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NOTES ON AUCTION BIDDING

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1. INTRODUCTION

In Chapter 19 of [1] (pp. 560-563), there is an interesting discussion of auction bidding for the case of two bidders and two items. The reasoning, however, appears to be incomplete, in the sense that it does not cover the all possible contingencies. Here we will try to make it complete.

As in [1], we consider the simple situation where two items of known values, V_1 and V_2, are to be auctioned, one at a time. We assume that there are only two bidders, A and B, who have S_A and S_B dollars, respectively, available for bidding. We assume also that A knows the amount of money B has available and vice versa. Bidder A wishes to know at what point he should stop bidding for the first item.

Here it is to be noted that, contrary to [1], we do not make such assumptions as $S_A < V_1 + V_2$, $S_B < V_1 + V_2$, $1/2 < S_A/S_B < 2$.

To solve the problem, A must first decide what his objective should be in the bidding. In Sections 3 and 4, we will consider A's objective to be the maximization of the difference, $R_A - R_B$, where R_A is A's return and R_B is B's return. In later Sections, we will consider that A's objective is the maximization of his own expected return.

We first derive in the next Section, however, a set of formulas applicable in both cases.

For the sake of simplicity, we take the smallest increment of bid Δ to be negligibly small, written hereafter as $+0$.

2. RETURNS

Suppose B wins the first item with the bid X. B will then have S_B-X dollars remaining for the second item. There are three possibilities which can happen for the second item, depending upon which one of the three quantities S_A, S_B-X, and V_2 is the smallest.

Case 1: S_A is the smallest, i. e.,

$$S_A<S_B-X, \qquad S_A<V_2. \tag{2.1}$$

In this case, B can win also the second item for S_A+0, so that

$$\left.\begin{aligned} &R_A=0, \\ &R_B=(V_1-X)+(V_2-S_A). \end{aligned}\right\} \tag{2.2}$$

Case 2: S_B-X is the smallest, i. e.,

$$S_B-X<S_A, \qquad S_B-X<V_2. \tag{2.3}$$

Here, A can win the second item for S_B-X+0, whence

$$\left.\begin{aligned} &R_A=V_2-(S_B-X), \\ &R_B=V_1-X. \end{aligned}\right\} \tag{2.4}$$

Case 3: V_2 is the smallest, i. e.,

$$V_2<S_A, \qquad V_2<S_B-X. \tag{2.5}$$

In this case, both A and B can and will bid up to V_2 and certainly will not bid more than that. Whichever may get the second item (for V_2), his return is zero for this item. Hence

$$\left.\begin{aligned} R_A&=0, \\ R_B&=V_1-X. \end{aligned}\right\} \tag{2.6}$$

Cases 1 and 3 can be summarized as giving

$$\left.\begin{aligned} R_A&=0, \\ R_B&=V_1-X+\max\ (V_2-S_A,\ 0) \end{aligned}\right\} \tag{2.7}$$

for

$$X<S_B-\min\ (S_A,\ V_2). \tag{2.8}$$

Case 2 gives

$$\left.\begin{aligned} R_A&=V_2-S_B+X, \\ R_B&=V_1-X \end{aligned}\right\} \tag{2.9}$$

for

$$X>S_B-\min\ (S_A,\ V_2). \tag{2.10}$$

If A wins the first item for X dollars, then similar situations will occur with A and B reversed.

3. MAXIMIZING DIFFERENCE IN RETURN

Let us consider A's objective to be the maximization of the difference R_A-R_B. Let us further assume that B tries to minimize the difference R_A-R_B. If B wins the first item for X dollars, then, from (2.7) and (2.9), R_A-R_B will be given by $f_1(X)$, where

$$f_1(X) = -V_1 + X - \max(V_2 - S_A,\ 0), \quad \text{for } X < S_B - \min(S_A,\ V_2)\ ; \tag{3.1}$$

$$= -V_1 + V_2 - S_B + 2X, \quad \text{for } X > S_B - \min(S_A,\ V_2). \tag{3.2}$$

If, on the other hand, A wins the first item for X dollars, then $R_A - R_B$ will be given by $f_2(X)$, where

$$f_2(X) = V_1 - X + \max(V_2 - S_B,\ 0), \quad \text{for } X < S_A - \min(S_B,\ V_2)\ ; \tag{3.3}$$

$$= V_1 - V_2 + S_A - 2X, \quad \text{for } X > S_A - \min(S_B,\ V_2). \tag{3.4}$$

Now, we can easily verify that $f_1(X)$ is monotone increasing for $0 < X < \infty$ (there is a jump at the boundary point $X = S_B - S_A$ when $S_A < V_2$ and $S_A < S_B$), and that $f_2(X)$ is monotone decreasing for $0 < X < \infty$ (there is a jump at the boundary point $X = S_A - S_B$ when $S_B < V_2$ and $S_B < S_A$). Hence A will not let B win the first item for X dollars as long as $f_1(X)$ is smaller than $f_2(X)$. And A will not bid up to such an X for which $f_1(X)$ is larger than $f_2(X)$. Therefore the critical value X_0 of X where A will stop bidding for the first item is given by the condition

$$f_1(X_0) = f_2(X_0) \tag{3.5}$$

with an obvious exception which may occur at the above-mentioned point of discontinuity.

There are four cases which we will consider in turn.

Case A1 : $S_A < S_B$, $S_A < V_2$.

In this case, the graphs of $f_1(X)$ and $f_2(X)$ will look like Fig. 1. The ordinates of points P, Q, R are given, respectively, by,

$$P:\quad f_2(S_B - S_A) = V_1 - V_2 + 3S_A - 2S_B, \tag{3.6}$$

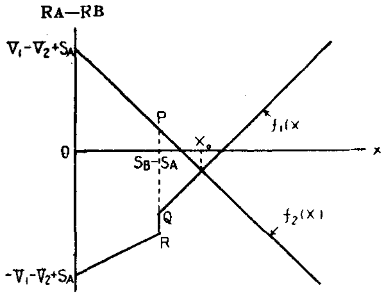

Fig. 1.
$R_A - R_B$ for Case AI.

$$Q: \quad f_1(S_B - S_A + 0) = -V_1 + V_2 - 2S_A + S_B, \tag{3.7}$$

$$R: \quad f_1(S_B - S_A - 0) = -V_1 - V_2 + S_B. \tag{3.8}$$

Point Q lies always above R. Point P lies either above Q, or between Q and R, or below R.

(i) If P lies above Q, i. e., if

$$2V_1 - 2V_2 + 5S_A - 3S_B > 0, \tag{3.9}$$

then the critical value X_0 will be given by the condition

$$-V_1 + V_2 - S_B + 2X_0 = V_1 - V_2 + S_A - 2X_0, \tag{3.10}$$

whence

$$X_0 = \frac{1}{4}(2V_1 - 2V_2 + S_A + S_B). \tag{3.11}$$

(ii) If P lies between Q and R, i. e., if

$$\left.\begin{aligned} &2V_1 - 2V_2 + 5S_A - 3S_B < 0, \\ &2V_1 + 3S_A - 3S_B > 0, \end{aligned}\right\} \tag{3.12}$$

then X_0 will be given by

$$X_0 = S_B - S_A. \tag{3.13}$$

(iii) If P lies below R, i. e., if

$$2V_1 + 3S_A - 3S_B < 0, \tag{3.14}$$

then X_0 will be given by the condition

$$-V_1-V_2+S_A+X_0=V_1-V_2+S_A-2X_0, \tag{3.15}$$

whence

$$X_0=\frac{2}{3}V_1. \tag{3.16}$$

In either case, if X_0 as given by the appropriate formula exceeds S_A, then X_0 should be taken as S_A.

These results can be summarized in a graph as in Fig. 2.

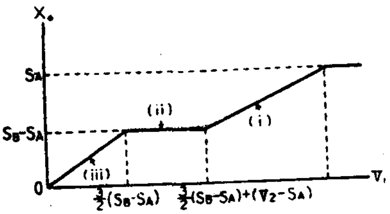

Fig. 2.
Results in Case A1.

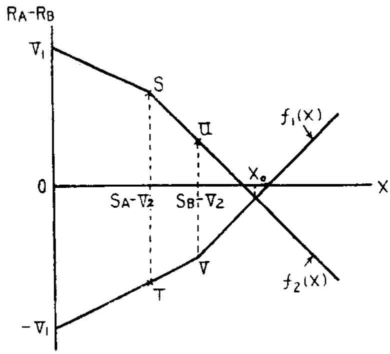

Fig. 3.
Case A2.

Case A2 : $S_A<S_B$, $V_2<S_A$.

In this case, the graphs of $f_1(X)$ and $f_2(X)$ will look like Fig. 3. The ordinates of points S, T, U, V are given, respectively, by

$$S:\quad f_2(S_A-V_2)=V_1+V_2-S_A, \tag{3.17}$$

$$T:\quad f_1(S_A-V_2)=-V_1-V_2+S_A, \tag{3.18}$$

$$U:\quad f_2(S_B-V_2)=V_1+V_2+S_A-2S_B, \tag{3.19}$$

$$V:\quad f_1(S_B-V_2)=-V_1-V_2+S_B. \tag{3.20}$$

(i) If U lies above V, i. e., if

$$2V_1+2V_2+S_A-3S_B>0, \tag{3.21}$$

then the critical value X_0 will be given by the condition

$$-V_1+V_2-S_B+2X_0=V_1-V_2+S_A-2X_0, \tag{3.22}$$

whence

$$X_0=\frac{1}{4}(2V_1-2V_2+S_A+S_B). \tag{3.23}$$

(ii) If U lies below V, and if S lies above T, i. e., if

$$\left.\begin{aligned} 2V_1+2V_2+S_A-3S_B<0, \\ V_1+V_2-S_A>0, \end{aligned}\right\} \tag{3.24}$$

then X_0 will be given by the condition

$$-V_1+X_0=V_1-V_2+S_A-2X_0, \tag{3.25}$$

whence

$$X_0=\frac{1}{3}(2V_1-V_2+S_A). \tag{3.26}$$

(iii) If S lies below T, i. e., if

$$V_1+V_2-S_A<0, \tag{3.27}$$

then X_0 will be given by the condition

$$-V_1+X_0=V_1-X_0, \tag{3.28}$$

whence

$$X_0=V_1. \tag{3.29}$$

In either case, if X_0 as given by the appropriate formula exceeds

S_A, then X_0 should be taken as S_A.

These results can be summarized in a graph as in Fig. 4.

Case B1 : $S_B < S_A$, $S_B < V_2$.

This case is similar to Case A1 and results can be easily obtained by exchanging S_A with S_B.

Case B2 : $S_B < S_A$, $V_2 < S_B$.

This case is similar to Case A2 and results can be easily obtained by exchanging S_A with S_B.

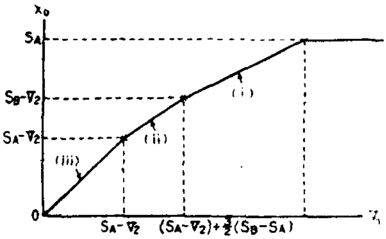

Fig. 4.
Results in Case A2.

4. EXAMPLES

Let us illustrate the results by a few examples.

Example I. Let $S_A=\$100$, $S_B=\$110$, $V_1=\$75$, $V_2=\$100$. This is the border-line case between Cases A1 and A2. In either case, since $V_1>(3/2)(S_B-S_A)$, (i) holds, and X_0 is given by (3.11) or (3.23) as

$$X_0=\frac{1}{4}[2(\$75-\$100)+(\$100+\$110)]=\$40.$$

Therefore A should not bid more than \$40 for item 1.

Example II. Let $S_A=\$100$, $S_B=\$110$, $V_1=\$12$, $V_2=\$105$. This belongs to Case A1. Since $V_1<(3/2)(S_B-S_A)$, (iii) holds, and X_0 is given by (3.16) as

$$X_0=\frac{2}{3}(\$12)=\$8.$$

Note that, if one used equation (3.11), one would get $X_0=(1/4)[2(\$12-\$105)+(\$100+\$110)]=\$6$. However, if A stops bidding at \$6, then B can win item 1 for \$6 and, as B has \$104 remaining, B can win also item 2 for \$100. Hence $R_A=0$ and $R_B=(\$12-\$6)+(\$105-\$100)=\$11$. On the other hand, if A bids up to \$8 and lets

B win item 1 for \$8, then B will have \$102 and will be able to win also item 2 for \$100. Hence $R_A=0$ and $R_B=(\$12-\$8)+(\$105-\$100)=\$9$. If A wins item 1 for \$8, then B will be able to win item 2 for only \$92 (A has only \$92 remaining). Hence $R_A=\$12-\$8=\$4$ and $R_B=\$105-\$92=\$13$. Thus $R_A-R_B=-\$9$ for the latter two cases, in contrast with $R_A-R_B=-\$11$ for the former case.

Example III. Let $S_A=\$100$, $S_B=\$110$, $V_1=\$20$, $V_2=\$30$. Case A2 (iii) holds. X_0 is given by (3.29) as $X_0=\$20$. Neither A nor B would stop below V_1 for item 1. If they bid up to V_1, the return will be virtually zero, whichever gets item 1. The same thing will happen for item 2. It follows that $R_A=R_B=0$.

5. MAXIMIZING EXPECTED RETURN

Let us now consider the case where A has as his objective the maximization of his own expected return R_A. Let us further assume that B tries to minimize R_A (this may not be a realistic assumption). If A lets B win the first item for X dollars, then R_A will be given by $g_1(X)$, where

$$g_1(X)=0, \quad \text{if } X<S_B-\min(S_A,\ V_2), \tag{5.1}$$

$$=V_2-S_B+X, \quad \text{if } X>S_B-\min(S_A,\ V_2). \tag{5.2}$$

If, on the other hand, A wins item 1 for X dollars, then R_A will be given by $g_2(X)$, where

$$g_2(X)=V_1-X+\max(V_2-S_B,\ 0), \quad \text{if } X<S_A-\min(S_B,\ V_2), \tag{5.3}$$

$$=V_1-X, \quad \text{if } X>S_A-\min(S_B,\ V_2). \tag{5.4}$$

As in Section 3, we will consider four cases in turn.

Case A1: $S_A<S_B$, $S_A<V_2$.

In this case, the graphs of $g_1(X)$ and $g_2(X)$ will look like Fig. 5.

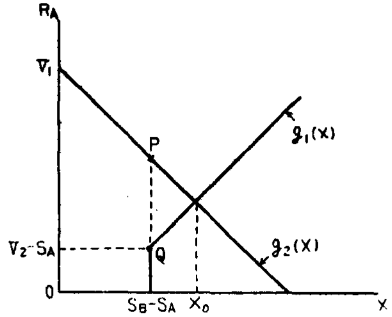

Fig.5.
Case A1.

The ordinates of points P, Q are given, respectively, by

$$P: \quad g_2(S_B-S_A)=V_1+S_A-S_B, \quad (5.5)$$

$$Q: \quad g_1(S_B-S_A+0)=V_2-S_A. \quad (5.6)$$

(i) If P lies above Q, i. e., if

$$V_1-V_2+2S_A-S_B>0, \tag{5.7}$$

then the critical value X_0 will be given by the condition

$$V_2-S_B+X_0=V_1-X_0, \tag{5.8}$$

whence

$$X_0=\frac{1}{2}(V_1-V_2+S_B). \tag{5.9}$$

(ii) If P lies below Q and above the horizontal axis, i.e., if

$$V_1-V_2+2S_A-S_B<0, \quad V_1+S_A-S_B>0, \tag{5.10}$$

then X_0 will be given by

$$X_0=S_B-S_A. \tag{5.11}$$

(iii) If P lies below the horisontal axis, i. e., if

$$V_1+S_A-S_B<0, \tag{5.12}$$

then X_0 will be given by the condition

$$V_1-X_0=0, \tag{5.13}$$

whence

$$X_0=V_1. \tag{5.14}$$

In either case, if X_0 as given by the appropriate formula exceeds S_A, then X_0 should be taken as S_A.

These results can be summarized in a graph asin Fig. 6.

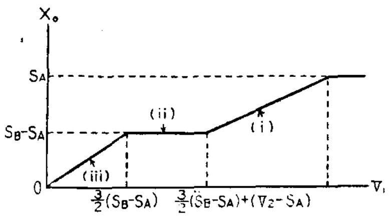

Fig. 6.
Results in Case A1.

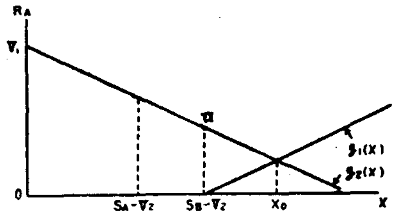

Fig. 7.
Case A2.

Case A 2: $S_A<S_B$, $V_2<S_A$.
In this case, the graphs of $g_1(X)$ and $g_2(X)$ will look like Fig. 7.
The ordinate of point U is given by

$$g_2(S_B-V_2)=V_1+V_2-S_B. \tag{5.15}$$

(i) If U lies above the horizontal axis, i. e., if

$$V_1+V_2-S_B>0, \tag{5.16}$$

then X_0 will be given by the condition

$$V_2-S_B+X_0=V_1-X_0, \tag{5.17}$$

whence

$$X_0=\frac{1}{2}\ (V_1-V_2+S_B). \tag{5.18}$$

(ii) If U lies below the horizontal axis, i. e., if

$$V_1+V_2-S_B<0, \tag{5.19}$$

then X_0 will be given by the condition

$$V_1-X_0=0, \tag{5.20}$$

whence

$$X_0=V_1. \tag{5.21}$$

In either case, if X_0 as given by the appropriate formula exceeds S_A, then X_0 should be taken as S_A.

These results can be summarized in a graph as in Fig. 8.

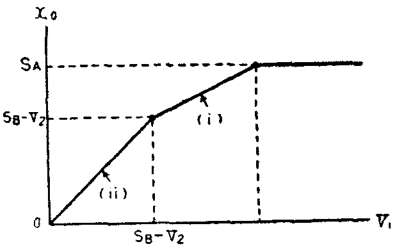

Fig. 8.
Results in Case A2.

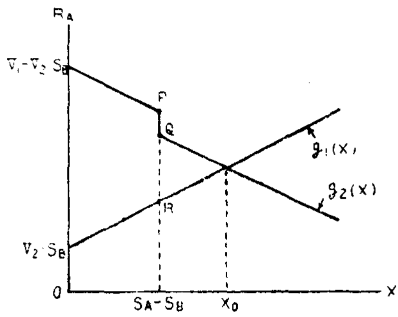

Fig. 9.
Case B1.

Case B1: $S_B<S_A,\ \ S_B<V_2.$

In this case, the graphs of $g_1(X)$ and $g_2(X)$ will look like Fig. 9. The ordinates of points P, Q, R are given, respectively, by

$$P: \quad g_2(S_A - S_B - 0) = V_1 + V_2 - S_A, \tag{5.22}$$

$$Q: \quad g_2(S_A - S_B + 0) = V_1 - S_A + S_B, \tag{5.23}$$

$$R: \quad g_1(S_A - S_B) = V_2 + S_A - 2S_B. \tag{5.24}$$

(i) If R lies below Q, i. e., if

$$V_1 - V_2 - 2S_A + 3S_B > 0, \tag{5.25}$$

then the critical value X_0 will be given by the condition

$$V_2 - S_B + X_0 = V_1 - X_0, \tag{5.26}$$

whence

$$X_0 = \frac{1}{2}(V_1 - V_2 + S_B). \tag{5.27}$$

(ii) If R lies between P and Q, i. e., if

$$V_1 - V_2 - 2S_A - 3S_B < 0, \quad V_1 - 2S_A + 2S_B > 0, \tag{5.28}$$

then X_0 will be given by

$$X_0 = S_A - S_B. \tag{5.29}$$

(iii) If R lies above P, i. e., if

$$V_1 - 2S_A + 2S_B < 0, \tag{5.30}$$

then X_0 will be given by the condition

$$V_2 - S_B + X_0 = V_1 + V_2 - S_B - X_0, \tag{5.31}$$

whence

$$X_0=\frac{1}{2}V. \tag{5.32}$$

In either case, if X_0 as given by the appropriate formula exceeds S_A, then X_0 should be taken as S_A.

These results can be summarized in a graph as in Fig. 10.

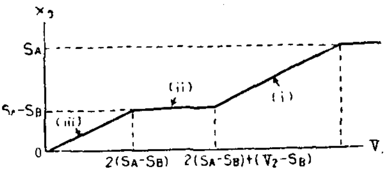

Fig. 10.
Results for Case B1.

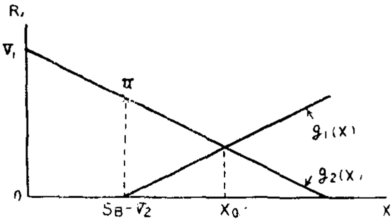

Fig. 11.
R_A for Case B2.

Case B 2 : $S_B<S_A$, $V_2<S_B$.

In this case, the graphs of $g_1(X)$ and $g_2(X)$ will look like Fig. 11. The ordinate of point U is given by

$$g_2(S_B-V_2)=V_1+V_2-S_B. \tag{5.33}$$

(i) If U lies above the horizontal axis, i. e., if

$$V_1+V_2-S_B>0, \tag{5.34}$$

then X_0 will be given by the condition

$$V_2-S_B+X_0=V_1-X_0, \tag{5.35}$$

whence

$$X_0=\frac{1}{2}(V_1-V_2+S_B). \tag{5.36}$$

(ii) If U lies below the horizontal axis, i. e., if

$$V_1+V_2-S_B<0, \tag{5.37}$$

then X_0 will be given by the condition

$$V_1-X_0=0, \tag{5.38}$$

whence

$$X_0=V_1. \tag{5.39}$$

In either case, if X_0 as given by the appropriate formula exceeds S_A, then X_0 should be taken as S_A.

These results can be summarized in a graph as in Fig. 12.

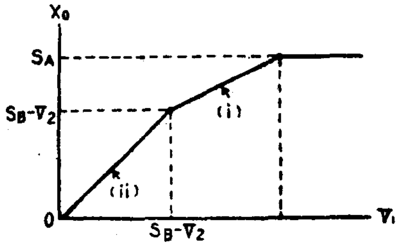

Fig. 12.
Results for Case B2.

6. EXAMPLES

Let us illustrate the use of the results obtained in Section 5 by a few examples.

Example I. Let $S_A=\$100$, $S_B=\$110$, $V_1=\$75$, $V_2=\$100$. This is the border-line case between Case A1 and A2. In either case, (i) holds, and X_0 is given by (5.9) as

$$X_0=\frac{1}{2}(\$75-\$100+\$110)=\$42.50.$$

Thus, A should not bid more than \$42.50 for item 1.

Remark 1. Compare this example with Example I of Section 4, where the critical value was \$40.

Remark 2. If B has as his objective the maximization of his own expected return R_B, and if he assumes that A will try to minimize R_B, then he will stop bidding at

$$X_0=\frac{1}{2}(V_1-V_2+S_A)$$

$$= \frac{1}{2}(\$75 - \$100 + \$100) = \$37.50.$$

A will then be able to win item 1 for \$37.50 and his return will be

$$R_A = \$75 - \$37.50 = \$37.50.$$

This is greater than the return he would get if the bidding went up to \$42.50 as given above. If A wants to maximize R_A and if B wants to maximize R_B, then the problem is one of non-zero-sum two-person game.

Example II. Let $S_A = \$100$, $S_B = \$110$, $V_1 = \$12$, $V_2 = \$105$. Case A1 (ii) holds, and X_0 is given by (5.11) as

$$X_0 = \$110 - \$100 = \$10.$$

A should stop bidding at \$10.

Remark 1. If B wins item 1 for \$11, say, then A will be able to win item 2 for \$100 and his return will be $R_A = \$105 - \$100 = \$5$. On the other hand, if B lets A win item 1 for \$10, then A's return will be $R_A = \$12 - \$10 = \$2$. Thus, he is assured of the return of at least \$2.

Remark 2. If B has as his objective the maximization of his own return R_B, and if he assumes that A will try to minimize R_B, then Case B1 (iii) holds (with A and B reversed). So B will stop bidding at

$$X_0 = \frac{1}{2}(\$12) = \$6.$$

A will then be able to win item 1 for little greater than \$6, thus getting the return of $R_A = \$12 - \$6 = \$6$. This is greater than the return he would get if the bidding went up to \$10. If A wants to maximize R_A, and if B wants to maximize R_B, then the problem is one of non-zero-sum two-person game.

Example III. Let $S_A = \$100$, $S_B = \$110$, $V_1 = \$20$, $V_2 = \$30$.

Case A2 (ii) holds. X_0 will be given by

$$X_0 = V_1 = \$20.$$

A should not stop bidding for item 1 below \$20. B would not either. Hence they will bid up to \$20 and the same thing will happen for item 2. The return will then be $R_A=0$.

REFERENCE

[1] C. W. CHURCHMAN, R. L. ACKOFF, and E. L. ARNOFF, *Introduction to Operations Research*, John Wiley and Sons, 1957.