AN EXTENSION OF N. WIENER'S PREDICTION THEORY

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SUMMARY

The prediction of a stationary time series is very important in the various fields of O. R. works. N. Wiener (1) completed an elegant theory on the prediction. He introduced an integral equation of a predictor $K(t)$ for a continuous time series $f(t)$. Since this equation is singular and of special type we have to use the so-called Wiener-Hoph technique or the factorization to find $K(t)$ from the equation. Sometimes to carry out this technique is very difficult when the auto-correlation function $\varphi(t)$ of $f(t)$ is not expressed in a simple form.

The present paper includes an extension of the Wiener's theory. The prediction of a time series $f(t)$ by an another time series $g(t)$ will be introduced, by taking notice of the cross-correlation between $f(t)$ and $g(t)$. We shall deduce an integral equation of $K(t)$. This time again, the integeral equation is singular, but since it will not need to apply the factorization technique, we could carry out $K(t)$ in general. The results will reduce to the Wiener's case when $f(t)\equiv g(t)$. Therefore this method is including the Wiener's prediction theory as a special case. This method will be modified for the cases of discrete time series, but the conventional method will be proposed for the practical purposes.

THE FUNDAMENTAL EQUATION OF THE PREDICTOR

Time series $f(t)$ and $g(t)$ are assumed to be continuous, bounded and their correlation functions exsist

$$\left.\begin{aligned}\varphi_{ff}(\tau)&=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}f(t+\tau)f(t)dt,\\ \varphi_{fg}(\tau)&=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}f(t+\tau)g(t)dt,\end{aligned}\right\}\qquad(1)$$

$$\varphi_{gg}(\tau)=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}g(t+\tau)g(t)dt,\Bigg|$$

where φ_{ff}, φ_{gg} are the auto-correlation functions of $f(t)$ and $g(t)$ respectively and φ_{fg} is the cross-correlation function of $f(t)$ and $g(t)$. The correlation functions $\varphi_{ff}(\tau)$, $\varphi_{fg}(\tau)$, and $\varphi_{gg}(\tau)$ are continuous and their Fourier transforms are supposed to exsist

$$\left.\begin{aligned}\Phi_{ffc}(\mu)&=\int_0^\infty\varphi_{ff}(\tau)\cos\pi\mu\tau\ \mathrm{d}\tau,\\ \Phi_{ffs}(\mu)&=\int_0^\infty\varphi_{ff}(\tau)\sin\pi\mu\tau\ \mathrm{d}\tau,\end{aligned}\right\}\tag{2}$$

We shall predict the future value of $f(t)$ after a lead of α units of time, $f(t+\alpha)$, $\alpha>0$, from the values of $f(t)$, $g(t)$ on the interval of $t(-\infty, t)$, where we assume the effects of $g(t-\tau)$ on $f(t+\alpha)$ are uniform in time and linear. Then we have the integral

$$\int_0^\infty g(t-\tau)K(\tau)d\tau$$

for the estimated value of $f(t+\alpha)$, introducing a linear predictor $K(\tau)$.

We shall determine $K(\tau)$ from $f(t)$ and $g(t)$ in the meaning of the theory of least squares. That is to find $K(\tau)$ minimizing the functional

$$I[K]\equiv\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\left\{f(t+\alpha)-\int_0^\infty g(t-\tau)K(\tau)d\tau\right\}^2dt.\tag{3}$$

This is a variational calculas. We have

$$\begin{aligned}I[K]&\equiv\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\left\{f(t+\alpha)-\int_0^\infty g(t-\tau)K(\tau)d\tau\right\}^2dt\\ &=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}f(t+\alpha)f(t+\alpha)dt\\ &\quad-2\int_0^\infty K(\tau)d\tau\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}f(t+\alpha)g(t-\tau)dt\\ &\quad+\int_0^\infty K(\tau)d\tau\int_0^\infty K(\sigma)d\sigma\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}g(t-\tau)g(t-\sigma)dt,\end{aligned}$$

from (2), $I[K]$ yields to be

$$I[K]=\varphi_{ff}(0)-2\int_0^\infty \varphi_{fg}(\alpha+\tau)K(\tau)d\tau$$
$$+\int_0^\infty K(\tau)d\tau\int_0^\infty \varphi_{gg}(\tau-\sigma)K(\sigma)d\tau.$$

Now we introduce an arbitrary function $M(\tau)$, assumed to be bounded variation and calculate $I[K]$ for functional space $K(\tau)+\varepsilon M(\tau)$ where ε is an arbitrary real number.

$$I[K+\varepsilon M]=I[K]-2\varepsilon\int_0^\infty \varphi_{fg}(\alpha+\tau)M(\tau)d\tau$$
$$+2\varepsilon\int_0^\infty M(\tau)d\tau\int_0^\infty \varphi_{gg}(\tau-\sigma)K(\sigma)d\sigma$$
$$+\varepsilon^2\int_0^\infty M(\tau)d\tau\int_0^\infty \varphi_{gg}(\tau-\sigma)M(\sigma)d\sigma. \qquad (4)$$

If $K(\tau)$ is the solution minimizing $I[K]$, it will be necessary to have

$$\left(\frac{\partial I[K+\varepsilon M]}{\partial \varepsilon}\right)_{\varepsilon=0}=0.$$

Since $M(\tau)$ is arbitrary, we have

$$\int_0^\infty\left[\varphi_{fg}(\tau+\alpha)-\int_0^\infty\varphi_{gg}(\tau-\sigma)K(\sigma)d\sigma\right]M(\sigma)d\sigma=0$$

or

$$\varphi_{fg}(\tau+\alpha)=\int_0^\infty \varphi_{gg}(\tau-\sigma)K(\sigma)d\sigma,\ \tau\geqq 0 \qquad (5)$$

as a necessary condition.

Next we shall show that the condition (5) is sufficient to give the minimum value of $I[K]$. If we have (5),

$$I[K+\varepsilon M]-I[K]=\varepsilon^2\int_0^\infty M(\tau)d\tau\int_0^\infty \varphi_{gg}(\tau-\alpha)M(\sigma)d\sigma.$$

but being we have

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\left[\int_0^\infty f(t-\tau)M(\tau)d\tau\right]^2dt=\int_0^\infty M(\tau)d\tau\int_0^\infty \varphi_{gg}(\tau-\sigma)M(\sigma)d\sigma,$$

we have

$$I[K+\varepsilon M]-I[K]\geqq 0$$

which means that $I[K+\varepsilon M]$ is minimum when $M\equiv 0$.

Hence we have for the predictor $K(\tau)$

$$\int_0^\infty \varphi_{gg}(\tau-\sigma)K(\sigma)d\sigma=\varphi_{fg}(\tau+\alpha),\ \tau\geqq 0.$$

This is an integral equation of $K(\tau)$, we have to find $K(\tau)$ knowing the correlation functions φ_{fg} and φ_{gg}.

When $f(t)$ coincides with $g(t)$, the cross correlaton function $\varphi_{fg}(\tau+\alpha)$ reduced to the auto-correlation function $\varphi_{ff}(\tau+\alpha)$, and (5) reduced to Wiener's equation.

THE SOLUTION OF THE INTEGRAL EQUATION

The fundamental equation of $K(\tau)$ is a singular integral equation of Fredholm's type of the first species. Multiplying $\cos\pi\mu(\tau-z)$ on both sides of the equation and integrating from 0 to ∞ by τ, we have

$$\int_0^\infty \cos\pi\mu(\tau-z)\varphi_{fg}(\tau+\alpha)d\tau$$
$$=\int_0^\infty \cos\pi\mu(\tau-z)d\tau\int_0^\infty \varphi_{gg}(\tau-\sigma)K(\sigma)d\sigma.$$

At the right side, changing the integral variable from τ to t by putting $\tau-\sigma=t$, we have

$$\int_0^\infty \cos\pi\mu(\tau-z)\varphi_{fg}(\tau+\alpha)d\tau$$
$$=\int_0^\infty \varphi_{gg}(t)\cos\pi\mu(t+\sigma-z)dt\int_0^\infty K(\sigma)d\sigma$$
$$=\Phi_{ggc}(\mu)\int_0^\infty \cos\pi\mu(\sigma-z)K(\sigma)d\sigma$$
$$-\Phi_{ggs}(\mu)\int_0^\infty \sin\pi\mu(\sigma-z)K(\sigma)d\sigma. \qquad (6)$$

Similary we have also

$$\int_0^\infty \sin\pi\mu(\tau-z)\varphi_{fg}(\tau+\alpha)d\tau$$

$$=\Phi_{ggs}(\mu)\int_0^\infty \cos \pi\mu(\sigma-z)K(\sigma)d\sigma$$

$$-\Phi_{ggc}(\mu)\int_0^\infty \sin \pi\mu(\sigma-z)K(\sigma)d\sigma. \qquad (6')$$

From these equations we shall eliminate the term containing $\int_0^\infty \sin \pi\mu(\sigma-z)K(\sigma)d\sigma$, then we have

$$\int_0^\infty \cos \pi\mu(\sigma-z)K(\sigma)d\sigma$$

$$=\int_0^\infty \frac{\Phi_{ggc}(\mu)\ \cos \pi\mu(\tau-z)+\Phi_{ggs}(\mu)\ \sin \pi\mu(\tau-z)}{[\Phi_{ggs}(\mu)]^2+[\Phi_{ggc}(\mu)]^2}\varphi_{fg}(\tau+\alpha)d\tau \qquad (7)$$

By the Fourier's integral theorem, if $K(t)$ satisfies the Dirichlet condition for $t>0$, and if

$$\int_0^\infty |K|\, dt$$

is bounded, then we have

$$\int_0^\infty d\mu\int_0^\infty \cos \pi\mu(t-\xi)K(\xi)d\xi=\begin{cases}K(t), & t>0\\ 0, & t<0.\end{cases}$$

Integrating both sides of (7) by μ from 0 to ∞, we have

$$K(t)=\int_0^\infty d\mu\int_0^\infty \frac{\Phi_{ggc}(\mu)\cos \pi\mu(\xi-t)+\Phi_{ggs}(\mu)\sin \pi\mu(\xi-t)}{[\Phi_{ggs}(\mu)]^2+[\Phi_{ggc}(\mu)]^2}$$

$$\times\varphi_{fg}(\xi+\alpha)d\xi, \qquad (8)$$

where z and τ are changed to t and ξ respectively. When $t<0$, the integral (8) vanishes, we do not need to apply the factorization technique. But if $\varphi_{fg}(\xi+\alpha)$ is $\varphi_{gg}(\xi+\alpha)$, the numerator of the integrand (8) yields to be 1 when $\alpha=0$ and the integral will diverge, therefore the solution given above does not apply to the case of Wiener's prediction.

CONCLUSIONS

The prediction theory of a stationary time series due to the cross-correlation of two time series is established. This is an extension of the Wiener's exterpolation theory. The solution for $K(\tau)$ is founded in a closed form. The factorization does not need in this case. This is the two stage prediction theory. The theory will be able to extend to the case of multistage prediction.

REFERECES

1) Wiener, N. : Extrapolation, Interpolation and Smoothing of Stationary Time Series with Engineering Applications. (1949) John Wiley & Sons.

2) Kondo, J. : A Linear Prediction of A Discrete Time Series. (in Japanese, Survey and Technique, Jan. 1960 pp. 3～12.)