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ON THE STABILITY OF TRAFFIC FLOW (REPORT-II)

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INTRODUCTION

In a previous paper (report-I), the movement of a queue of vehicles and the stability conditions under a disturbance have been investigated.[1] It was assumed that a spacing between successive vehicles was given only by the linear function of velocity of the following vehicle. It was shown theoretically that the indicial responses of a following were unstable for $n \leqq 2/\pi$, stable with cycling for $2/\pi < n < e$ or without cycling for $n \geqq e$, and, provided that $n > 2 \sin \omega T/\omega T$, the propagation of a sinusoidal disturbance was stable. However, the actual spacing between two successive vehicles in queue should depend on both the velocities of the previous vehicle and its following one. In this report, we will consider the stability problems from the standpoint mentioned above.

BASIC EQUATIONS

Let a queue of individual vehicle be traveling to the right as is shown in Fig. 1. Let $x_k(t)$ and $v_k(t)$ be the coordinate and the velocity

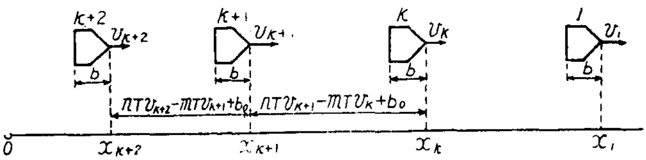

Fig. 1.

of the kth vehicle countered from the leading vehicle at a certain time t respectively, and T the reaction time of driver.

The velocity of the $(k+1)$th vehicle at the time t, $v_{k+1}(t)$ is determined as the result of reflection of both the clearance spacing $x_k(t-T)-x_{k+1}(t-T)$ and the velocity $v_k(t-T)$ just before the reaction time T of the driver, because there will be a delay of at least the reaction time to respond to the variation of the motion of the preceding vehicle.

As described in the report-I, the basic equation is

$$x_k(t-T)-x_{k+1}(t-T)=F[v_k(t-T),\ v_{k+1}(t)], \tag{1}$$

where F is a function which may be determined by taking the experience of drivers into account.

To simplify the analysis, let us assume F be a linear function of v_k and v_{k+1}, that is, the following relationship be assumed,

$$x_k(t-T)-x_{k+1}(t-T)=\alpha v_k(t-T)+\beta v_{k+1}(t)+b_0, \tag{2}$$

where α, β and b_0 are constants.

Differentiating both sides of Eq. (2) with respect to t, we have

$$v_k(t-T)-v_{k+1}(t-T)=\alpha\dot{v}_k(t-T)+\beta\dot{v}_{k+1}(t). \tag{3}$$

However, since it is evident that α is negative and β is positive in actual vehicular traffic, we may put

$$\alpha=-mT, \qquad \beta=nT, \tag{4}$$

where m and n are constants and T is the reaction time.

Hence the dynamic equation of traffic flow, which should be treated in this paper, becomes

$$v_k(t-T)-v_{k+1}(t-T)=-mT\dot{v}_k(t-T)+nT\dot{v}_{k+1}(t), \tag{5}$$

for $m=0$ this basic equation corresponds to that presented by R. E. Chandler, R. Herman and E. W. Montroll.[2)]

INDICIAL RESPONSES OF A FOLLOWING VEHICLE

Applying the Laplace transformation to Eq. (5), we have

$$V_k(s)e^{-Ts} - V_{k+1}(s)e^{-Ts}$$
$$= -mTsV_k(s)e^{-Ts} + nTsV_{k+1}(s) + mTe^{-Ts}v_k(0) - nTv_{k+1}(0)$$

or

$$V_{k+1}(s) = \frac{1+mTs}{nTs+e^{-Ts}} e^{-Ts} V_k(s)$$
$$- \frac{mTe^{-Ts}}{nTs+e^{-Ts}} v_k(0) + \frac{nT}{nTs+e^{-Ts}} v_{k+1}(0), \quad (6)$$

where

$$V_k(s) = \int_0^\infty v_k(t)e^{-ts}dt, \qquad V_{k+1}(s) = \int_0^\infty v_{k+1}(t)e^{-ts}dt,$$

and $v_k(0)$ and $v_{k+1}(0)$ denote the initial velocities of the vehicles at $t=0$.

The motions of the following vehicles are found by Eq. (6) if the movements of the preceding vehicles are known. When the leading vehicle moves with an acceleration or deceleration of the delta function type, we shall call the movement of the following vehicle the indicial response.

Now let us consider the case in which all the vehicles are standing still and the leading vehicle starts at a constant velocity v_0 with an impulsive acceleration.

Since the initial conditions are

$$v_1(t) = v_0 \quad \text{for } t \geqq 0 \quad \text{or} \quad V_1(s) = v_0/s$$
$$v_{k+1}(0) = 0, \qquad (k=1, 2, 3, \cdots)$$

we have by repeated applications of Ep. (6)

$$V_2(s)=\frac{e^{-Ts}}{nTs+e^{-Ts}}\cdot\frac{v_0}{s}.$$

$$V_3(s)=\left(\frac{e^{-Ts}}{nTs+e^{-Ts}}\right)^2(1+mTs)\cdot\frac{v_0}{s}.$$

....

$$V_{k+1}(s)=\left(\frac{e^{-Ts}}{nTs+e^{-Ts}}\right)^k(1+mTs)^{k-1}\cdot\frac{v_0}{s}. \tag{7}$$

To compute the inverse transforms of Eq. (7) for $k=1,\ 2,$ $V_2(s)$ and $V_3(s)$ are expanded in the power series as follows,

$$V_2(s)/v_0=\frac{e^{-Ts}}{nTs^2}-\frac{e^{-2Ts}}{n^2T^2s^3}+\frac{e^{-3Ts}}{n^3T^3s^4}-\frac{e^{-4Ts}}{n^4T^4s^5}+\frac{e^{-5Ts}}{n^5T^5s^6}-\cdots,$$

$$V_3(s)/v_0=\frac{e^{-2Ts}}{n^2T^2s^3}-\frac{2e^{-3Ts}}{n^3T^3s^4}+\frac{3e^{-4Ts}}{n^4T^4s^5}-\frac{4e^{-5Ts}}{n^5T^5s^6}+\cdots$$
$$+mT\left[\frac{e^{-2Ts}}{(nTs)^2}-\frac{2e^{-3Ts}}{(nTs)^3}+\frac{3e^{-4Ts}}{(nTs)^4}-\frac{4e^{-5Ts}}{(nTs)^5}+\cdots\right]. \tag{8}$$

Then the indicial responses required are given by

$$\frac{v_2(t)}{v_0}=\frac{1}{n}\left(\frac{t}{T}-1\right)-\frac{1}{2n^2}\left(\frac{t}{T}-2\right)^2+\frac{1}{6n^3}\left(\frac{t}{T}-3\right)^3$$
$$-\frac{1}{24n^4}\left(\frac{t}{T}-4\right)^4+\frac{1}{120n^5}\left(\frac{t}{T}-5\right)^5$$
$$-\frac{1}{720n^6}\left(\frac{t}{T}-6\right)^6+\frac{1}{5040n^7}\left(\frac{t}{T}-7\right)^7-\cdots, \tag{9}$$

$$\frac{v_3(t)}{v_0}=\frac{1}{2n^2}\left(\frac{t}{T}-2\right)^2-\frac{1}{3n^3}\left(\frac{t}{T}-3\right)^3+\frac{1}{8n^4}\left(\frac{t}{T}-4\right)^4$$
$$-\frac{1}{30n^5}\left(\frac{t}{T}-5\right)^5+\frac{1}{144n^6}\left(\frac{t}{T}-6\right)^6-\frac{1}{840n^7}\left(\frac{t}{T}-7\right)^7$$
$$+\cdots+m\left[\frac{1}{n^2}\left(\frac{t}{T}-2\right)-\frac{1}{n^3}\left(\frac{t}{T}-3\right)^2+\frac{1}{2n^4}\left(\frac{t}{T}-4\right)^3\right.$$

$$-\frac{1}{6n^5}\left(\frac{t}{T}-5\right)^4+\frac{1}{24n^6}\left(\frac{t}{T}-6\right)^5-\frac{1}{120n^7}\left(\frac{t}{T}-7\right)^6+\cdots\Big]. \tag{10}$$

As is known from Eq. (9), $v_2(t)$ is independent of m, accordingly Eq. (9) coincides with Eq. (10) of the report-I.

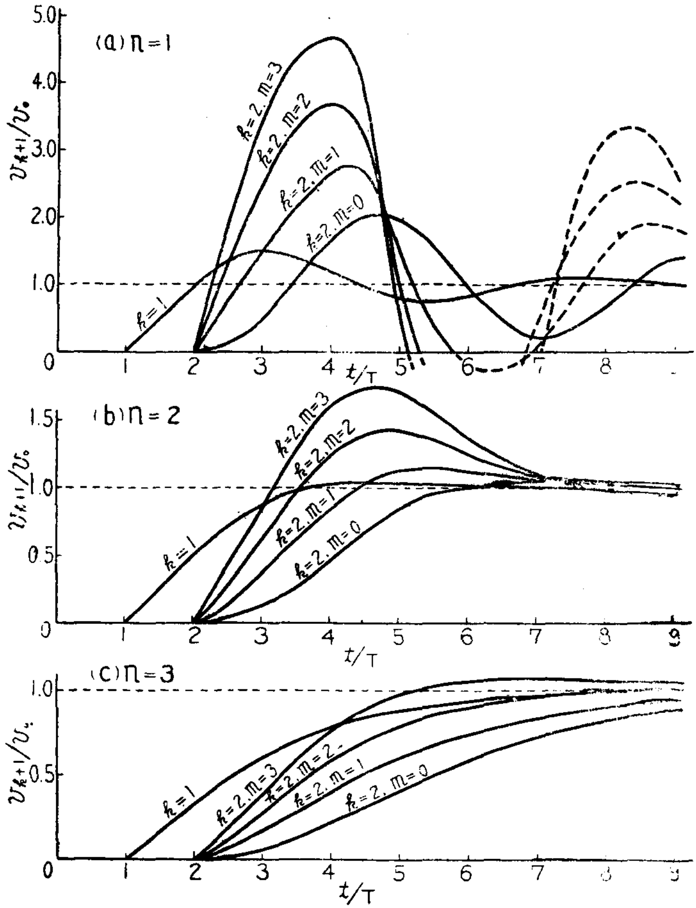

Fig. 2.

Fig. 2 shows the indicial responses calculated by the use of Eq. (9) and Eq. (10). As illustrated in Fig. 2, the amplitude of response increases as m becomes larger or n becomes smaller.

Now we proceed to consider the behaviour of a queue of traffic after a sudden stop of the top vehicle by the use of Eq. (6). Since all the vehicles in the queue of traffic are initially moving with a uniform velocity v_0, the initial conditions required are

$$v_1(t)=0 \quad \text{for} \quad t \geqq 0, \qquad v_{k+1}(0)=v_0, \qquad (k=1,\ 2,\ 3,\ \cdots).$$

Hence we obtain from Eq. (6)

$$V_2(s)=v_0F(s),$$

$$V_3(s)=V_2E(s)+v_0D(s)=v_0E(s)F(s)+v_0D(s)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\begin{aligned} V_{k+1}(s) &= V_kE(s)+v_0D(s)=V_{k-1}E^2(s)+v_0E(s)D(s)+v_0D(s) \\ &= \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &= v_0E^{k-1}(s)F(s)+v_0E^{k-2}(s)D(s)+v_0E^{k-3}(s)D(s) \\ &\quad +\cdots+v_0D(s), \end{aligned} \tag{11}$$

where

$$D(s)=(nT-mTe^{-Ts})/(nTs+e^{-Ts}),$$

$$E(s)=(1+mTs)e^{-Ts}/(nTs+e^{-Ts}),$$

$$F(s)=nT/(nTs+e^{-Ts}).$$

Replacing v in Eq. (7) by v_{start} and v in Eq. (11) by v_{stop} as described in the report-I, we have

$$\begin{aligned} V_{k+1,\ stop}/v_0 &= E^{k-1}(s)F(s)+D(s)[E^{k-2}(s)+E^{k-3}(s)+\cdots+E(s)+1] \\ &= E^{k-1}(s)[D(s)+mTG(s)]+D(s)[E^{k-2}(s) \end{aligned}$$

$$+E^{k-3}(s)+\cdots+E(s)+1]$$

$$=mTG(s)E^{k-1}(s)+D(s)[E^{k-1}(s)+E^{k-2}(s)$$

$$+E^{k-3}(s)+\cdots+E(s)+1]$$

$$=mTG(s)E^{k-1}(s)+\frac{1-E^{k}(s)}{1-E(s)}D(s), \tag{12}$$

where

$$G(s)=e^{-Ts}/(nTs+e^{-Ts}).$$

However, since

$$1-E(s)=1-G(s)-mTsG(s)$$

$$=s[F(s)-mTG(s)]$$

$$=sD(s),$$

Eq. (12) becomes

$$V_{k+1,\,stop}/v_0=mTG(s)E^{k-1}(s)+[1-E^{k}(s)]/s$$

$$=G(s)E^{k-1}(s)/s+mTG(s)E^{k-1}(s)+[1-E^{k}(s)]/s-V_{k+1,\,start}/v_0$$

$$=G(s)E^{k-1}(s)\cdot\frac{1+mTs}{s}+\frac{1}{s}-\frac{E^{k}(s)}{s}-\frac{V_{k+1,\,start}}{v_0}$$

$$=\frac{1}{s}-\frac{V_{k+1,\,start}}{v_0}. \tag{13}$$

Hence we obtain the following relation

$$v_{k+1,\,stop}=v_0-v_{k+1,\,start}. \tag{14}$$

It is noted that the relation of Eq. (14) is also sustained in this case as well as in the report-I.

Hence, even if a distance between two successive vehicles is a linear function of the velocities of each vehicle, the validity of Eq. (14) is also hold. Therefore, if we know either v_{start} or v_{stop}, we can calculate another unknown.

By the use of Eq. (14), the responses of the following vehicles are given as is shown in Fig. 3.

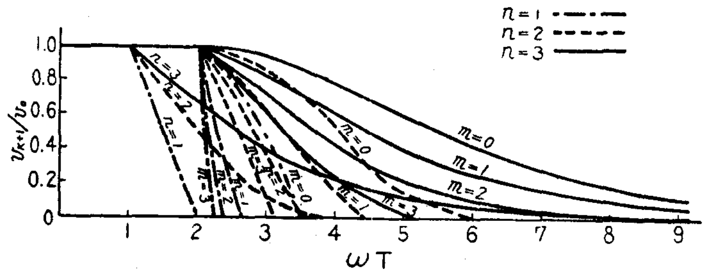

Fig. 3.

STABILITY OF INDICIAL RESPONSE

The analysis in this paragraph follows closely that used in the report-I.

The transfer function of Eq. (6) is

$$E(s)=\frac{(1+mTs)e^{-Ts}}{nTs+e^{-Ts}}. \tag{15}$$

However, since the characteristic equation of Eq. (15) is given by

$$nTs+e^{-Ts}=0, \tag{16}$$

which is just the same as that of $G(s)=e^{-Ts}/(nTs+e^{-Ts})$ in the report-I, so the indicial response of the $(k+1)$th vehicle is given as follows

$$v_{k+1,\,start}(t)=\frac{v_0}{2\pi j}\int_{c-j\infty}^{c+j\infty}\frac{e^{-kTs}e^{st}}{s(nTs+e^{-Ts})^k}(1+mTs)^{k-1}ds$$

$$= v_0 \sum \text{Residues of } [K(s) e^{st}], \tag{17}$$

where

$$K(s) = \frac{e^{-kTs}(1+mTs)^{k-1}}{s(nTs+e^{-Ts})^k}.$$

Hence, we have in general

$$\frac{v_{k-1,\,start}}{v_0} = 1 + \sum_{i=1}^{\infty} \Big\{ \frac{t^{k-1} e^{S_i t}}{(k-1)!} [(s-S_i) K(s)]_{s=S_i}$$

$$+ \frac{t^{k-2} e^{S_i t}}{(k-2)!} \Big[\frac{d}{ds} (s-S_i) K(s) \Big]_{s=S_i}$$

$$+ \frac{t^{k-3} e^{S_i t}}{2!(k-3)!} \Big[\frac{d^2}{ds^2} (s-S_i) K(s) \Big]_{s=S_i}$$

$$+ \cdots + \frac{e^{S_i t}}{(k-1)!} \Big[\frac{d^{k-1}}{ds^{k-1}} (s-S_i) K(s) \Big]_{s=S_i} \Big\}, \tag{18}$$

where

$$S_i = -\sigma_i \pm j\omega_i, \tag{19}$$

which are the roots of Eq. (16) and can be calculated dy

$$\sigma_i = \omega_i / \tan \omega_i T, \qquad \omega_i = \sqrt{e^{2\sigma_i T}/(nT)^2 - \sigma_i^2} \tag{20}$$

As is known from Eq. (18), the indicial response has the components of the form $t^{k-1} e^{S_i t}$. As time elapses, the amplitude of these components will be damped if $\sigma_i > 0$, but be increased if $\sigma_i < 0$. Therefore the critical condition of stability is given by $n = 2/\pi$ as mentioned in the previous report. On the other hand, since, if ω_i exists these components accompany cycling, then the critical condition of cycling is given by $n = e$. It is very important that the stability conditions are dependent merely on n and not of m.

STABILITY OF PROPAGATION OF A SINUSOIDAL DISTURBANCE

We will consider the character of a wave of the disturbance which would travel down the queue of vehicles caused by some disturbance of the top vehicle in a queue.

Now let us consider a simple sinusoidal disturbance

$$a = A \sin \omega t. \tag{21}$$

From Eq. (6) the transfer function of the $(k+1)$th vehicle is given as follows,

$$E_{k+1}(s) = [E(s)]^k = \left[\frac{e^{-Ts}(1+mTs)}{nTs+e^{-Ts}}\right]^k, \tag{22}$$

then the behaviour of the $(k+1)$th vehicle is represented by

$$z = A[|E(j\omega)|]^k \sin[\omega t + \arg E_{k+1}(j\omega)]. \tag{23}$$

Fig. 4 shows a block diagram of a queue of traffic.

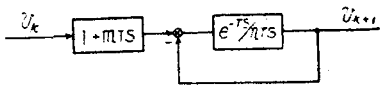

Fig. 4.

The initial sinusoidal oscillation is transmitted to the following vehicles, and is amplified provided that $|E(j\omega)|>1$ and is damped provided that $|E(j\omega)|<1$. Hence the stability condition of the propagation is

$$\left|\frac{e^{-j\omega t}(1+jm\omega T)}{jn\omega T+e^{-j\omega T}}\right| < 1. \tag{24}$$

However, since

$$|E(j\omega)| = \sqrt{\frac{1+m^2\omega^2T^2}{1+n^2\omega^2T^2-2n\omega T\sin\omega T}},$$

$$\arg E(j\omega)=\tan^{-1}\frac{m\omega T-n\omega T\cos\omega T-mn\omega^2T^2\sin\omega T}{1-n\omega T\sin\omega T+mn\omega^2T^2\cos\omega T},\tag{25}$$

the stability condition is given by

$$\frac{n^2-m^2}{n}>\frac{2\sin\omega T}{\omega T}.\tag{26}$$

If $m=0$, the stability condition becomes $n>2\sin\omega T/\omega T$ as described in the previous report.

Since all the values of $\sin\omega T/\omega T$ are less than or equal to 1, the stability criterion for all frequency of the disturbance is

$$n>1+\sqrt{1+m^2}\tag{27}$$

The stability criterion of Eq. (27) is shown in Fig. 5.

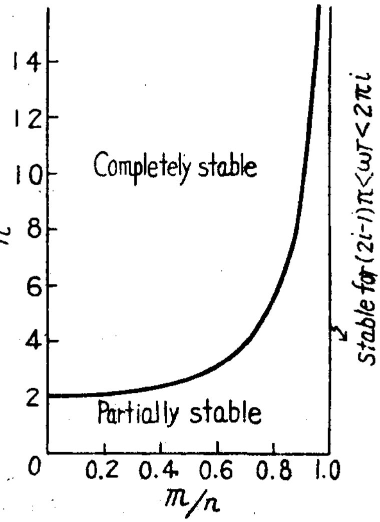

Fig. 5.

A case of $n=m$ is more interesting. In this case the stability condition is described by the following relation,

$$(2i-1)\pi<\omega T<2\pi i,$$

$$(i=1,\ 2,\ 3,\ \cdots)\tag{28}$$

which is independent of the values of n or m.

Fig. 6 shows the amplitudes of the frequency responses versus ωT. As illustrated in Fig. 6, the amplitude of response increases as m becomes larger or n becomes smaller. The amplitudes of frequency responses for $n=m$ is shown in Fig. 7.

As is known from Eq. (28) each response in Fig. 7 intersects at a point of $\omega T=\pi$, and the left region from this point is unstable.

The sufficient condition under which the propagation of a

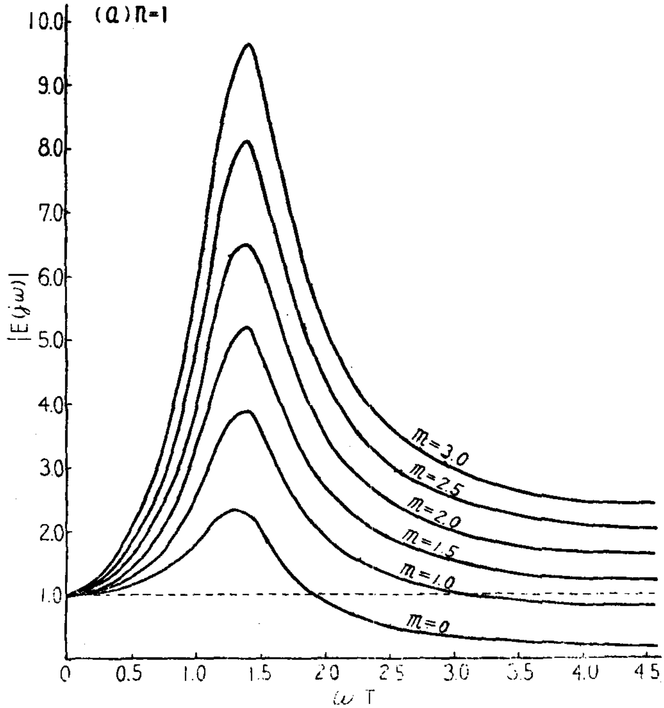
(a) n=1
10.0
9.0
8.0
7.0
6.0
5.0
4.0
3.0
2.0
1.0
|E(jω)|
m=3.0
m=2.5
m=2.0
m=1.5
m=1.0
m=0
0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
4.5
ωT

Fig. 6. (a)

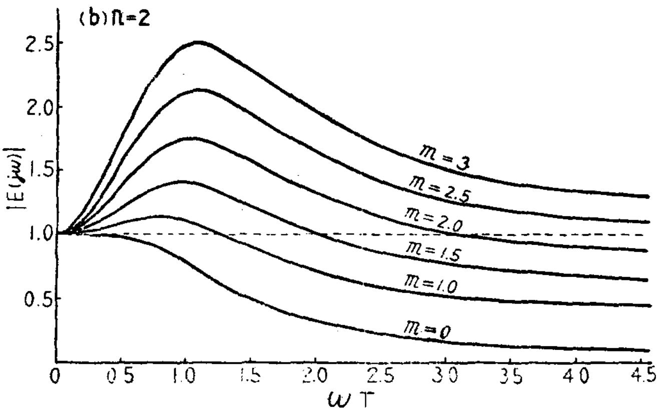
(b) n=2
2.5
2.0
1.5
1.0
0.5
|E(jω)|
m=3
m=2.5
m=2.0
m=1.5
m=1.0
m=0
0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
4.5
ωT

Fig. 6. (b)

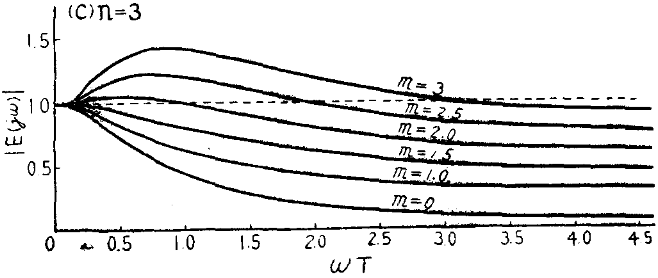

Fig. 6. (c)

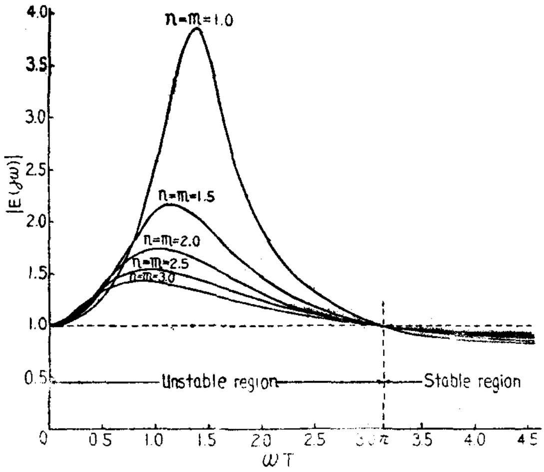

Fig. 7.

disturbance is stable for all ωT may be written briefly from Eq. (27)

$$n-m>2, \tag{29}$$

because of $1+m^2<(1+m)^2$.

Let us consider the actual traffic flow. A clearance spacing of a queue of vehicles traveling with a uniform velocity v_0 is represented by $(n-m)Tv_0+b_0-b$ as is shown in Fig. 1. On the main street in

Kumamoto City, we have observed the clearance spacings depending on velocity to determine the value of $n-m$. The result of this field observations shown in Fig. 8 suggests that the clearance spacings are not represented by the linear function of the velocities. However, this linearity holds for the range of velocities of 20~40 km per hour, accordingly we find $(n-m)T=1.10$, $b_0-b=2.48$ from Fig. 8. However, since the reaction time observed was 1.13 seconds, $n-m=0.973$ is obtained. This value of $n-m$ does not satisfy Eq. (27), so that there is no safety condition for all ωT in actual traffic.

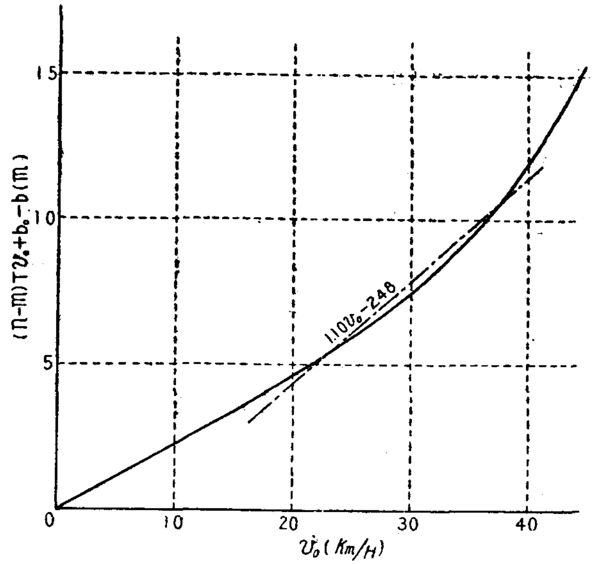

Fig. 8.

FLUCTUATION OF CLEARANCE SPACING

When all the vehicles in a queue are initially moving with a uniform velocity v_0, the movement of the top vehicle under a sinusoidal disturbance is described by

$$v_1(t) = v_0 - A\sin\omega t \quad \text{for} \quad t \geqq 0,$$

$$= v_0 \qquad \text{for} \quad t<0. \tag{30}$$

Then, after a sufficient time the response of the 2nd vehicle is given by

$$v_2(t) = v_0 - A|E(j\omega)|\sin[\omega t + \arg E(j\omega)]. \tag{31}$$

A clearance spacing between the top vehicle and its following vehicle is given by

$$\gamma(t) = x_1(0) - x_2(0) - b + \int_0^t v_1(t)\,dt - \int_0^t v_2(t)\,dt,$$

or

$$\gamma(t) = (n-m)Tv_0 + b_0 - b + \int_0^t v_1(t)\,dt - \int_0^t v_2(t)\,dt, \tag{32}$$

where b is a car length.

However, since Eq. (31) is the relation for the steady state, we can use Eq. (31) to evaluate Eq. (32).

If we place

$$\Gamma(t) = \int_0^t v_1(t)\,dt - \int_0^t v_2(t)\,dt, \tag{33}$$

we have

$$\Gamma(s) = [V_1(s) - V_2(s)]/s, \tag{34}$$

where

$$\Gamma(s) = \int_0^\infty \Gamma(t)e^{-st}dt.$$

However, since we have from Eq. (6)

$$V_2(s) = \frac{e^{-Ts}(1+mTs)}{nTs+e^{-Ts}}V_1(s) - \frac{mTe^{-Ts}}{nTs+e^{-Ts}}v_1(0) + \frac{nT}{nTs+e^{-Ts}}v_2(0),$$

we obtain

$$V_1(s)-V_2(s)=\frac{nT-mTe^{-Ts}}{nTs+e^{-Ts}}sV_1(s)$$

$$+\frac{mTe^{-Ts}}{nTs+e^{Ts}}v_1(0)-\frac{nT}{nTs+e^{-Ts}}v_2(0). \qquad (35)$$

Therefore Eq. (34) becomes

$$\Gamma(s)=\frac{nT-mTe^{-Ts}}{nTs+e^{-Ts}}V_1(s)+\frac{mTe^{-Ts}v_1(0)}{s(nTs+e^{-Ts})}-\frac{nTv_2(0)}{s(nTt+e^{-Ts})}. \qquad (36)$$

In this case the initial conditions required are

$$v_1(0)=v_2(0)=v_0, \qquad V_1(s)=\frac{v_0}{s}-\frac{A\omega}{s^2+\omega^2}. \qquad (37)$$

Hence we have from Eq. (36)

$$\Gamma(s)=-\frac{A\omega(nT-mTe^{-Ts})}{(s^2+\omega^2)(nTs+e^{-Ts})}\equiv-A\omega\frac{P(s)}{Q(s)}, \qquad (38)$$

or

$$\Gamma(t)=-\frac{A\omega}{2\pi j}\int_{-j\infty}^{+j\infty}\frac{nT-mTe^{-Ts}}{(s^2+\omega^2)(nTs+e^{-Ts})}\cdot e^{st}ds \qquad (39)$$

The characteristic roots of Eq. (38) are given by $s=S_l=-\sigma_l\pm j\omega_l$ and $s=\pm j\omega$.

Since the residues of $e^{st}P(s)/Q(s)$ at the simple poles $s=S_l$ are represented by

$$e^{S_lt}P(S_l)/Q'(S_l)=\frac{1+mTS_l}{(S_l^2+\omega^2)(1+TS_l)}\cdot e^{S_lt},$$

the component responses of the characteristic roots S_l appeared in Eq.

(39) are damped as time elapses provided that $\sigma_i > 0$.

On the other hand, the residue at the simple $s = j\omega$ is expressed by

$$Re^{j\omega t} = \frac{nT(-n\omega T - \sin\omega T + m\omega T\cos\omega T) + jT(m - mn\omega T\sin\omega T - n\cos\omega T)}{2\omega(1 + n^2\omega^2T^2 - 2n\omega T\sin\omega T)} \cdot e^{j\omega t} \tag{40}$$

and the residue at the simple pole $s = -j\omega$ is given by

$$R'e^{-j\omega t} = \frac{nT(-n\omega T + \sin\omega T + m\omega T\cos\omega T) - jT(m - mn\omega T\sin\omega T - n\cos\omega T)}{2\omega(1 + n^2\omega^2T^2 - 2n\omega T\sin\omega T)} \cdot e^{-j\omega t} \tag{41}$$

However, since

$$|R| = |R'|, \quad \arg R = -\arg R',$$

we can derive from Eq. (40) and (41)

$$\Gamma(t) = -2A\omega|R|\cos(\omega t + \arg R) \tag{42}$$

If we put

$$U = 2\omega R, \tag{43}$$

Eq. (42) becomes

$$\Gamma(t) = -A|U|\cos(\omega t + \arg U), \tag{44}$$

where

$$|U| = T\sqrt{\frac{n^2 + m^2 - 2mn\cos\omega T}{1 + n^2\omega^2T^2 - 2n\omega T\sin\omega T}},$$

$$\arg U = \tan^{-1} \frac{-m + mn\omega T \sin \omega T + n \cos \omega T}{n(n\omega T - \sin \omega T - m\omega T \cos \omega T)}. \tag{45}$$

The aspect of $|U|$ for the various values of m and n is illustrated in Fig. 9. Fig. 9 shows that the maximum amplitude of fluctuation increases as n becomes smaller or as the difference between n and m becomes greater.

From Eq. (45), we can find the following relation,

$$\lim_{\omega T \to 0} |U(\omega T)| = (n-m)T. \tag{46}$$

Hence we obtain by substituting Eq. (44) into Eq. (32)

$$\gamma(t) = (n-m)Tv_0 + b_0 - b - A|U| \cos(\omega t + \arg U). \tag{47}$$

Eq. (47) expresses the fluctuation of the clearance spacing under a sinusoidal disturbance.

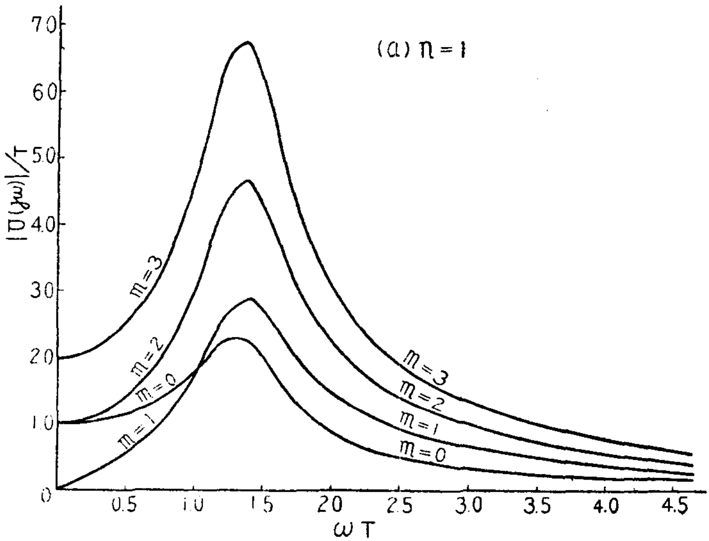

Fig. 9. (a)

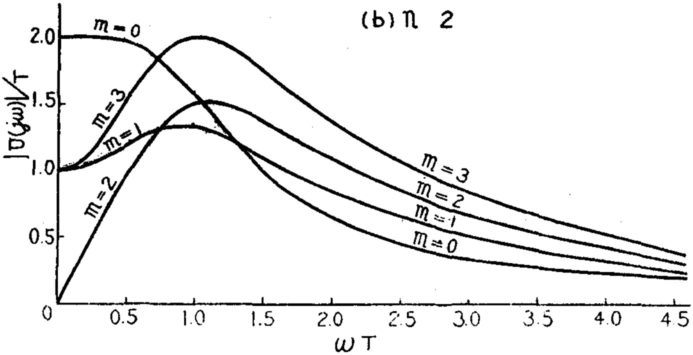

Fig. 9. (b)

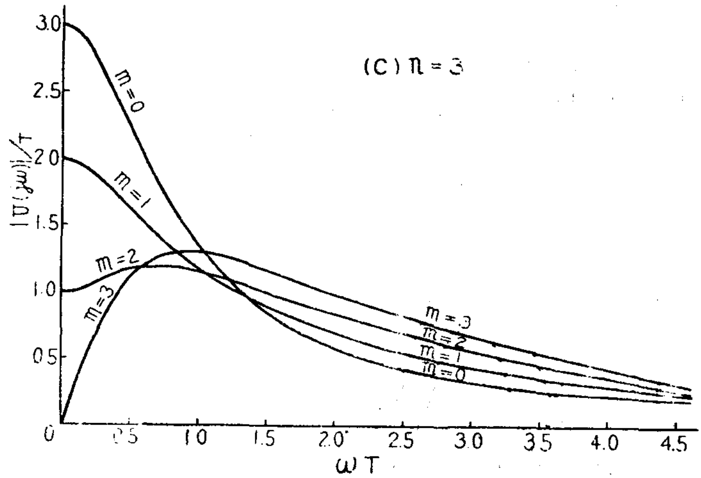

Fig. 9. (c)

By the way, we will consider about the safety of the spacing. If $\gamma(t)$ were negative, the following vehicle would collide with the previous vehicle. Therefore, the safety condition is given by

$$(n-m)Tv_0+b_0-b-A|U(j\omega)|>0 \tag{48}$$

Eq. (48) indicates that, as the value of $|U|$ increases, a queue of traffic will be led into dangerous state under the fixed values of m and n, even though the amplitude of a disturbance be kept constant at A.

As is shown in Fig. 9, the fluctuation of a spacing under a disturbance of high frequency are reduced enough to keep traffic safely. Therefore it is a disturbance of low frequency that leads traffic into dangerous state.

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