative decision criteria have been suggested. Among them, Savage's minimax regret principle is the most interesting one.

The same criticism has been applied also to our previous studies [12]~[14] and the application of the minimax regret principle has been suggested. It is the purpose of this note to analyse the same problem applying the minimax regret principle. The results will be also compared to the maximin ordering policy which we presented previously. It may be reminded that Morris [11] already presented an interesting study of optimal ordering problem under several decision criteria. It is noteworthy, however, that Morris' model can be regarded as a special case of our model because, if we restrict our model to the static case and neglect some parameters, our model coincides with his model.

2. ECONOMIC MEANING AND METHODS OF EVALUATION OF REGRET IN MULTI-STAGE CASE

We have no intention of discussing here the general properties of the minimax regret principle, because they were studied in many literatures (e. g., [4]~[9]). However, it must be noted here that there exists a conceptual correspondence between the minimax regret principle and the opportunity cost doctrine in the following sense. Opportunity cost can be defined as the forgone profit due to the selection of particular course of action. More precisely, if we assume a situation where we are given two alternative courses of action, A_1 and A_2, and we can obtain the profit P_1 and P_2 respectively, the opportunity cost due to the selection of A_1 is equal to the forgone profit P_2. On the other hand, the regret r_{ij} is defined by the form

$$(2.\ 1) \qquad r_{ij}=\operatorname*{Max}_{k} a_{kj}-a_{ij}$$

where our choice is the i^{th} alternative and the nature realizes the state j and a_{ij} is the real payoff in money or utility. Comparing these two definitions, we can easily find the fact that the term Max a_{kj} in (2. 1) is the possible maximum opportunity cost when the state of the nature is j. Hence, the regret is a concept which may also be defined as the opportunity loss. The essential meaning of the minimax regret principle, consequently, can be restated as the minimization of the possible maximum opportunity loss.

In our problem of optimal ordering, the one-stage profit is a function, $P(x, y, z)$, of the initial stock x, starting stock y, and demand z at the stage. Hence, the one stage regret function $R(x, y, z)$ is given by

$$(2.\ 2) \qquad R(x, y, z)=\operatorname*{Max}_{y'} P(x, y', z)-P(x, y, z)$$

and we can consider a function $f_1(x)$ such as

$$(2.\ 3) \qquad f_1(x)=\text{Val } R(x, y, z).$$

(Here it must be noted, however, that the decision-maker or the management is the minimizing player and the nature is the maximizing player in this case.)

But we are confronted with a difficulty in multi-stage case. Let $R_n(x, y, z)$ denote the total regret when the initial stock, starting stock, and demand at the first stage are x, y and z respectively, and an optimal policy is used for the subsequent $n-1$ stages. Then, we must determine what formula we will use to evaluate $R_n(x, y, z)$. Although several methods for the evaluation of $R_n(x, y, z)$ can be considered, we will discuss the following two methods.

[METHOD I] $R_n(x, y, z)=R(x, y, z)+\alpha f_{n-1}\{\text{Max}(y-z, 0)\}$

where $f_{n-1}(x)$ denotes the total regret value of the game starting with initial stock x and using the optimal policy under the assumption of this evaluation method in $n-1$ stages.

[METHOD II] $R_n(x, y, z)=\text{Max}[P(x, y', z)-\alpha f_{n-1}\{\text{Max}(y'-z, 0)\}]$
$-[P(x, y, z)-\alpha f_{n-1}\{\text{Max}(y-z, 0)\}]$

where $f_{n-1}(x)$ denotes the total regret value of the game starting with initial stock x and using the optimal policy under the assumption of this evaluation method in $n-1$ stage.

In either method, α is a suitable discount factor. Eihter method is subject to a boundary condition $R_1(x, y, z)=R(x, y, z)$. Which of these two methods is more valid is a somewhat troublesome problem, although it may be a matter of subjective preference of the decision maker. However, we can find fortunately in most cases the following properties. First of all, in our problem,

(2. 4) $$\operatorname*{Max}_{y} P(x, y, z)=P(x, z, z)$$

and, accordingly,

(2. 5) $$R(x, y. z)=P(x, z, z)-P(x. y. z).$$

Furthermore, in many cases,

(2. 6) $$\operatorname*{Max}_{y}[P(x, y, z)-\alpha f_{n-1}\{\mathrm{Max}(y-z, 0)\}]$$
$$=P(x, z, z)-\alpha f_{n-1}(0)$$

Hence,

(2. 7) $$\operatorname*{Max}_{y}[P(x, y, z)-\alpha f_{n-1}\{\mathrm{Max}(y-z, 0)\}]$$
$$-[P(x, y, z)-\alpha f_{n-1}\{\mathrm{Max}(y-z, 0)\}]$$
$$=P(x, z, z)-P(x, y, z)-\alpha f_{n-1}(0)+\alpha f_{n-1}\{\mathrm{Max}(y-z, 0)\}$$
$$=R(x, y, z)+\alpha f_{n-1}\{\mathrm{Max}(y-z, 0)\}-\alpha f_{n-1}(0).$$

As $\alpha f_{n-1}(0)$ is a certain constant, if, for n=2, 3, ……successively, (2. 6) is satisfied, the optimal strategy for the decision maker is by no means affected by the selection of the evaluation method.

In the following considerations, we will adopt the method I mainly. Althogh some of the conceptual difficulties may be said to be remaining, the following results will be sufficiently suggestive in practice especially with regard to the character of the minimax regret principle as a decision criterion. It is noteworthy that, although the minimization of mean regret all over the stages can be proposed as another criterion for multi-stage cases, the structure of the optimal policy for the method is also similar.

3. SOLUTION TO THE CASE WHEN BACKORDERING IS ACCEPTED AT THE PURCHASING PRICE

In our previous papers [12] [13] somewhat rigid condition was placed that the returning (backordering) or the disposal of the merchandise is accepted at the purchasing price. In such a simple case we can obtain the minimax regret solution quite easily.

As in [12] [13], let us consider a one-stage profit function such as

$$(3.\ 1)\qquad P(x,\ y,\ z)=\begin{cases}(p+b)z-(a+b)y+ax & (\text{when } y\geqq z)\\ -cz+(p+c-a)y+ax & (\text{when } y\leqq z)\end{cases}$$

where

p : retail or sales price

a : wholesale or purchasing price

b : storage or holding cost per unit

c : penalty per unit of shortage.

By a simple observation, it is clear that

$$(3.\ 2)\qquad \operatorname*{Max}_{y} P(x,\ y,\ z)=P(x,\ z,\ z)$$
$$=pz-a(z-x)$$
$$=(p-a)z+ax.$$

Thus

$$(3.\ 3)\qquad R(x,\ y,\ z)=\begin{cases}(a+b)(y-z) & (\text{when } y\geqq z)\\ (p+c-a)(z-y) & (\text{when } y\leqq z).\end{cases}$$

Noting that, in this case, $R(x,\ y,\ z)$ can be regarded as convex in y for every x and z, and that the decision maker is the minimizing player, the starting stock level $y_1{}^*$ which is optimal in the sense of the minimax regret principle in the one-stage model will be found as

$$(3.\ 4)\qquad y_1{}^*=\frac{(a+b)z_{\min}+(p+c-a)z_{\max}}{p+b+c}$$

and

$$(3.\ 5)\qquad f_1(x)=(a+b)(y_1{}^*-z_{\min})$$
$$=\frac{(a+b)(p+c-a)}{p+b+c}(z_{\max}-z_{\min})$$

where $z_{\min}$ and $z_{\max}$ are the lower and the upper limit of the demand in each stage, respectively.

It is intresting to compare this result to the maximin policy. The corresponding maximin policy is found in our previous papers. The difference between the minimax regert starting stock and the maximin starting stock is

$$(3.\ 6)\qquad \frac{(a+b)z_{\min}+(p+c-a)z_{\max}}{p+b+c}-\frac{(p+b)z_{\min}+cz_{\max}}{p+b+c}$$

$$=\frac{p-a}{p+b+c}(z_{max}-z_{min})$$

and obviously the minimax regret solution is larger than the maximin solution because $p>a$ in the case of sound enterprise. Furthermore, the difference is proportional to the difference between sales price and the purchasing price and to the difference between z_{max} and z_{min}, and moreover the smaller the parameters b and c, the larger the difference between the solutins.

Another intresting fact is that $f_1(x)$ in this case is independent of x, i. e., a constant. Accordingly, let us set $f_1(x)=f_1$. Then, by the method I of the multi-stage regret evaluation, clearly,

(3. 7) $$R_2(x, y, z)=R(x, y, z)+\alpha f_1$$

and the optimal strategy in this two-stage case is all the same as obtained before, and

(3. 8) $$f_2(x)=f_1+\alpha f_1=(1+\alpha)f_1.$$

By the similar argument we can obtain a recurrence relation such as

(3. 9) $$f_n(x)=f_n=f_1+\alpha f_{n-1}$$

and consequently

(3. 10) $$f_n-f_{n-1}=\alpha(f_{n-1}-f_{n-2}).$$

Hence the series $\{f_n\}$ converges as $0<\alpha<1$. Let f denote the limit, then

(3. 11) $$f=\frac{f_1}{1-\alpha}=\frac{1}{1-\alpha}\cdot\frac{(a+b)(p+c-a)}{p+b+c}(z_{max}-z_{min}).$$

On the other hand, if we adopt the method II of the multi-stage regret evaluation, we can find, for every n, that

(3. 12) $$R_n(x, y, z)=R(x, y, z)$$

and the optimal starting stock for each stage is equal to the level given by (3. 4) regardless of the number of stages allowed to be considered, and $f_n(x)=f_1$ for every n.

4. ONE-STAGE SOLUTION TO THE CASE OF DISCOUNTED BACKORDERING PRICE

Removing the rigid condition imposed on the above-mentioned model, let us assume, as in the case A in [14], a discounted price in case of returning (backordering) or disposal a' such as $a'\leqq a$. For convenience of analysis, let us consider, in this section, a truncated one-stage model. One-stage profit function in such case is, for $y\geqq x$,

$$(4.\ 1)\quad P(x,\ y,\ z)=\begin{cases}(p+b)z-(a+b)y+ax & (\text{when } y\geqq z)\\ -cz+(p+c-a)y+ax & (\text{when } y\leqq z)\end{cases}$$

and, for $y\leqq x$,

$$(4.\ 2)\quad P(x,\ y,\ z)=\begin{cases}(p+b)z-(a'+b)y+a'x & (\text{when } y\geqq z)\\ -cz+(p+c-a')y+a'x & (\text{when } y\leqq z).\end{cases}$$

By a simple observation we can see that max $P(x,\ y,\ z)=P(x,\ z,\ z)$. Here we can decompose the regret function as follows,

$$(.\ 3)\quad \begin{aligned}R(x,\ y,\ z)&=P(x,\ z,\ z)-P(x,\ y,\ z)\\ &=[pz-K(z,\ x)]-[P(y,\ z)-K(y,\ x)]\end{aligned}$$

where $K(z,\ x)$ and $K(y,\ x)$ are the costs incurred by the transformations from x to z and y, respectively, and $P(y,\ z)$ denotes the profit obtained by the combination of y and z excluding the cost $K(y,\ z)$ from the consideration. Now, from (4. 3),

$$(4.\ 4)\quad R(x,\ y,\ z)=[pz-P(y,\ z)]-K(z,\ x)+K(y,\ x).$$

Let us define here.

$$(4.\ 5)\quad R'(x,\ y,\ z)=[pz-P(y,\ z)]-K(z,\ x).$$

The explicit expression of (4. 5) is

$$(4.\ 6)\quad R'(x,\ y,\ z)=\begin{cases}b(y-z)-K(z,\ x) & (\text{when } y\geqq z)\\ (p+c)(z-y)-K(z,\ x) & (\text{when } y\leqq z).\end{cases}$$

By the similar argument to the section 3 in [14], i. e., by the examination of the slope of the security level, the optimal starting stock can be found directly from (4. 6) instead of the original regret function $R(x,\ y,\ z)$. Moreover, it will be also clear that the optimal stock level will be a certain number in the interval $[z_{\min}, z_{\max}]$.

Now, consider the case when $x\leqq z_{\min}$. In this case it should be noted that $K(z,\ x)$ is the purchasing cost for the quantity $z-x$, i. e., $K(z,\ x)=a(z,\ x)$. The optimal starting stock for this case is clearly

$$(4.\ 7)\quad y_1{}^*=\frac{(a+b)z_{\min}+(p+c-a)z_{\max}}{p+b+c}.$$

For $x\geqq z_{\max}$, $K(z,\ x)$ is the revenue due to the returning or disposal, i. e., $K(z,\ x)=a'(z-x)$, a negative cost. The optimal stock level is

$$(4.\ 8)\quad y_1{}^*=\frac{(a'+\mathrm{b})z_{\min}+(p+c-a')z_{\max}}{p+b+c}.$$

For $z_{\min}\leqq x\leqq z_{\max}$, $K(z_{\max},\ x)$ is the purchasing cost and $K(z_{\min},\ x)$ is the revenue due to the returning.

Hence,

$$(4.\ 9)\quad \operatorname*{Max}_{z} R'(x,\ y,\ z)=\begin{cases} by-(a'+b)z_{\min}+a'x & (\text{when } y\geqq z)\\ (p+c-a)z_{\max}-(p+c)y+ax & (\text{when } y\leqq z)\end{cases}$$

and

$$(4.\ 10)\qquad y_1{}^*=\frac{(a'+b)z_{\min}+(p+c-a)z_{\max}+(a-a')x}{p+b+c}$$

which is a function of x. The relationship of the optimal starting stock level to the initial stock x is shown in Fig. 1.

We can also consider the "optimal initial stock" in the sense that $y_1{}^*=x$, which is obtained from (4. 10) as

$$(4.\ 11)\qquad x_1{}^*=\frac{(a'+b)z_{\min}+(p+c-a)z_{\max}}{p+b+c+a'-a}$$

In order to determine $f_1(x)$, it suffices for us to consider the fact that $K(y_1{}^*,\ x)$ is the purchasing cost when $x\leqq x_1{}^*$ and the returning revenue when $x\geqq x_1{}^*$. The explicit form of $f_1(x)$ is obtained as follows.

$$(4.\ 12)\qquad \begin{aligned}
f_1(x)&=\frac{(a+b)(p+c-a)}{p+b+c}(z_{\max}-z_{\min}) \qquad for\ \ x\leqq z_{\min}\\
f_1(x)&=\frac{p+c-a}{p+b+c}\{(a+b)z_{\max}-(a'+b)z_{\min}-(a-a')x\}\\
&\qquad\qquad for\ \ z_{\min}\leqq x\leqq x_1{}^*\\
f_1(x)&=\frac{a'+b}{p+b+c}\{(p+c-a)z_{\max}-(p+c-a')z_{\min}+(a-a')x\}\\
&\qquad\qquad for\ \ x_1{}^*\leqq x\leqq z_{\max}\\
f_1(x)&=\frac{(a'+b)(p+c-a')}{p+b+c}(z_{\max}-z_{\min}) \qquad for\ \ x\geqq z_{\max}
\end{aligned}$$

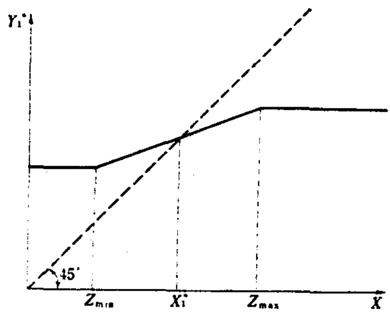

Fig. 1. Relation of the optimal starting stock to the initial stock

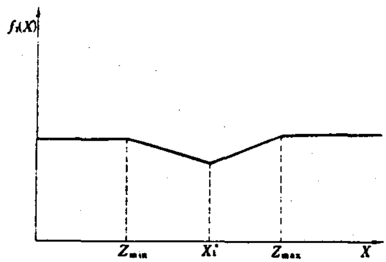

Fig. 2. Structure of $f_1(x)$

Fig. 2 shows the form of this solution function. However, it must be remarked that, among the four expressions which value of the first and the last is larger depends upon the parameters.

5. A SIMPLE OBSERVATION ON THE PROPERTIES OF THE SOLUTION

Several interesting properties can be found with regard to the solution. Firstly, the optimal starting stock is independent of x when $x \leqq z_{min}$ and $x \geqq z_{max}$. Computationally, it is because the term of x vanishes during the computation of $y_1{}^*$. However, the following significance must be remarked. The independence of $y_1{}^*$ from x when $x \leqq z_{min}$ and when $x \geqq z_{max}$ is due to the proportionality of the purchasing cost. For instance, consider two stock levels x' and x'' such as $x' < x'' < z_{min}$. Then, in each case of $x = x'$ *and* $x = x''$, $K(z_{min}, x)$ *of* $K(z_{min}, x'')$ is precisely offset, and the diference of x' and x'' by no means affects the value of the solution. this property will be found also in the multi-stage cases. But, if the purchasing cost or the returning revenue is proportional, it is not the case.

Another interesting property of $y_1{}^*$ is that it is a nondecreasing function of x, and especialy in the interval $[z_{min}, z_{max}]$ it is slowly and linearly increasing with x. On the other hand, the maxmin policy which was previously obtained is completely independent of x. Here, we think that we can see the less pessimistic character of the minimax regret principle compared with the minimax principle.

It will be needless to be mentioned that the static solution in the section 3 is only a special case of the solution in the section 4.

Consider, now, the dynamic cases on the basis of the solution in the previous section. As was mentioned above, $f_n(x)$ for each n will be independent of x for $x \leqq z_{min}$ and $x \geqq z_{max}$ if the purchasing cost and the returning revenue are both proportional. Here we are confronted with a difficulty, however, that the pure strategy solution may not be optimal from the mathematical viewpoint for $n=2, 3, \cdots$ because of the form of $f_1(x)$. (Necessary condition for the optimality of the pure strategy solution was given in [14]). But fortunately there exist cases where the pure strategy solutions are optimal under a certain condition, and furthermore, if the difference between a and a' is relatively small, the pure strategy solutin obtained by

(5. 1) $$f_n(x)=\operatorname*{Min}_{y}\operatorname*{Max}_{y}[R(x,\ y,\ z)+\alpha f_{n-1}\{\mathrm{Max}(y-z,\ 0)\}]$$

will be sufficiently good approximation to the mixed strategy solution. More details of them will be discussed in succeeding sections. From the viewpoint of management practice, the randomization of policies is undesirable.

6. INFINITE-STAGE SOLUTION WHEN $z_{\max}\leqq 2z_{\min}$

Multi-stage cases can be analysed by the method of successive approximations. But, for the present, let us consider the infinite-stage solution directly.

Let us assume that the soluton $f(x)$ of

(6. 1) $$f(x)=Val[R(x,\ y,\ z)+\alpha f\{\mathrm{Max}(y-z,\ 0)\}]$$

is of a similar form to Fig. 2, i. e., a contant for $x\leqq z_{\min}$, linearly decreasing in x for $z_{\min}<x<x^*$, linearly increasing in x for $x^*<x<z_{\max}$, and a constant for $x>z_{\max}$, where y and z are the control variables of the minimizing player and the maximizing player, respectively. Then, we have an optimal pure strategy solution if $z_{\max}<2z_{\min}$, because, for, $z_{\min}\leqq y\leqq z_{\max}$, $f\{\mathrm{Max}(y-z,\ 0)\}=f(0)$,

for $x\leqq z_{\min}$,

(6. 2) $$\operatorname*{Max}_{z}[R(x,\ y,\ z)+\alpha f\{\mathrm{Max}(y-z,\ 0)\}]$$

$$=\begin{cases}(a+b)(y-z_{\min})+\alpha f(0) & (\text{when } y\geqq z)\\ (p+c-a)(z_{\max}-y)+\alpha f(0) & (\text{when } y\leqq z)\end{cases}$$

and the optimal stock level will be

(6. 3) $$y^*=\frac{(a+b)z_{\min}+(p+c-a)z_{\max}}{p+b+c}$$

By the similar argument, for $x\geqq z_{\max}$,

(6. 4) $$y^*=\frac{(a'+b)z_{\min}+(p+c-a')z_{\max}}{p+b+c}$$

For $z_{\min}\leqq x\leqq z_{\max}$, if $y>x$,

(6. 5) $$\operatorname*{Max}_{z}[R(x,\ y,\ z)+\alpha f\{\mathrm{Max}(y-z,\ 0)\}]$$

$$=\begin{cases}b(y-z_{\min})-a'(z_{\min}-x)+a(y-x)+\alpha f(0) & (\text{when } y\geqq z)\\ (p+c-a)(z_{\max}-y)+\alpha f(0) & (\text{when } y\leqq z)\end{cases}$$

and we have

(6. 6) $$y^*=\frac{(a'+b)z_{\min}+(p+c-a)z_{\max}+(a-a')x}{p+b+c}$$

The same solution can be obtained also under assumption $y \leqq x$. Now, by

$$(6.\ 7) \qquad x^* = \frac{(a'+b)z_{\min}+(p+c-a)z_{\max}}{p+b+c+a'-a}$$

the optimal initial stock in the sense that $y^*=x$ is given.

We can determine $f(x)$ as follows. For $x \leqq z_{\min}$,

$$(6.\ 7) \qquad f(x)=(p+c-a)(z_{\max}-y^*)+\alpha f(0)$$

where y^* is given by (6. 3). By the assumption, (6. 7) must be a equal to $f(0)$, and consequently

$$(6.\ 8) \qquad f(x)=f(0)=\frac{1}{1-\alpha}\cdot\frac{(p+c-a)(a+b)}{p+b+c}(z_{\max}-z_{\min}).$$

For $z_{\min} \leqq x \leqq z^*$,

$$(6.\ 9) \qquad f(x)=(p+c-a)(z_{\max}-y^*)+\alpha f(0)$$

where y^* is given by (6. 6) and consequently $f(x)$ is linearly decreasing. For $x^* \leqq x \leqq z_{\max}$,

$$(6.\ 10) \qquad f(x)=(a'+b)(y^*-z_{\min})+\alpha f(0)$$

where y^* is given by (6. 6) and $f(x)$ is linearly increasing. Similarly for $x \geqq z_{\max}$,

$$(6.\ 11) \qquad f(x)=(a'+b)(y^*-z_{\min})+\alpha f(0)$$

where y^* is given by (6. 4) and accordingly $f(x)$ is independent of x. It must be noted that these results have an interesting property of correspondence to the static solution which was mentioned in the section 4. The property is due to the condition $z_{\max} \leqq 2z_{\min}$.

7. NOTE ON INFINITE-STAGE SOLUTION WHEN $z_{\max} > 2z_{\min}$

When $z_{\max} > 2z_{\min}$, we have not the optimal pure strategy solution. However, as was mentioned before, the pure strategy solution will be a sufficiently good approximation for the optimal solution if the difference between a and a' is small, because $f_n(x)$ for each n is nearly independent of x if the difference is small.

When we intend to obtain the pure strategy solution, it suffices to consider the equation

$$(7.\ 1) \qquad f(x)=\operatorname*{Min}_{y}\operatorname*{Max}_{z}[R(x,\ y,\ z)+\alpha f\{\operatorname{Max}(y-z,\ 0)\}].$$

Again let us place the similar assumption about the form of $f(x)$ as in the analysis of the equation (6. 1). Then, for $x \leqq z_{\min}$,

(7. 2) $$y^*=\frac{(a+b)z_{\min}+(p+c-a)z_{\max}+\alpha\{f(0)-f(y^*-z_{\min})\}}{p+b+c}$$

(7. 3) $$f(x)=f(0)=(p+c-a)(z_{\max}-y^*)+\alpha f(0)$$

and, for $x\geqq z_{\max}$,

(7. 4) $$y^*=\frac{(a'+b)z_{\min}+(p+c-a')z_{\max}+\alpha\{f(0)-f(y^*-z_{\min})\}}{p+b+c}$$

(7. 5) $$f(x)=(p+c-a')(z_{\max}-y^*)+\alpha f(0)$$

either solution of which is subject to the constraint

(7. 6) $$\frac{\partial}{\partial y}[R(x,\ y,\ z)+\alpha f\{\text{Max}(y-z)\}]>0 \qquad (\textit{for } y\geqq z)$$

which is satisfied in most cases. Furthermore, by the similar analysis for $z_{\min}\leqq x\leqq z_{\max}$,

(7. 7) $$y^*=\frac{(a'+b)z_{\min}+(p+c-a)z_{\max}+(a-a')x+\alpha\{f(0)-f(y^*-z_{\min})\}}{p+b+c}$$

However it must be noted that the problem has not yet completely solved because the values of $f(0)$ and $f(y^*-z_{\min})$ for each case of y^* not yet known. In order to obtain the complete solution, it is helpful, first of all, to assume that $z_{\min}>y^*-z_{\min}$ for each case, because $f(y^*-z_{\min})=f(0)$ if the assumption is valid, and the solution is immediately found. (But we must not forget to check whether the solution satisified the assumption.) If the assumption $z_{\min}>y^*-z_{\min}$ is not satisfied, we use the method of unknown coefficient. For example, if $z_{\min}<y^*-z_{\min}<x^*$ for some x, it suffices to set

(7. 8) $$\frac{\partial}{\partial x}f(x)=-A$$

for $z_{\min}<x<x^*$ where x^* is the initial stock level which is "optimal" in the sense as was mentioned before. Noticing that, for $z_{\min}<x<x^*$.

(7. 9) $$f(x)=(p+c-a)(z_{\max}-y^*)+\alpha f(0)$$

where y^* is given by (7. 7) and, accordingly, is a funcion of x and A, we can derive an equation for the unknown coefficient A because the coefficient of x in (7. 9) must be equal to-A. The case $y^*-z_{\min}>x^*$ is rare even if x is considerably large, but the similar method of analysis can be applied also to such case. (Note that $f(y^*-z_{\min})$ is assumed to be linearly increasing when $y^*-z_{\min}$ exceeds x^*.) In either case, we must try to find the consistent solution by the above mentioned computational techniques.

Now, let us turn to the consideration on the properties of the mixed strategy solution, though the randomization of the policies may be practically unpreferable. The analysis becomes quite complicated, but we can find some cues for numerical analysis. When we consider the mixed strategy solution, we can not impose the assumption of the form of $f(x)$, similar to the previous considerations. At least it is obvious, however, that $f(x)$ is independent of x when $x \leqq z_{\min}$ and when $x \geqq z_{\max}$. The remaining problem for us is to determine the form of $f(x)$ for $z_{\min} < x < z_{\max}$. Empirically it has become clear that the least favourable distribution on z is a certain two-point distribution on $z_{\min}$ and $z_{\max}$, and the optimal strategy for the decision maker (the management) is, in most cases, to randomize two stock levels out of $y = z_{\min}$, $y = x$, and $y = z_{\max}$, or to randomize one of these three levels with another proper level which also lies in the interval $[z_{\min}, z_{\max}]$. Hence, it is fruitful for us, in order to estimate the form of $f(x)$, to study a situation where the optimal strategy for the decision maker is a certain randomization of two stock levels y' and y'' such as $z_{\min} \leqq y' \leqq y'' \leqq z_{\max}$ and the nature randomize $z_{\min}$ and $z_{\max}$. If $y' < x < y''$, the following game should be considered.

$$\text{(7. 10)} \qquad \begin{pmatrix} M_1 + (a-a')x & M_2 \\ M_3 & M_4 - (a-a')x \end{pmatrix}$$

where the 1st and the 2nd rows correspond to the nature's pure strategies $z_{\max}$ and $z_{\min}$ respectively, and the 1st and the 2nd columns correspond to the decision maker's pure strategies y' and y'' respectively. M_i's are the suitable constants. As it is our purpose, for the present, to clarify the effect of x, the terms other than the term of x may be regarded as constants in the multi-stage regret payoff function. However, the following relations may be assumed.

$$M_1 + (a-a')x > M_2,\ M_3$$
$$M_4 - (a-a')x > M_3,\ M_2.$$

Hence, the value of the above game is

$$\text{(7. 11)} \qquad \frac{\{M_1 + (a-a')x\}\{M_4 - M_2 + (a-a')x\} + M_3\{M_1 - M_3 + (a-a')x\}}{M_1 - M_3 + M_4 - M_2}$$

and $M_1 + M_4 - M_3 - M_2 > 0$ the value is covex in x (more precisely, a parabola which is convex downward).

By the similar considerations, it can be easily found that, for $x' < y'$ and for $x > y''$, the value is linearly decreasing in x. These results will be fairly suggestive if we are to analyse a numerical problem. However, the randomization of the policies, is, as has been already mentioned, unpreferable from the practical viewpoint and the pure strategy solution is fairly good approximation. Thus, we should like to omit the further discussions.

8. SUMMARY

In spite of some conceptual difficulties, operationally meaningful results were obtained. It should be noted that the dynamization of the model hat only a slight effect, if any, on the optimal strategy in fairly general cases.

In the section 3, we have analysed the case when the returning of the merchandise is permitted at the purchasing (or wholesale) price. The optimal stock level which is optimal in the sense of the minimax regret principle is, in such a case, equal to the number which divides the interval between the lower and the upper limit of the anticipated demand at the stage by the ratio $(p+c-a):(a+b)$. The larger the penalty c the larger the stock and the less the holding cost the larger the stock. The solution is by no means affected by the dynamization. Compared to the corresponding maximin solution for static case, the minimax regret stock level is somewhat larger.

Generalizing the condition, we analysed the case where the returning of the merchandise is permitted at a discounted price a'. In the truncated one-stage model, the optimal stock level is found to be equal to the number which divides the interval between the lower and the upper limit of the anticipated demand by the ratio $(p+c-a):(a+b)$ if the initial stock is smaller than the lower limit and by the ratio $(p+c-a'):(a'+b)$ if the initial stock is larger than the upper limit of the demand. If the initial stock is the number between the two limits of anticipated demand, the optimal starting stock level is a linear increasing function of the initial stock, connecting the two optimal levels mentioned above. Here, again we can see the somewhat more optimistic nature than the maximin policy. (Section 4)

Even when the model is dynamized the results mentioned above is valid if the anticipated upper limit of the demand is not more than the double of the lower limit. (Section 6)

When the upper limit is larger than the double of the lower limit, the randomized policy is required from the mathematical point of view or from the viewpoint of game theory, in the ordinary sense. However, if the price level difference $(a-a')$ is sufficiently small, the pure strategy solution is fairly good approximation. This will be very helpful fact for the practice. If we confine our consideration to the pure strategy solution, the stock level mentioned above or the somewhat larger level is optimal. (Section 7)

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