α INVENTORY POLICY——SCHEDULING MODEL.

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1. INTRODUCTION.

In order to determine the economic lot size which minimizes the total combined costs composed of ordering costs and carrying costs, we can use the well-known formula, [1], [2]:

$$Q=\sqrt{\frac{2YK}{IC}} \tag{1}$$

where Q=economic lot size,
Y=expected demand per year,
K=cost of making one order,
IC=carrying cost per year.

In the recent article entitled 'Economic Lot Sizes with Seasonal Demand' [3], Mark B. Schupack pointed out three conditions, which must be hold when we use the above formula (1), that is,

1. Future demand is known with certainty.
2. Demand is constant over time.
3. No shortages are allowed.

The purpose of Schupack's paper was to obtain a method how to calculate economic lot sizes in the cases when the condition 2 above does not hold and demand seasonally fluctuates. In this paper, I shall deal with another method of calculation in the case when the condition 2 is relaxed.

2. α LEVEL OF INVENTORY.

In this simplified problem of optimal inventory policy, I shall use the measure:

$$\alpha=\frac{nK}{IC} \quad (n=12), \tag{2}$$

where n represents the number of unit periods. In this paper, let us suppose unit period is month.

For example, if $C=$ ¥125.00, $I=4\%$ and $K=$ ¥2,400.00,

$$\alpha=\frac{2{,}400\times 12}{0.04\times 125}=5{,}760. \tag{3}$$

Clearly $\frac{IC}{n}=\frac{0.04\times 125}{12}=\frac{5}{12}$ is carrying cost per one unit over a month. If we assume that the inventory level at the beginning of and at the end of i-th month are respectively I_{i-1} and I_i, the average inventory is

$$\bar{I}_i=\frac{I_{i-1}+I_i}{2}. \tag{4}$$

Now let us consider the total cost over i-th month assuming that order must be made at the end of i-th month and $I_i>\alpha$. Then we can compare the inventory policy (I_{i-1}, I_i) with the policy $(I_{i-1}-I_i, 0)$, where the former means that no order is placed at the end of i-th month and the latter an order is placed at the end of i-th month. If we represent these total costs over i-th month by L_i and L_α, it is shown policy $(I_{i-1}-I_i, 0)$ is better than policy (I_{i-1}, I_i) as follows.

$$L_i-L_\alpha=\left(\frac{I_{i-1}+I_i}{2}\right)\cdot\frac{IC}{n}-\left(\frac{I_{i-1}-I_i}{2}\cdot\frac{IC}{n}+\frac{IC\alpha}{n}\right)=\frac{IC}{n}(I_i-\alpha)>0. \tag{5}$$

If $I_i<\alpha$, on the contrary, then

$$L_i-L_\alpha<0. \tag{6}$$

From these results (5) and (6), it can be said that, whenever the level I_i is higher than level α, we must take the policy $(x_i, 0)$, that is, we must place an order at the end of i-th month. However, this criterion can give us only the optimal policy for unit period, then we must consider how to obtain the optimal policy over n periods, say, over two months or over three months.

Let us assume the demand of i-th month x_i $(i=1, 2, \cdots\cdots, 12)$ is known with certainty in the following manner,

$$(x_i)=(x_1, x_2, \cdots\cdots, x_{12}). \tag{7}$$

To develop our method of quasi-optimal policy, let us proceed by working backwards from 12-th month (December) to 1-st month (January). [4] The problem to which we are referring here is a typical one of multistage decision process which consists of 12 stages, and therefore it is necessary to decide successively 12 decision variables I_i $(i=12, 11, \cdots\cdots, 2, 1)$ step by step so as to minimize the combined total cost. In this method of solution, it should be noted that, in order to decide the inventory level I_i, we must determine the optimal policy over the last

$(n-i)$ periods, that is, the optimal policy over $(i+1)$-th, ……, and n-th periods.

However, we cannot find any clear-cut method of obtaining the optimal policy because of a number of possibilities in alternative combinations of our policies. Therefore we are compelled to proceed step by step through trial and error approach. The first step is to obtain the optimal solution for 12-th period, and after that we can gain the optimal policy over 11-th and 12-th period, and so on. For simplicity we assume $I_{12}=0$, and then the optimal level I_{11} is always x_{12}. And the next step is to decide the level of I_{10} in terms of the policy over the last two periods, which is obviously stated as follows:

$$\left.\begin{array}{l} \text{if } I_{11}=x_{12}<\alpha, \text{ then } I_{10}=x_{11}+x_{12}, \text{ and} \\ \text{if } I_{11}=x_{12}\geq\alpha, \text{ then } I_{10}=x_{11}. \end{array}\right\} \qquad (8)$$

If we take the policy $I_{10}=x_{11}+x_{12}$, the total cost is given by

$$L'_{11,12}=\frac{1}{2}hx_{12}+h\left(x_{12}+\frac{1}{2}x_{11}\right)=\frac{h}{x}(x_{12}+x_{11})+hx_{12},$$

where $nh=12h=IC$. On the other hand, if we take the policy $I_{10}=x_{11}$, the total cost is

$$L''_{11,12}=\frac{1}{2}hx_{12}+K+\frac{1}{2}hx_{11}=\frac{h}{2}(x_{11}+x_{12})+K.$$

Furthermore, the total cost, in the case when we take the policy $I_{10}=x_{11}+I'_{11}$ $(0<I'_{11}<x_{12})$, is given by

$$L'''_{11,12}=\frac{1}{2}hx_{12}+K+\frac{h}{2}(2I'_{11}+x_{11})=\frac{1}{2}h(x_{12}+x_{11})+K+hI'_{11}.$$

When $I_{11}=x_{12}<\alpha$, $L'_{11,12}<L''_{11,12}<L'''_{11,12}$ is obvious, since $K=\alpha h$ by equation (2).

On the contrary, if $I_{11}=x_{12}\geq\alpha$, we have inequalities $L'_{11,12}>L''_{11,12}$ and $L'''_{11,12}>L''_{11,12}$. Consequently the optimal one becomes the policy (8).

Now we proceed to the next stage of our solution which is to designate the optimal policy for the last three months. In this case there are such four alternative policies (a), (b), (c) and (d) as listed below.

	I_{12}	I_{11}	I_{10}	I_9
a	0	x_{12}	$x_{11}+x_{12}$	$x_{10}+x_{11}+x_{12}$

b	0	x_{12}	$x_{11}+x_{12}$	x_{10}
c	0	x_{12}	x_{11}	$x_{10}+x_{11}$
d	0	x_{12}	x_{11}	x_{10}

For example, the policy (b) in the above table represents the policy that we place an order at the end of 10-th period. If the total costs of these four policies are respectively L_a, L_b, L_c and L_d, then they are given by

$$\left.\begin{aligned} L_a &= h(2x_{12}+x_{11}) \\ L_b &= hx_{12}+K \\ L_c &= hx_{11}+K \\ L_d &= 2K \end{aligned}\right\} \qquad (9)$$

Let us introduce the measure:

$$\frac{\alpha}{2}=\frac{nK}{2IC}=\frac{K}{2h} \qquad (10)$$

Using this measure, it is shown that policy (c) is better than policy (a) when $\alpha<2I_{12}<2\alpha$, because

$$L_a-L_c=2hx_{12}-K=h(2I_{12}-\alpha)>0. \qquad (11)$$

This argument leads us to the following four cases.

Case I. $x_{12}<\alpha$ and $I_{10}=x_{11}+x_{12}<\alpha$.
Case II. $x_{12}<\alpha$ and $I_{10}=x_{11}+x_{12}>\alpha$.
Case III. $x_{12}>\alpha$ and $x_{11}<\alpha$.
Case IV. $x_{12}>\alpha$ and $x_{11}>\alpha$.

Case I. If $\frac{\alpha}{2}<x_{12}<\alpha$, obviously policy (c) is optimal, since

$L_a>L_c$,
$L_a-L_b=h(x_{11}+x_{12})-K=h(x_{11}+x_{12})-h\alpha<0.$
$L_c-L_d=hx_{11}-K=h(x_{11}-\alpha)<0.$

Conversely, if $x_{12}<\frac{\alpha}{2}$, policy (a) is optimal.

Case II. $L_a-L_b=h(x_{11}+x_{12})-h\alpha>0,$
$L_b-L_d=hx_{12}-K=h(x_{12}-\alpha)<0.$

If $x_{12}<\frac{\alpha}{2}$, then $L_a-L_c=h(2x_{12}-\alpha)<0.$

That is, policy (b) is optimal. (It must be noticed that in this case $x_{12}<\frac{\alpha}{2}$ and $x_{11}>\frac{\alpha}{2}$, therefore $x_{11}>x_{12}$.)

If $\frac{\alpha}{2}<x_{12}<\alpha$, (c) is optimal when $x_{11}<x_{12}$, and (b) is optimal when $x_{11}>x_{12}$, since

$$L_b-L_c=h(x_{12}-x_{11})$$

Case III. Similarly, it is shown that policy (c) is optimal in this case.
Case IV. Obviously, (d) is optimal.

Thus the above obtained results are summed up as follows.

$$\left.\begin{array}{l}\text{If } x_{11}<\frac{\alpha}{2} \text{ and } x_{12}<\frac{\alpha}{2}, \text{ (a) is}\\ \text{If } \frac{\alpha}{2}<x_{12}<\alpha \text{ and } x_{11}+x_{12}<\alpha, \text{ (c) is}\\ \text{If } x_{12}<\alpha,\ x_{11}+x_{12}>\alpha \text{ and } x_{11}>x_{12}, \text{ (b) is}\\ \text{If } x_{12}<\alpha,\ x_{11}+x_{12}>\alpha \text{ and } x_{11}<x_{12}, \text{ (c) is}\\ \text{If } x_{12}>\alpha \text{ and } x_{11}>\alpha, \text{ (d) is}\\ \text{If } x_{12}>\alpha \text{ and } x_{11}<\alpha, \text{ (c) is}\end{array}\right\} \text{optimal policy.} \quad (11)$$

The next step of our solution is to obtain the set of optimal policies over the last four months. Similarly to the former case, we have the possible alternatives of our possible policies and the respective loss functions in the following manner.

	I_{12}	I_{11}	I_{10}	I_9	I_8
a	0	x_{12}	$x_{11}+x_{12}$	$x_{10}+x_{11}+x_{12}$	$x_9+x_{10}+x_{11}+x_{12}$
b	0	x_{12}	$x_{11}+x_{12}$	$x_{10}+x_{11}+x_{12}$	x_9
c	0	x_{12}	$x_{11}+x_{12}$	x_{10}	x_9+x_{10}
d	0	x_{12}	x_{11}	$x_{10}+x_{11}$	$x_9+x_{10}+x_{11}$
e	0	x_{12}	x_{11}	$x_{10}+x_{11}$	x_9
f	0	x_{12}	$x_{11}+x_{12}$	x_{10}	x_9
g	0	x_{12}	x_{11}	x_{10}	x_9+x_{10}
h	0	x_{12}	x_{11}	x_{10}	x_9

$$L_a=h(3x_{12}+2x_{11}+x_{10})$$
$$L_b=h(2x_{12}+x_{11})+K$$

$$\left.\begin{aligned} L_c&=h(x_{12}+x_{10})+K \\ L_d&=h(2x_{11}+x_{10})+K \\ L_e&=hx_{11}+2K \\ L_f&=hx_{12}+2K \\ L_g&=hx_{10}+2K \\ L_h&=3K \end{aligned}\right\} \quad (12)$$

Again introducing the measure:

$$\frac{\alpha}{3}=\frac{nK}{3IC}=\frac{K}{3h}, \quad (13)$$

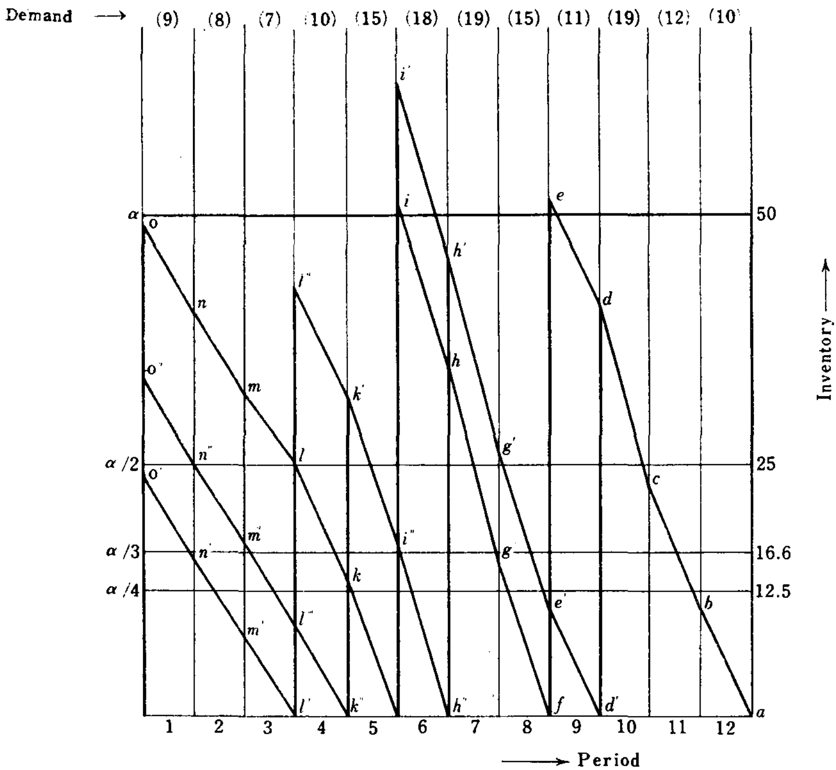
we can gain the decision table for the set of optimal policies without logical difficulty albeit the procedure is considerably cumbrous. Theoretically speaking, along this way of solution we must complete the decision table over 12 months. However, the table will become too complicated one to utilize efficiently. Then, let us turn our attention to another method of solution, that is, the method of graphical solution.

3. GRAPHICAL SOLUTION.

To render our procedure less difficult, let us assume the demand table is given as below.

MONTH	DEMAND	MONTH	DEMAND
1	9	7	19
2	8	8	15
3	7	9	11
4	10	10	19
5	15	11	12
6	18	12	10

Suppose that $I=24\%$, $K=$ ¥1,000.00 and $C=$ ¥1,000.00, then it follow $\alpha=50$.



In this section, I would like to show a graphical method in which α-levels of inventory are used as decision criteria. Although there is no assurance that we can attain to the optimal inventory policy by this method, pretty quickly we can approach to the optimal solution making use of measure α/i (i=1, 2, 3,……).

In our example, the first step of solution is represented by the policy

(I) (a, b, c, d, e, f, g, h, i, j, k, l, m, n, o),

shown in the figure above. However, that the policy

(II) (a, b, c, d, e, f, g, h, i, j, k, l′, m′, n′, o′)

is better than policy (I), is verified. Furethermore, in the above figure, $d=I_9=x_{10}+x_{11}+x_{12}=41>3e'=3x_9=33$, and also $e'<\alpha/4$. Then, if we consider two policies (d, e, f, g, h, i) and (d′, e′, g′, h′, i′), difference between those two kinds of costs is $(hd-3e'h)=(41-33)h=8h$. Therefore, the policy

(III) (a, b, c, d, d′, e′, g′, h′, i′, j, k, l, l′, m′, n′, o′)

is realized in lower cost than (II). Similarly, the policy

(IV) (a, b, c, d, d′, e′, g′, h′, h″, i″, k′, l″l′, m′, n′, n′o′)

is more preferable than (III), and also

(V) (a, b, c, d, d′, e′, g′, h′, h″, i″, k′, k″, l‴, m″, n″, o″)

is better than (IV).

The total cost of these five alternative policies can be respectively gained as follows.

(I) ¥ 8,210.00
(II) ¥ 7,710.00
(III) ¥ 7,550.00
(IV) ¥ 7,370.00
(V) ¥ 7,310.00

4. GENERALIZATIONS.

(1) From the viewpoint of general theory, it is of course desirable that α-method of inventory control should be generalized into the form, in which three conditions described in section 1 are all relaxed. If we intend to develop the generalized theory of α inventory control method, we may refer to the so-called Arrow-Harris-Marschack model or Dvoretzky-Kiefer-Wolfowitz model. Admittedly, the method developed in this paper has a close connection with that of dynamic programming, though the case to which we were referred was rather simple. It is well-known we can determine reorder point s by (S, s) policy, and this level might be recognized as the lower safety bound against depletion. And, in the generalized stochastic theory of α inventory policy, we will be able to check excess-inventory by α level.

(2) Furthermore, in this paper, instead of the three conditions above-mentioned, we assumed that:

1. Future demand is known with certainty.
3. No shortages are allowed.
4. Order is made at the beginning of any required month.

Suppose that n is not a constant and is a variable, i. e.,

$$\alpha=\alpha(n)=\frac{nK}{IC}, \tag{14}$$

and that we take the ordering policy

$$n\alpha(n)=Y, \tag{15}$$

then the total variable cost is given by

$$L=\frac{Y}{\alpha(n)}\cdot K+\frac{\alpha(n)}{2}\cdot IC. \tag{16}$$

Differentiating (14) and (16) by n, we have

$$\frac{d\alpha}{dn}=\frac{K}{IC},$$

$$\frac{dL}{dn}=-\frac{YK}{\alpha^2}\cdot\frac{d\alpha}{dn}+\frac{IC}{2}\cdot\frac{d\alpha}{dn}=0, \text{ and}$$

$$\frac{d^2L}{dn^2}=\frac{2YIC}{n^3}>0.$$

Therefore,

$$-\frac{YIC}{n^2}+\frac{K}{2}=0.$$

Thus,

$$n=\sqrt{\frac{2YIC}{K}}. \tag{17}$$

Substituting (17) into (14), we obtain

$$\alpha(n)=\sqrt{\frac{2YK}{IC}}, \tag{18}$$

which is in the same form as formula (1).

(3) In the case when future demand is unknown and demand fluctuates over time, it is theoretically inconsistent to use formula (1), though it might give us a rule of thumb. For example, in simulation model of inventory control, where all conditions except 4 do not hold, we can use formula (1) from the practical point of view to determine when we should place order in what amount. However, it can be shown that α method reveals more reasonable policy in such simulation experiments in which all conditions except 4 are relaxed.

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