TRAFFIC DYNAMICS: ANALYSIS AS SAMPLED-DATA CONTROL SYSTEMS

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(Received Nov. 18, 1959)

The behavior of a line of cars is investigated by assuming individual vehicle as a sampled-data control system, which might be used as a method of automatic controlling of vehicles in the future. The case of relative velocity control is worked out in this paper, and the system is found to be stable if two system parameters satisfy some inequality.

INTRODUCTION

Recently, theories and experiments on the traffic dynamics were developed at General Motors Corporation and at Kyoto University[(1)~(7)]. In these theories, vehicles following each other on a highway without passing are considered as a linear array of idealized vehicls, each of which is assumed to move with acceleration proportional to the difference between its velocity and that of preceding one at the instant Δ see. Before, i. e., the velocity of the kth car, $u_k(t)$, is assumed to obey the following equation

$$T\frac{d}{dt}u_k(t)=u_{k-1}(t-\Delta)-u_k(t-\Delta),\ k=1,\ 2,\ \cdots \qquad (1)$$

where T and Δ are constants with the dimension of time, and the latter is the response time lag of the driver-car system.

It might be supposed that the behavior of the actual line of vehicles should be far different from the above model because of the individuality or characteristic of the drivers and the vehicles. However, Chandler et. al.[(1)] discovered considerably good agreements between experiments and the theory stated above, and could determine the mean value of T and of Δ with rather small variance.

This fact means that the actual driver-vehicle system can be considered as an automatic control system with its velocity and that of its leading car as the output and input respectively, the block diagram of which is as shown in Fig. 1. The chain of vehicles is then considered as a set of these systems connected in cascade.

In the future, if the vehicle should be controlled automatically, the theory of traffic dynamics will have more realistic importance, as vehicles will be controlled mechanically in accordance with some definite rule of motion, and the constants such as T and Δ will be less ambiguous than those of present human-vehicle systems.

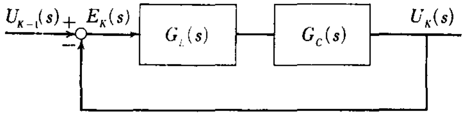

Fig. 1. The block diagram of the kth diver-vehicle system.

There might be a number of rules for controlling vehicle speed, but one of the simplest rule of use is possibly the equation (1). The relative velocity is measured by means of the microwave or ultrasonic emitted at regular intervals, and the force proportional to the relative velocity is automatically applied to the vehicle. If the intervals between successive pulses are fairly short, the existing theories are applicable. On the other hand, however, if the intervals are long compared to the significant time constant of the system, this control system must be reconsidered rather distinctly, i. e., as a sampled-data control system. The anaysis of the latter case, especially the investigation of the stability condition is the purpose of our paper.

SAMPLED-DATA CONTROL SYSTEM

Let us consider the control system of the kth vehicle. The input and the output of the system are the velocities of the preceding vehicle, $u_{k-1}(t)$, and own $u_k(t)$, with Laplace transforms $U_{k-1}(s)$ and $U_k(s)$ respectively. The output is feeded back to the summing point and compared to the input, and the error $e_k(t)=u_{k-1}(t)-u_k(t)$ is used as the actuating signal, the Laplace transform of which is denoted by $E_k(s)$. This continuous signal is sampled at regular intervals of time, every τ sec. Therefore the sampler output $e^*_k(t)$ is a train of regularly spaced unit impulses modulated by $e_k(t)$, viz.

$$e^*_k(t)=\sum_{n=0}^{\infty} e_k(n\tau)\delta(t-n\tau) \qquad (2)$$

where $\delta(t-n\tau)$ is the well known delta function or impulse function occurring at $t=n\tau$, and the Laplace transform of (2) is given by

$$E_k^*(s)=\sum_{n=0}^{\infty} e_k(n\tau)e^{-n\tau s} \tag{3}$$

For this trains of pulses can be used to control the output of the whole system, the high frequency components introduced by the sampler must be removed before the signal reaches the output. A large portion of the required smoothing is usually accomplished by the mechanical components in the system, but in many cases more complete filtering is accomplished by introducting an additional circuit.

One of the simplest filtering circuit is the holding circuit, in which the value of sampling pulse is held until the arrival of the next pulse. The transfer function of this filter is given by

$$G_H(s)=\frac{1-e^{-\tau s}}{s}, \tag{4}$$

and, for convenience, this circuit is introduced in our system as shown in Fig. 2.

Next, the signal is usually transported with time lag Δ, more or less, and this effect is expressed by inserting the transfer function $G_L(s)$,

$$G_L(s)=e^{-\Delta s}. \tag{5}$$

Finally, the transfer function of the control circuit is

$$G_C(s)=\frac{1}{TS}, \tag{6}$$

where T is a time constant.

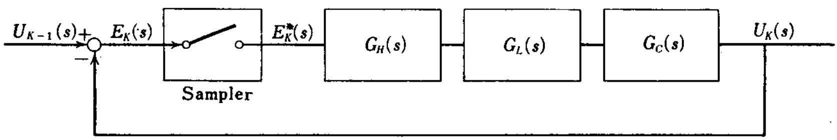

Fig. 2. The block diagram of the kth sampled-data control system.

The over-all system is visualized by the block diagram shown in Fig. 2.

If the time solutions are to be described only at the sampling instants, the z-transform method is much of use[(8)].

Generally, the z-transform $F(z)$ of a time function $f(t)$ is a generating function of its sampled data $f(n\tau)$, and is defined as

$$F(z)=\sum_{n=0}^{\infty} f(n\tau)z^{-n}. \tag{7}$$

For example, $E_k(z)$, the z-transform of $e_k(t)$, is nothing but the formula

(3) with $e^{\tau s}$ replaced by z.

Let the z-transforms corresponding to $U_{k-1}(s)$, $U_k(s)$ and $G(s)=G_H(s)G_L(s)G_C(s)$ be $U_{k-1}(z)$, $U_k(z)$ and $G(z)$ respectively. Then the output is found to be related to the input by the following formura:

$$U_k(z)=\frac{G(z)}{1+G(z)}U_{k-1}(z),\ k=1,\ 2,\ \cdots \tag{8}$$

where zero initial conditions are assumed, and $G(z)$ can be derived if we apply the z-transformation on the inverse Laplace transform of $G(s)$, and is expressed as

$$\begin{aligned} G(z)&=\sum_{n=0}^{\infty}\left[\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}G(s)e^{ts}ds\right]_{t=n\tau}z^{-n} \\ &=\frac{(\tau-\Delta')z+\Delta'}{Tz^{n+1}(z-1)} \end{aligned} \tag{9}$$

with $\qquad \Delta=n\tau+\Delta',\ 0\leqq\Delta'<\tau,\ \mathrm{n}=0,\ 1,\ 2,\ \cdots$

CRITERION FOR STABILITY

The closed-loop system shown in Fig. 2. is stable if the denominator of (8) possesses no zeros outside the unit circle, $|z|=1$.

Denoting

$$\frac{\tau}{T}=\mu,\ \frac{\Delta}{T}=\lambda=n\mu+\lambda' \quad \text{with } 0\leqq\lambda'<\mu \tag{10}$$

we have the characteristic equation

$$p(z)=z^{n+1}(z-1)+(\mu-\lambda')z+\lambda'=0, \tag{11}$$

and the stability condition for $\lambda\leqq\mu$ is easily derived because (11), quadratic in z for $n=0$, has no roots outside the unit circle if $|p(0)|<1$, $p(1)>0$ and $p(-1)>0$. Thus the criterion for stability is

$$\begin{cases} \lambda<1, & \text{(12a)} \\ & \text{when } \lambda\leqq\mu, \\ \mu<2(\lambda+1). & \text{(12b)} \end{cases}$$

The $\lambda>\mu$ case is somewhat complicated to be analyzed, and for practical purpose, it might be sufficient to know the criterion only for $\lambda=n\mu$, $n=1$, 2, $\cdots$. Therefore only this special case will be investigated.

If we introduce ζ,

$$\zeta=\frac{1}{z} \tag{13}$$

the required condition is that the equation

$$1-\zeta+\mu\zeta^{n+1}=0 \tag{14}$$

has no roots inside the unit circle, $|\zeta|=1$, and the method similar to that of Satche[(9)(10)] is successfully applied to obtain this condition.

Plot two functions

$$f(\zeta)=\mu\zeta^{n+1}$$

and

$$g(\zeta)=\zeta-1 \tag{15}$$

in the complex plane for ζ tracing the contour $|\zeta|=1$. If the vector $f(\zeta)-g(\zeta)$ does not make complete rotations around the origine, there are no roots inside the unit circle.

Fig. 3. illustrates the contours of $f(\zeta)$ and $g(\zeta)$. The angles φ and ψ in this figure are

$$\begin{cases} \varphi=\tan^{-1}\dfrac{\mu\sqrt{1-\dfrac{\mu^2}{4}}}{1-\dfrac{\mu^2}{2}} \\ \psi=\tan^{-1}\dfrac{2\sqrt{1-\dfrac{\mu^2}{4}}}{-\mu} \end{cases} \tag{16}$$

respectively, and the rule stated above requires that the representative point of $f(\zeta)$ starting from the point $(\mu, 0)$must not reach the intersection point Q, before the representative point of $g(\zeta)$ reach this point, namely,

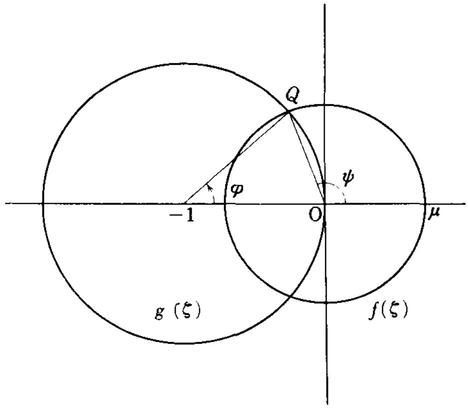

Fig. 3. Paths of $f(\zeta)$ and $g(\zeta)$ in the complex plane.

$$(n+1)\varphi<\psi. \tag{17}$$

This formula together with (16) yields, for example

$$\mu<1 \quad \text{as} \quad \lambda=\mu,$$

$$\mu<\frac{\sqrt{5}-1}{2} \quad \text{as} \quad \lambda=2\mu, \tag{17'}$$

and

$$\mu<\frac{\pi}{2}$$

as $\mu\to 0$ and $n\to\infty$, with finite $\lambda=n\mu$, which is the condition for the case of continuous control, already pointed out by Herman et. al.[(2)] and Kometani and Sakaki[(5)(6)] independently with each other.

These criteria are shown in Fig. 4. If the set of parameters λ and μ is inside the domain enclosed by dotted lines and coordinate axes, the system is stable, viz., the system output is damping with or without oscillation.

The condition (17) is more readily obtained by the following intuitive method. Inserting

$$\zeta=\rho e^{i\varphi} \tag{18}$$

into the characteristic equation (14), we get

$$\begin{cases} 1-\rho\cos\varphi+\mu\rho^{n+1}\cos(n+1)\varphi=0 \\ -\rho\sin\varphi+\mu\rho^{n+1}\sin(n+1)\varphi=0 \end{cases} \tag{19}$$

and the same condition as (17) is easily derived if we take the limit $\rho\to 1$, and eliminate φ from these two equations.

CRITERTION FOR DAMPING WITHOUT OSCILLATION

The above method can be used to determine the condition for critical damping. From the equation (19), after, taking the limit $\varphi\to 0$ and eliminating ρ from these formulae, we get the condition for critical damping

$$\mu=\frac{1}{n}\left(\frac{n}{n+1}\right)^{n+1}$$

and the response of the system to a stimulus dies down without any oscillations if the stimulus is removed, when the next inequality is satisfied by the system:

$$\mu\leqq\frac{1}{n}\left(\frac{n}{n+1}\right)^{n+1},\quad \text{as } \lambda=n\mu, \quad n=1,2,\cdots \tag{20}$$

If $\mu\to 0$, and $n\to\infty$ with finite $\lambda=n\mu$, this condition approaches the known result(2)(5)(6)

$$\mu\leqq e^{-1}. \tag{20'}$$

As to the $\lambda\leqq\mu$ case, the characteristic equation (11) is quadratic in z, and its roots are real and positive when

$$\mu<1-2\sqrt{\lambda}+\lambda \quad \text{as } \lambda\leqq\mu \tag{21}$$

Namely, (21) is the required condition for $\lambda\leqq\mu$. The domain enclosed by the solid line and the coordinate axes in Fig. 4 corresponds to inequalities (20)and (21).

CRITERION FOR ASYMPTOTIC STABILITY

If the velocity of the leading vehicle deviates from the normal value, the fluctuation propagates down the chain with increasing or decreasing amplitudes. When the line of vehicles is fairly long, this fluctuation must not grow as it propagates down the line, and in such a case, rather severe condition, named the condition for asymptotic stability(1), must be fulfilled by the control systems.

This condition is written as

$$\left|\frac{U_k(z)}{U_{k-1}(z)}\right| = \frac{1}{\left|\dfrac{1}{G(z)}+1\right|} < 1 \qquad (22)$$

for

$$z = e^{i\omega\tau}$$

As before, the cases of $\lambda \leqq \mu$ and $\lambda > \mu$ will be treated separately. If $\lambda \leqq \mu$, (22) becomes

$$|(\mu-\lambda)z+\lambda| < |z^2-(1+\lambda-\mu)z+\lambda|,$$

or

$$1+2\lambda-\mu > \lambda\frac{1-\cos 2\omega\tau}{1-\cos\omega\tau} \qquad (23)$$

The right hand side has a maximum value 4λ as $\omega \to 0$, therefore λ must satisty the inequality,

$$\mu < 1-2\lambda \quad \text{as } \lambda \leqq \mu \qquad (24)$$

in order that the relation (23) is held for any value of ω.

When $\lambda = n\mu$, $n=1, 2, \cdots$, (22) is written as

$$\mu < |z^n(z-1)+\mu|$$

or

$$\frac{1}{\mu} > \frac{\cos n\omega\tau - \cos(n+1)\omega\tau}{1-\cos\omega\tau}, \quad \lambda = n\mu \qquad (25)$$

and the required condition is

$$\mu < \frac{1}{2n+1} = 1-2\lambda \quad \text{as } \lambda = n\mu \qquad (26)$$

$$n = 1, 2, \cdots$$

because the maximum value of the right hand side of the inequality(25) is $2n+1$. This condition is held in the limit of $\mu\to 0$ and $n\to\infty$ with finite $\lambda=n\mu$, which coincides the known result also[1].

The condition for asymptotic stability is shown by the interior of the dashed line in Fig. 4. If we have a glance at this figure, we can conclude that the condition for damping output is most weak, that for asymptotic stability is the next, and the condition for damping without oscillation is the most strong limitation of all. Therefore, if the condition for nonoscillatory and damping, (20) or (21), is satisfied by the system, it is stable in any way, i. e., the vehicle can safely follow the leading car.

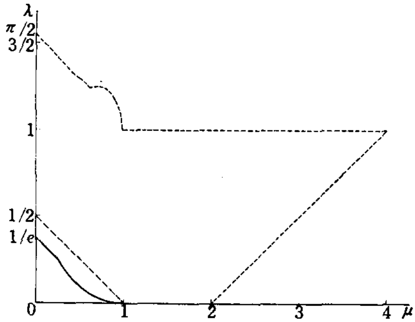

Fig. 4. Graphical representation of the conditions necessary for the system.

MOVEMENT OF THE KTH VEHICLE

Given the motion of the leading vehicle, we can easily obtain the expression for the velocity of the kth vehicle, $u_k(t)$, by means of the inverse z-transformation. For example, let us consider a line of cars initially at rest, supposing the controlling system of individual vehicle is free of response lag, $\Delta=0$, and supposing the sampling instants of every systems are perfectly coincide, though the latter is by no means practical assumption.

If the leading vehicle starts at $t=\delta$ (assumed $0<\delta\leqq\tau$), with a velocity of step function type, the z-transform of the velocity of the kth vehicle is given by

$$U_k(z)=\left[\frac{\mu}{z-(1-\mu)}\right]^k U_0(z),\ k=1,\ 2,\ \cdots \tag{27}$$

where

$$U_0(z)=\frac{1}{z-1}, \tag{28}$$

and the relative velocity $V_k(z)$ is

$$V_k(z)=U_{k-1}(z)-U_k(z)$$
$$=\frac{1}{\mu}\left[\frac{\mu}{z-(1-\mu)}\right]^k$$
$$=\frac{\mu^{k-1}}{(k-1)!}\sum_{n=k}^{\infty}\frac{(n-1)!}{(n-k)!}(1-\mu)^{n-k}\frac{1}{z^n}. \tag{29}$$

Therefore the inverse z-tranform of $V_k(z)$ is found to be

$$U_{k-1}(n\tau)-U_k(n\tau)$$
$$=\begin{cases}\dfrac{(k-1)!}{(n-1)!(n-k)!}(1-\mu)^{n-k}\mu^{k-1} & n\geqq k\\ 0 & n\leqq k-1\end{cases} \tag{30}$$

and finally we get

$$u_k(n\tau)=\begin{cases}1-\sum_{m=1}^{k}\dfrac{(n-1)!}{(n-m)!(m-1)!}(1-\mu)^{n-m}\mu^{n-1}, & n\geqq k\\ 0 & n\leqq k-1\end{cases} \tag{31}$$

for $k=1, 2, \cdots$

If the sampling period τ is extremely short, i. e. $\mu\to 0$ and $n\to\infty$ with finite $n\mu=t/T$, the above expression goes over to

$$u_k(t)=1-\sum_{m=1}^{k}\frac{e^{-\frac{t}{T}}}{(m-1)!}\left(\frac{t}{T}\right)^{m-1} \qquad k=1, 2, \cdots \tag{31'}$$

which is just the same formula already given by Pipes[11].

The formula (31) tells us only the values of the output at discrete sampling instants. The behavior of $u_k(t)$ between sampling instants, however, is easily found to be linear in t, as the acceleration of a vehicle in the chain is constant between successive sampling instants. In Fig. 5, the curves of several u_ks for $\mu=\frac{1}{2}$ are plotted together with the corres-

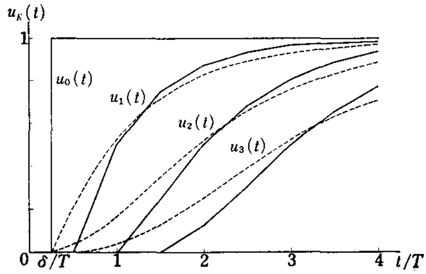

Fig. 5. $u_k(t)$ for several values of k. The solid lines represent $\mu=1/2$ case and the dotted lines are corresponding curves for $\mu=0$

ponding curves of the continuous-control-case investigated by Pipes.

For completion of our present study, it is necessary to analyze separation distances between cars, the criterion for the local stability and so on. Similar discussions with those in the previous sections can be also carried out as to the case of other rules for controlling cars, for instance, constant speacing and California Code control case[1]. Such investigations are now being prepared, and will appear in future publications.

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