CHARACTERISTICS OF DYNAMIC MAXIMIN ORDERING POLICY

HIROSHI KASUGAI AND TADAMI KASEGAI

Waseda University

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1. INTRODUCTION

It is the purpose of this article to attempt a further extension of our previous studies [11] [12]. The problem which is the concern of our study is the following type. There is a certain one-stage profit function $P(x, y, z)$, where x, y, and z denote the initial stock, starting stock, and demand of the merchandise, respectively. The exact probability distribution on z is unknown, but z can be assumed to exist in the closed interval $[z_{\min}, z_{\max}]$. The demand interval assumed is identical at any stage in future. Under the assumption of maximin profit principle, our problem is reduced to one of the multi-stage games, and the solution to our problem is to solve the functional equation of the form

(1.1) $$f(x)=\mathrm{Val}[P(x, y, z)+\alpha f\{\mathrm{Max}(y-z, 0)\}]$$

where $f(x)$ is the total present value of securable profit starting with initial stock x and using a maximin ordering policy in an unlimited number of stages, and α is a discount factor such that $0\leq\alpha\leq 1$. (By a maximin ordering policy we mean an ordering policy which is optimal in the sense of the maximin profit principle, i. e., an ordering policy which maximizes the securable profit assuming that the least favourable situation will occur under the prescribed assumptions.)

The method of successive approximations is fully utilized to obtain the solution of this equation. Hence, we analyse the following sequence of equations.

(1.2) $$f_1(x)=\mathrm{Val}P(x, y, z)$$

(1.3) $$f_n(x)=\mathrm{Val}[P(x, y, z)+\alpha f_{n-1}\{\mathrm{Max}(y-z, 0)\}]$$

for $n=2, 3,\cdots\cdots$, where $f_n(x)$ is the total present value of securable profit starting with initial stock x and using a maximin ordering policy when only n stages of operations are allowed.

The main part of this paper will be concerned with the cases with linear profit functions, and the follwing notations will be used;

p : retail price of the merchandise per unit

a : wholesale price of the merchandise per uinit (ordering cost)

a' : returning (or disposal) price of the merchandise per unit (revenue due to returning or disposal)

b : storage cost per unit of remainder

c : penalty cost per unit of shortage.

In general, it is considered that $p>a>0$ and a', b, $c\geq 0$. Throughout this paper (except section 7), it is assumed that there is no constant cost due to ordering, returning (or disposal), remainder or shortage. These assumptions, we think, cause neither loss of generality nor loss of practicability. Moreover, in the mathematical analysis it is the great benefit of the assumption of non-existence of constant costs that $P\ (x, y, z)$ becomes a continuous function.

In our previous studies [11] [12], we treated a special case where $a=a'$ and the interval of z is $[0, z_{\max}]$. The solution to the case was obtained as follows.

$$f_1(x)=ax-\frac{(a+b)c}{p+b+c}z_{\max} \tag{1.4}$$

$$y_1{}^*=\frac{cz_{\max}}{p+b+c} \tag{1.5}$$

$$f_n(x)=ax-\alpha^{n-1}\frac{(\mathrm{a}+\mathrm{b})c}{p+b+c}z_{\max}-\frac{1-\alpha^{n-1}}{1-\alpha}\cdot\frac{\{a(1-\alpha)+b\}c}{p+b+c-\alpha a}z_{\max} \tag{1.6}$$

$$y_n{}^*=\frac{cz_{\max}}{p+b+c-\alpha a} \tag{1.7}$$

for $n=2, 3,\cdots\cdots$, where y^*_n is the optimal starting stock for the first stage when only n stages of operations are allowed. It is interesting to note that $y_n{}^*$ is identical at any n except $n=1$. The convergence of $f_n(x)$, as $n\to\infty$, is apparent and the limit

$$\lim_{n\to\infty} f_n(x)=ax-\frac{1}{1-\alpha}\cdot\frac{\{a(1-\alpha)+b\}c}{p+b+c-\alpha a}z_{\max} \tag{1.8}$$

satisfies the equation (1.1).

Here we intend to analyse more general cases. Several decision criteria for general cases described in [11] [12] (especially in Part II of [12]) are fully utilized in the present analysis as shortways to the solutions.

2. GENERAL PROPERTIES OF OPTIMAL ORDERING POLICY

In this section, we show that our problem is a special class of "multi-stage games of timing with concave payoff functions", and the management has an optimal strategy which is pure (nonrandomized ordering policy) although the optimal strategy is not unique.*

Let

$$P(x, y, z)=\begin{cases} K(x, y, z) & (\text{when } y \geq z) \\ M(x, y, z) & (\text{when } y \leq z). \end{cases} \tag{2.1}$$

Then, if the following requirements [R. 1. 1]~[R. 1. 6] are satisfied, there is an optimal non-randomized ordering policy when only one stage is allowed, and the optimal starting stock is the solution of the equation of the form

$$K(x, y, z_{\min})=M(x, y, z_{\max}). \tag{2.2}$$

[R. 1. 1] Continuity.

[R. 1. 2] $\frac{\partial}{\partial z} K(x, y, z)>0.$

[R. 1. 3] $\frac{\partial}{\partial z} M(x, y, z)<0..$

[R. 1. 4] $\frac{\partial}{\partial y} K(x, y, z)<0.$

[R. 1. 5] $\frac{\partial}{\partial y} M(x, y, z)>0.$

[R. 1. 6] Non-Convexity.**

* The least favourable probability distribution of demand will be a certain two-point distribution on $z_{\min}$ and $z_{\max}$. Such probability distribution and the optimal (maximin) ordering policy constitute an equilibrium pair of strategies in our problem when it is viewed as zero-sum two-person game. However, as it is our main purpose to find the optimal strategy for the management (the maximin ordering policy) and, furthermore, as the market is the "Nature" in our problem, we will omit the detailed description of such distribution of demand and the equilibrium.

** Note that, if we drop the requirement [R. 1. 6], there is a possibility of existence of better mixed strategies than the pure strategy y given by (2. 2). The reason for the necessity of [R. N. 6] is similar.

Here, [R. 1. 1] requires that $P(x, y, z)$ is continuous in y and z, and [R. 1. 6] requires that any part of $P(x, y, z)$ is not convex in y.

Furthermore, if the following requirements [R. N. 1]~[R. N. 6] are satisfied, there is an optimal non-randomized ordering policy for the first stage when only n stages of operations are considered, and the optimal starting stock for the first stage is the solution of the equation

(2.3) $$K(x, y, z_{\min})+\alpha f_{n-1}(y-z_{\min})=M(x, y, z_{\max})+\alpha f_{n-1}(0).$$

[R. N. 1] Continuity.

[R. N. 2] $\frac{\partial}{\partial z}\{K(x, y, z)+\alpha f_{n-1}(y-z)\}>0.$

[R. N. 3] $\frac{\partial}{\partial z}\{M(x, y, z)+\alpha f_{n-1}(0)\}<0.$

[R. N. 4] $\frac{\partial}{\partial y}\{K(x, y\ z)+\alpha f_{n-1}(y-z)\}<0.$

[R. N. 5] $\frac{\partial}{\partial y}\{M(x, y, z)\ ,+\alpha f_{n-1}(0)\}>0.$

[R. N. 6] Non-Convexity.

Here [R. N. 1] requires that $P(x, y, z)+\alpha f_{n-1}\{\text{Max},(y-z, 0)\}$ is continuous in y and z, and [R. N. 6] requires that any part of the function $P(x, y, z)+\alpha f_{n-1}\{\text{Max}(y-z, 0)\}$ is not convex in y.

The cases with linear profit functions mentioned in the previous section satisfy the requirement [R. N. 1] because of the assumption of non-existence of constant costs. It must be noted, in addition, that all the requirements are satisfied in both of the following two cases;

case A : returning (or disposal) is permitted and $a \geq a'$,

case B : neither returning nor disposal is permitted.

This can be shown as follows.

Case A. The explicit form of one-stage profit function is as follows. When $x \leq y$(i. e., the case of ordering),

(2.4) $$\begin{aligned}K(x, y, z)&=pz-a(y-x)-b(y-z)\\&=(p+b)z-(a+b)y+ax\end{aligned}$$

(2.5) $$\begin{aligned}M(x, y, z)&=py-a(y-x)-c(z-y)\\&=-cz+(p+c-a)y+ax\end{aligned}$$

and when $x \geq y$ (i. e., the case of returning or disposal, or we may say, the case of "negative ordering"),

(2.6) $$K(x, y, z)=pz+a'(x-y)-b(y-z)$$

$$=(p+b)z-(a'+b)y+a'x$$

$$M(x,\ y,\ z)=py+a'(x-y)-c(z-y) \tag{2.7}$$
$$=-cz+(p+c-a')y+a'x.$$

That [R. 1. 1]$\sim$[R. 1. 5] are satisfied is obvious. From the assumption $a\geq a'$, it can be seen that $p+c-a\leq p+c-a'$. Hence, [R. 1. 6] holds.

It is also apparent that $f_1(x)$ is a function of the form

$$f_1(x)=v_1+\begin{cases}-a(y_1{}^*-x) & (\text{when } x\leq y_1{}^*)\\ a'(x-y_1{}^*) & (\text{when } x\geq y_1{}^*)\end{cases} \tag{2.8}$$

where v_1 is a certain constant and $y_1{}^*$ is the optimal starting stock when only one stage is considered. (Note that $\underset{z}{\text{Min}}\ P(x,\ x,\ z)$ is less than $\text{Min}\ P(x,\ y_1{}^*,\ z)$ unless $x=y_1{}^*$.)

Now, suppose that

$$f_{n-1}(x)=v_{n-1}+\begin{cases}-a(y_{n-1}{}^*-x) & (\text{when } x\leq y_{n-1}{}^*)\\ a'(x-y_{n-1}{}^*) & (\text{when } x\geq y_{n-1}{}^*)\end{cases} \tag{2.9}$$

where $y_{n-1}{}^*$ is the optimal starting stock for the first stage when $n-1$ stages are allowod. Then.

$$\frac{\partial}{\partial x}f_{n-1}(x)=\begin{cases}a & (\text{when } x\leq y_{n-1}{}^*)\\ a' & (\text{when } x\geq y_{n-1}{}^*).\end{cases} \tag{2.10}$$

It is apparent that [R. N. 1], [R. N. 3] and [R. N. 5.] hold. For [R. N. 2], since

$$\frac{\partial}{\partial z}K(x,\ y,\ z)=p+b \tag{2.11}$$

and

$$\alpha\frac{\partial}{\partial z}f_{n-1}(y-z)=\begin{cases}-\alpha a & (\text{when } y-z\leq y_{n-1}{}^*)\\ -\alpha a' & (\text{when } y-z\geq y_{n-1}{}^*),\end{cases} \tag{2.12}$$

thus [R. N. 2] holds. For [R. N. 4],

$$\frac{\partial}{\partial y}K(x,\ y,\ z)=\begin{cases}-(a'+b) & (\text{when } x\geq y)\\ -(a+b) & (\text{when } x\leq y)\end{cases} \tag{2.13}$$

and

$$\alpha\frac{\partial}{\partial y}f_{n-1}(y-z)=\begin{cases}\alpha a & (\text{when } y-z\leq y_{n-1}{}^*)\\ \alpha a' & (\text{when } y-z\gtrless y_{n-1}{}^*).\end{cases} \tag{2.14}$$

Hence, if $-(a'+b)+\alpha a<0$, [R. N. 4] holds. It is, now, also clear that [R. N. 6] is satisfied. The solution $f_n(x)$ is a function of the form

$$f_n(x)=v_n+\begin{cases}-a(y_n{}^*-x) & (\text{when } x\leq y_n{}^*)\\ a'(x-y_n{}^*) & (\text{when } x\geq y_n{}^*)\end{cases} \tag{2.15}$$

where v_n is a certain constant and y_n^* is the optimal starting stock for the first stage when n stages of operations are considered. According to the mathematical induction mentioned above, it has been verified that an optimal non-randomized ordering policy for the first stage exists regardless of the number of stages considered, and, for $n=1, 2, 3, \cdots\cdots$, the solutions are the functions of the form (2.15).

Case B. The one-stage profit function of this case is all the same as that of the case A mentioned before when $y \geq x$. But it is impossible to let $y \leq x$.

It is apparent that there is a certain initial stock level x_1^* which is "optimal" in the sense that

$$\text{(2.16)} \qquad \operatorname*{Min}_z P(x_1^*, x_1^*, z) > \operatorname*{Min}_z P(x, x, z)$$

for any x such thal $x \neq x_1^*$. It is also clear that, for $x < x_1,^*$

$$\text{(2.17)} \qquad \operatorname*{Min}_z P(x, x_1^*, z) > \operatorname*{Min}_z P(x, x, z)$$

and, for $x > x_1^*$,

$$\text{(2.18)} \qquad \operatorname*{Min}_z P(x, x, z) > \operatorname*{Min}_z P(x, y, z)$$

where $y > x$. That [R. 1. 1]~[R. 1. 6] are satisfied is obvious and the proof is omitted here. The solution $f_1(x)$ is a function of the form

$$\text{(2.19)} \qquad f_1(x) = v_1 - \begin{cases} a(x_1^* - x) & \text{(when } x \leq x_1^*) \\ b(x - x_1^*) & \text{(when } x \geq x_1^*) \end{cases}$$

where v_1 is a certain constant and the optimal starting stock is $y_1^* = x_1^*$ if $x \leq x_1^*$ and $y_1^* = x$ if $x \geq x_1^*$.

Now, suppose that

$$\text{(2.20)} \qquad f_{n-1}(x) = v_{n-1} - \begin{cases} a(x_{n-1}^* - x) & \text{(when } x \leq x_{n-1}^*) \\ b(x - x_{n-1}^*) & \text{(when } x \geq x_{n-1}^*) \end{cases}$$

where v_{n-1} is a certain constant and x_{n-1}^* is the "optimal" initial stock for the first stage when $n-1$ stages are considered. It is apparent that [R. N. 1], [R. N. 3] and [R. N. 5] are satisfied. That [R. N. 2] holds is verified because

$$\text{(2.21)} \qquad \frac{\partial}{\partial z} K(x, y, z) = p + b$$

and

$$\text{(2.22)} \qquad \alpha \frac{\partial}{\partial z} f_{n-1}(y-z) = \begin{cases} -\alpha b & \text{(when } y - z \geq x_{n-1}^*) \\ -\alpha a & \text{(when } y - z \leq x_{n-1}^*). \end{cases}$$

Further,

$$\frac{\partial}{\partial y}K(x, y, z)=-(a+b) \tag{2.23}$$

if and only if $y \geq x$. (Note that it is impossible to be $y<x$.) And it is also easily seen that

$$\alpha\frac{\partial}{\partial y}f_{n-1}(y-z)=\begin{cases}\alpha a & (\text{when } y-z\leq x_{n-1}^*)\\ -\alpha b & (\text{when } y-z\geq x_{n-1}^*).\end{cases} \tag{2.24}$$

Hence, that [R. N. 4.] is satisfied had become clear. It is, now, also clear that [R. N. 6] is satisfied.

Here, it must be noted, however, that the profit functions of these cases are linear in y, and not "strictly" concave. Thus the optimal strategy for the management is not unique although there is one optimal strategy which is pure.

In the succeeding two sections, we will obtain the exact solutions for the cases discussed here.

3. SOLUTION FOR THE CASE A

Suppose that the profit function is given by (2.4)$\sim$(2.7) and the demand z in any stage will be any number in the closed interval $[z_{\min}, z_{\max}]$. By the previous discussion, it is apparent that

$$y_1^*=\frac{(p+b)z_{\min}+cz_{\max}}{p+b+c} \tag{3.1}$$

and

$$f_1(x)=v_1+\begin{cases}-a(y_1^*-x) & (\text{when } x\leq y_1^*)\\ a'(x-y_1^*) & (\text{when } x\geq y_1^*).\end{cases} \tag{3.2}$$

The explicit expression of v_1 in terms of p, a, a', b, c, $z_{\min}$ and $z_{\max}$ is, of cause obtainable, but it may be omitted here because it is not so important.

Now, suppose that

$$f_{n-1}(x)=v_{n-1}+\begin{cases}-a(y_{n-1}^*-x) & (\text{when } x\leq y_{n-1}^*)\\ a'(x-y_{n-1}^*) & (\text{when } x\geq y_{n-1}^*).\end{cases} \tag{3.3}$$

For the convenience of calculation, it is valid to consider

$$K(x, x, z_{\min})+\alpha f_{n-1}(x-z_{\min}) \tag{3.4}$$

in stead of $K(x, y, z_{\min})+\alpha f_{n-1}(y-z_{\min})$. (The reasoning will be easily found by examining the slope of the security level.)

The explicit form of (3.4) is, for $x-z_{\min}\leq y_{n-1}^*$,

$$pz_{\min}-b(x-z_{\min})+\alpha v_{n-1}-\alpha a y_{n-1}^*+\alpha a(x-z_{\min})$$

(3.5) $\quad =(p+b-\alpha a)z_{\min}-(b-\alpha a)x+\alpha v_{n-1}-\alpha a y_{n-1}^*$

and, for $x-z\geq y_{n-1}^*$,

(3.6) $\quad (p+b-\alpha a')z_{\min}-(b-\alpha a')x+\alpha v_{n-1}-\alpha a y_{n-1}^*$.

Consider, in addition,

(3.7) $\quad M(x,\ x,\ z_{\max})+\alpha f_{n-1}(0)$

the explicit form of which is

(3.8) $\quad (p+c)x-cz_{\max}+\alpha v_{n-1}-\alpha a y_{n-1}^*$.

When $x=z_{\min}+y_{n-1}^*$, (3.5) and (3.6) are equal, but the slope of (3.5) and (3.6) are different. Hence, one and only one of (3.5) and (3.6) intersects (3.8).

If we assume that (3.5) and (3.8) intersect, the x-coordinate of the intersection x_n^* is

(3.9) $$x_n^*=\frac{(p+b-\alpha a)z_{\min}+cz_{\max}}{p+b+c-\alpha a}$$

which is apparently not less than x_1^*. It is worth noticing that, for a certain n, if (3.9) is valid, i. e., (3.5) and (3.8) intersect, the limitting solution x^* as $n\rightarrow\infty$ is equal to (3.9) because $x_{r+1}^*-z_{\min}<x_r^*$ and $x_r^*=x_n^*$ for $r\geq n$.

If we assume that (3.6) and (3.8) intersect, the x-coordinate of the intersection x_n^* is

(3.10) $$x_n^*=\frac{1}{p+b+c-\alpha a'}\{(p+b-\alpha a')z_{\min}+cz_{\max}+\alpha(a-a')x_{n-1}^*\}.$$

Thus, we can easily obtain, from (3.10),

(3.11) $$x_{n+1}^*-x_n^*=\frac{\alpha(a-a')}{p+b+c-\alpha a'}(x_n^*-x_{n-1}^*).$$

For

(3.12) $$0<\frac{\alpha(a-a')}{p+b+c-\alpha a'}<1$$

the series $\{x_n\}$ converges as $n\rightarrow\infty$ The limit is easily found to be

(3.13) $$x^*=\frac{(p+b-\alpha a')z_{\min}+cz_{\max}}{p+b+c-\alpha a}$$

(Note that, if $a>a'$, (3.13) is less than (3.9).) However, it must be noted that, from (3.13),

(3.14) $$x^*-z_{\min}=\frac{\{\alpha(a-a')-c\}z_{\min}+cz_{\max}}{p+b+c\alpha a}<x^*$$

as $(p+b-\alpha a')>\{\alpha(a-a')-c\}$. Hence, contrary to the assumption, (3.6) does not intersect (3.8) when $n \to \infty$. Thus, we can conclude

$$x^* = \frac{(p+b-\alpha a)z_{\min} + cz_{\max}}{p+b+c-\alpha a} \tag{3.15}$$

The series $\{y_n^*\}$ of the optimal starting stock for the first stage when n stages are considered is all the same as $\{x_n^*\}$. (The reason is easily seen by subtracting the cost $a(y_n^* - x)$ or adding the revenue $a'(x - y_n^*)$ and examining the slope of the security level.)

For sufficiently large n, we can set

$$\begin{aligned} v_{n+1} &\fallingdotseq -cz_{\max} + (p+c)x^* + \alpha v_n - \alpha a x^* \\ &= -cz_{\max} + (p+c-\alpha a)x^* + \alpha v_n. \end{aligned} \tag{3.16}$$

Hence,

$$v_{n+1} - v_n = \alpha(v_n - v_{n-1}). \tag{3.17}$$

If $\alpha = 1$, the series $\{v_n\}$ approaches to a certain arithmetical series. For $0 < \alpha < 1$, the series converges. Let v denote the limit of $\{v_n\}$, then, from (3.16),

$$v = \frac{1}{1-\alpha}\{(p+c-\alpha a)x^* - cz_{\max}\}. \tag{3.18}$$

where x^* is given by (3.15).

Summarizing our conclusions in this section, the optimal starting stock for the first stage is given by

$$y_1^* = \frac{(p+b)z_{\min} + cz_{\max}}{p+b+c} \tag{3.19}$$

if only one stage is allowed to be considered. However, as the number of stages considered increases, the optimal starting stock for the first stage also gradually increases, and approaches to the limit y^* such that

$$y^* = \frac{(p+b-\alpha a)z_{\min} + cz_{\max}}{p+b+c-\alpha a} \tag{3.20}$$

The solution $f_n(x)$ is, for any n, the function of the form

$$f_n(x) = v_n + \begin{cases} -a(y_n^* - x) & (\text{when } x \leq y^*_n) \\ a'(x - y^*_n) & (\text{when } x \geq y^*_n) \end{cases} \tag{3.21}$$

and the series $\{v_n\}$ also converges to a certain limit.

4. SOLUTION FOR THE CASE B

Consider the case B where neither returning nor disposal is permitted. Here we can't reduce the stock except by selling the merchandise to the customers.

From the discussion in the section 2, it is already clear that

$$x_1^* = \frac{(p+b)z_{\min} + cz_{\max}}{p+b+c} \tag{4.1}$$

$$y_1^* = \begin{cases} x_1^* & (\text{when } x \leq x_1^*) \\ x & (\text{when } x \geq x_1^*) \end{cases} \tag{4.2}$$

and

$$f_1(x) = v_1 - \begin{cases} a(x_1^* - x) & (\text{when } x \leq x_1) \\ b(x - x_1^*) & (\text{when } x \geq x_1). \end{cases} \tag{4.3}$$

Now, suppose that

$$f_{n-1}(x) = v_{n-1} - \begin{cases} a(x_{n-1}^* - x) & (\text{when } x \leq x_{n-1}^*) \\ b(x - x_{n-1}^*) & (\text{when } x \geq x_{n-1}^*) \end{cases} \tag{4.4}$$

and consider, for the convenience of calculation,

$$K(x,\ x,\ z_{\min}) + \alpha f_{n-1}(x - z_{\min}) \tag{4.5}$$

the explicit expression of which is, for $x - z_{\min} \leq x_{n-1}^*$,

$$(p+b-\alpha a)z_{\min} - (b - \alpha a)x + \alpha v_{n-1} - \alpha a x_{n-1}^* \tag{4.6}$$

and, for $x - z_{\min} \geq x_{n-1}^*$

$$(p+b+\alpha b)z_{\min} - (b+\alpha b)x + \alpha v_{n-1} + \alpha b x_{n-1}^*. \tag{4.7}$$

Consider, in addition,

$$M(x,\ x,\ z_{\max}) + \alpha f_{n-1}(0) \tag{4.8}$$

the explicit expression of which is

$$(p+c)x - cz_{\max} + \alpha v_{n-1} - \alpha a x_{n-1}^*. \tag{4.9}$$

Again, it is apparent that one and only one of (4.6) and (4.7) intersects (4.9) by the similar reason to the previous case.

Suppose, then, (4.6) intersects (4.9). The x-coordinate of the intersection is

$$x_n^* = \frac{(p+b-\alpha a)z_{\min} + cz_{\max}}{p+b+c-\alpha a} \tag{4.10}$$

which is apparently larger than x_1^*. If, for a certain n, (4.10) is valid, i. e., (4.6) intersects (4.9), the limitting solution as $n \to \infty$ is equal to (4.10) because $x^*_{r+1} - z_{\min} < x_r^*$ and $x_r^* = x^*_n$ for any $r \geq n$.

Suppose, on the other hand, (4.7) intersects (4.9). Then, x-coordinate of the intersection is given by

$$x_n^* = \frac{1}{p+b+c+\alpha b}\{(p+b+\alpha b)z_{\min} + cz_{\max} + \alpha(a+b)x_{n-1}^*\} \tag{4.11}$$

Here, we can obtain

(4.12) $$x_{n+1}^*-x_n^*=\frac{\alpha(a+b)}{p+b+c+\alpha b}(x_n^*-x_{n-1}^*)$$

and it is clear that the series $\{x_n^*\}$ converges, because

(4.13) $$0<\frac{\alpha(a+b)}{p+b+c+\alpha b}<1.$$

The limit x^* is the solution of the equation

(4.14) $$x^*=\frac{1}{p+b+c+\alpha b}\{(p+b+\alpha b)z_{\min}+cz_{\max}+\alpha(a+b)x^*\}.$$

Hence,

(4.15) $$x^*=\frac{(p+b+\alpha b)z_{\min}+cz_{\max}}{p+b+c-\alpha a}$$

which is larger than (4.10). However, $x^*-z_{\min}<x^*$where x^* is given by (4.15). Thus, contrary to the assumption, (4.7) does not intersect (4.9) when n is sufficiently large. We can conclude that the limitting solution is given by

(4.16) $$x^*=\frac{(p+b-\alpha a)z_{\min}+cz_{\max}}{p+b+c-\alpha a.}$$

Here, it must be noted that

(4.17) $$y_n^*=\begin{cases}x_n^* & (\text{when } x\leq x_n^*)\\ x & (\text{when } x\geq x_n^*)\end{cases}$$

and the limitting solution

(4.18) $$y^*=\begin{cases}x^* & (\text{when } x\leq x^*)\\ x & (\text{when } x\geq x^*).\end{cases}$$

The reason can be easily found by examining the security level.

Now, for sufficiently large n,

(4.19) $$v_n\doteqdot(p+c)x^*-cz_{\max}+\alpha v_{n-1}-\alpha ax^*$$

and we can see

(4.20) $$v_{n+1}-v_n=\alpha(v_n-v_{n-1}).$$

Hence, if $\alpha=1$, the series $\{v_n\}$ approaches to a certain arithmetical series. For $0<\alpha<1$, the series $\{v_n\}$ converges to a limit v given by

(4.21) $$v=\frac{1}{1-\alpha}\{(p+c-\alpha a)x^*-cz_{\max}\}$$

where x^* is given by (4.16). The function $f_n(x)$ is of the form

(4.22) $$f_n(x)=v_n-\begin{cases}a(x_n^*-x) & (\text{when } x\leq x_n^*)\\ b(x-x_n^*) & (\text{when } x\geq x^*).\end{cases}$$

5. COMPARISON OF SOLUTIONS

As was mentioned before, the case example we discussed in our previous papers [11] [12] was the one under the conditions of $a=a'$ and the demand interval $[0, z_{\max}]$. Hence, it is a special case of the one which we have analysed in the section 3 of the present article. In order to compare the solutions, it is interesting to set $a=a'$ and $z_{\min}=0$ in the solutions obtained in the section 3.

Then, we obtain, from (3.19),

$$y_1{}^* = \frac{cz_{\max}}{p+b+c}, \tag{5.1}$$

and, from, (3.20),

$$y^* = \frac{cz_{\max}}{p+b+c-\alpha a}, \tag{5.2}$$

which correspond to (1.5) and (1.6), respectively. From (3.18) and (3.19), the explicit expression of the limit v is obtained as

$$v = \frac{1}{(1-\alpha)(p+b+c-\alpha a)}\{(p+c-\alpha a)(p+b-\alpha a)z_{\min}+bcz_{\max}\}. \tag{5.3}$$

Setting $z_{\min}=0$ in (5.3) and substituting into (3.21),

$$f(x) = \frac{-bcz_{\max}}{(1-\alpha)(p+b+c-\alpha a)} + \begin{cases} -a'\ (y^*-x) \\ a(x-y^*). \end{cases} \tag{5.4}$$

Here, we obtain, for $a=a'$,

$$\begin{aligned} f(x) &= \frac{-bcz_{\max}}{(1-\alpha)(\mathrm{p}+b+c-\alpha a)} - ay^* + ax \\ &= ax - \frac{\{a(1-\alpha)+b\}cz_{\max}}{(1-\alpha)(p+b+c-\alpha a)} \end{aligned} \tag{5.5}$$

where y^* is given by (5.2). Note that (5.5) is the same solution as (1.8).

Now. let us turn to the comparison of the case A and case B. First of all, it must be noted that we can find a corresponding property in the one-stage solutions of these two cases, i. e., the stock level given by the formula

$$x_1{}^* = \frac{(p+b)z_{\min}+cz_{\max}}{p+b+c} \tag{5.6}$$

has a critical meaning in both cases. In the case A, the optimal ordering policy is to adjust the stock level to $x_1{}^*$ given by (5.6), i. e., to order x^*-x if $x \leq x^*$ and to return (or dispose) $x-x_1{}^*$ if $x \geq x_1{}^*$. In the case B, the optimal ordering policy is to order x_1^*-x if $x \leq x_1^*$ and not to order if $x \geq x_1^*$. For the limit, as $n \to \infty$, of the n-stage solutions, the stock level given by

$$x^*=\frac{(p+b-\alpha a)z_{\min}+cz_{\max}}{p+b+c-\alpha a} \tag{5.7}$$

has a critical meaning in both cases in the following sense. The optimal ordering policy for the case A is to order x^*-x if $x \leq x^*$ and to return (or dispose) $x-x^*$ if $x \geq x^*$. On the other hand, the optimal ordering policy for the case B is to order x^*-x if $x \leq x^*$ and not to order if $x \geq x^*$. The correspondence existing for any n is similarly clear.

That the limitting solution in each cases satisfies the equation (1.1) will be also clear. This will be easily verified by substituting $f(x)$ and y^* in each case to the equation (1.1).

6. NOTES ON PRACTICAL INTERPRETATIONS

Two points must be noted here as to the interpretation of our study from the practical viewpoint.

Firstly, the solution y_n^* (or x_n^*) is, in any case, the optimal starting (or initial) stock "for the first stage when n stages are allowed to be considered". It is the fundamental doctorine of our study and also of the dynamic programming approach pioneered by Richard Bellman and others to make decision for the present stage considering succeeding stages.

It seems for us, however, that many of the mathematicians who are interested in the theory of multi-stage decision processes lay too much stress on the limitting solution as $n \to \infty$ because of the theoretical interest. From the point of view of practical businesses, on the other hand, it has little meaning to consider too many stages under the assumption of stationary conditions. Thus, we should like to suggest to pay more attention to the series y_1^*, y_2^*, y_3^*, ... (or the series x_1^*, x_2^*, x_3^*,......) with a clear recognition of the assumptions of the number of stages under consideration. The observed tendencies of the series will be suggestive in making practical decisions.

Hence, such formulae as (3.1), (3.9) and (3.10) are particularly important in the case A, where y_1^* is given by (3.1), and y_n^* is given by either (3.9) or (3.10), especially by (3.9) for sufficiently large n.

(Note that, in case A, $y_n^*=x_n^*$.)

Similarly, such formulae as (4.1), (4.10) and (4.11) are particularly important in the case B, where the critical stock level x_1^* is given by (4.1), and x_n^* is given by either (4.10) or (4.11), especially by (4.10) for sufficiently large n. (Note that, in case B, $y_n^*=x_n^*$ if $x \leq x_n^*$, and $y_n^*=x$ if $x>x_n^*$.)

The second point which we intend to emphasize is the relation of solutions to the assumption of demand interval $[z_{\min}, z_{\max}]$. Under the assumption of identical demand interval $[z_{\min}, z_{\max}]$ (at any stage in future), the solutions have been obtained as was already shown. Thus, these solutions are particularly valid when the following conditions are satisfied; (1) the demand in one period (stage) cannot be less than $z_{\min}$, but can be $z_{\min}$, (2) the demand in one period (stage) cannot be more than $z_{\max}$, but can be $z_{\max}$, and (3) the probability distribution on z in the interval $[z_{\min}, z_{\max}]$ is completely unknown. The reason why such conditions as that "the demand can be $z_{\min}$" or "the demand can be $z_{\max}$" are needed is that the least favourable probability distribution on z (the minimax strategy of the market) under the assumption is a certain two-point distribution on $z_{\min}$ and $z_{\max}$.

Unfortunately, the conditions are not always fully satisfied. In many practical cases, decision-makers may be too ignorant to assume z to be a number in a closed interval $[z_{\min}, z_{\max}]$. However, the solutions we have obtained have quite simple structures, i. e., the quantity given by (3.1) or (4.1) corresponds to the point which divides $z_{\min}$ and $z_{\max}$ internally by the ratio $c : (p+b)$, and similarly the quantity given by (3.20) or (4.16) corresponds to the point which divides $z_{\min}$ and $z_{\max}$ internally by the ratio $c : (p+b-\alpha a)$. Thus, the solutions corresponding to various assumptions of the numbers $z_{\min}$ and $z_{\max}$ can be instantly found, and the effect of the assumption $[z_{\min}, z_{\max}]$ on the solution is easily observed. Therefore, even if we are under the situation of nearly complete ignorance, we may be able to find our ways. On the other hand, if the decision-maker has any information about the probability distribution on z, our model is not adequate because, if it is applied to such cases, the information is ignored. For such cases, the stochastic models [6] [7] [9] [10] will be more suggestive.

7. NOTES ON OTHER CASES

In this section, notes on several miscellaneous problems which have not been referred to in the above discussions will be given.

In the case A, we assumed $a \geq a'$. Then, how about the case if we assume, on the contrary, $a < a'$? In such case, we cannot guarantee the concavity of the payoff function, and more complicated analyses will be required. However, the problem itself is quite nonsensical if $a < a'$, because it may be the best policy "to buy more and still more and to return more and still more". The reason is that the returning is profitable in this case. Hence, we omitted the analysis of this case.

In some cases which we have not mentioned, we often have to consider the existence of "constant costs" as to ordering, returning, shortage or storage. For example, an additional constant cost, such as the procedure cost, may be incurred beside the wholesale price multiplied by the quantity ordered. If such constant costs exist, the function $P(x, y, z)$ is discontinuous and accordingly the problem is hard to be analysed. In order to find our way out of such difficulty, we may reconstruct the model as a finite game.

Here, we can consider the multi-stage decision process of finite type which is closely relating to our study hitherto discussed. There is a payoff matrix $[a_{ij}(x_0)]$ which is specified by the initial state x_0. A row and a column must be chosen by the decision-maker and by the nature, respectively, and, for each pair of strategy choices i and j, the initial state x_1 of the second stage is determined by a rule ϕ, that is, $x_1 = \phi\ (i, j)$. Hence, under the assumption of the maximin principle, our problem is to solve the functional equation of the form

$$f(x_0) = \mathrm{Val}[a_{ij}(x_0) + \alpha f\{\phi(i, j)\}] \tag{7.1}$$

where $f(x_0)$ is the total present value of payoff starting from initial state x_0 and using the maximin principle in an unlimitted number of stages. The method of successive approximations is usable also in this case. Here, note that, if the decision-maker and the nature have m and n alternatives (strategies), respectively, it suffices for us to consider m n initial states (or at most $mn+1$ initial states). Hence, we have to consider mn (or at most $mn+1$) matrices simultaneously. One merit of such a finite type approach is the possibility of utilization of incomplete information about the probability distribution on the state of the nature by analysing the problem as a constrained game.

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