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ON THE STABILITY OF TRAFFIC FLOW (REPORT-I)

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INTRODUCTION

According to the tremendous increase of traffic on the roads in our country, traffic congestions and accidents have been rapidly promoted. As the first step of the theoretical studies to prevent accidents caused by improper following, we will consider the stability of vehicular traffic flow in this paper.

The theory of traffic flow may be classified in three types, the first is the stochastic approaches to the characterization of random flow, the second is the hydrodynamic expressions performed by P. I. Richards,[1] and the third is the traffic dynamics by L. A. Pipes.[2]

After investigating the dynamic equation of Pipes, we have introduced in the equation the transfer lag due to reaction time only. In the following discussions a fundamental equation to traffic dynamics will be drived in which the transfer delay of reaction time is being considered. Thus the stability of traffic flow will be given by the cycling phenomena not mentioned in Pipes' traffic dynamics and by the propagation of a sinusoidal disturbance.

FUNDAMNTAL EQUATIONS OF TRAFFIC DYNAMICS

A model of traffic flow which several vehicles are traveling in queue is shown in Fig. 1. Each driver will drive his vehicle with a spacing that will avoid colliding his car ahead. But there is no absolute proof that his car never collides if it keeps that spacing. The final factor of safety depends on the judgement of the driver attained through his experience. In fact, an improper following due to his misjudgement often leads to accidents.

As shown in Fig. 1 consider the queue of individual vehicle

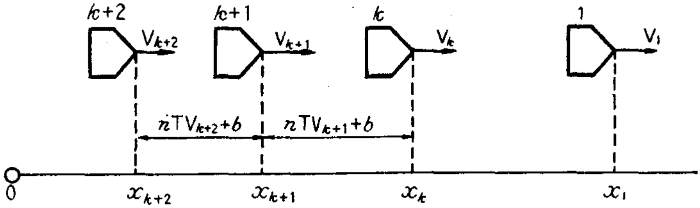

Fig. 1.

traveling to the right. Let $x_k(t)$ and $v_k(t)$ be the coordinate and the velocity of the k th vehicle from the leading vehicle at a certain time t respectively, and T, the reaction time of driver. The velocity of the $(k+1)$th vehicle at the time t, $v_{k+1}(t)$ is attained as the result of reflection of the clearance spacing $x_k(t-T)-x_{k+1}(t-T)$ just before the reaction time T of the driver. It is because there will be a delay of at least the reaction time to response to the variation of the motion of the preceding vehicle. Since the spacing between two successive vehicles in queue is expected to depend on the velocities of those vehicles, the following relation is given,

$$x_k(t-T)-x_{k+1}(t-T)=F[v_k(t-T),\ v_{k+1}(t)], \tag{1}$$

where F is a function fixed by the experience of drivers.

To simplify the analysis F is assumed as a linear function of v_k and v_{k+1}, then

$$F[v_k(t-T),\ v_{k+1}(t)]=\alpha v_k(t-T)+\beta v_{k+1}(t)+b, \tag{2}$$

where α, β, and b are constants. In this paper (report I), we consider the simplest case that F is the function of v_{k+1}, namely

$$F[v_k(t-T),\ v_{k+1}(t)]=\beta v_{k+1}(t)+b, \tag{3}$$

and the case of Eq. (2) will be discussed later in the paper (report II).

The dynamic equation of traffic flow treated in this report becomes

$$x_k(t-T)-x_{k+1}(t-T)=\beta v_{k+1}(t)+b, \quad (k=1,\ 2,\ 3,\ \cdots), \tag{4}$$

or differentiating both sides with respect to t, it becomes

$$v_k(t-T)-v_{k+1}(t-T)=\beta v_{k+1}(t), \quad (k=1,\ 2,\ 3,\ \cdots). \qquad (5)$$

MOVEMENT OF A QUEUE OF VEHICLES

Applying the Laplace transformation to Eq. (5) and putting $\beta=nT$, we have

$$V_k(s)e^{-Ts}-V_{k+1}(s)e^{-Ts}=nTsV_{k+1}(s)-nTv_{k+1}(0),$$

or

$$V_{k+1}(s)=\frac{e^{-Ts}}{nTs+e^{-Ts}}V_k(s)+\frac{nTv_{k+1}(0)}{nTs+e^{-Ts}}, \quad (k=1,\ 2,\ 3,\ \cdots) \qquad (6)$$

where

$$V_k(s)=\int_0^\infty v_k(t)e^{-ts}dt, \qquad V_{k+1}(s)=\int_0^\infty v_{k+1}(t)e^{-ts}dt,$$

and $v_{k+1}(0)$ denotes the initial velocities of the vehicles at $t=0$. Eq. (6) gives the motions of the following vehicles if the movements of the preceding vehicles are known.

Suppose now all the vehicles are standing still and the leading vehicle starts at a constant velocity v_0 with an impulsive acceleration of the delta function type.

Now let us consider the movements of all the following vehicles. The motion of the following vehicle under such an input of the step function type may be called indicial response as generally termed in the field of automatic controls.

In this case, since the initial conditions of Eq. (6) are

$$v_1(t)=v_0 \text{ for } t\geqq 0, \quad v_{k+1}(0)=0, \qquad (k=1,\ 2,\ 3,\ \cdots)$$

so

$$V_1(s)=v_0/s, \quad V_{k+1}(s)=0, \qquad (k=1,\ 2,\ 3,\ \cdots).$$

The indicial responses of the $(k+1)$th vehicle are obtained by repeated application of Eq. (6) as follows.

$$V_{k+1}(s)=\left(\frac{e^{-Ts}}{nTs+e^{-Ts}}\right)^k\frac{v_0}{s},\qquad (k=1,\ 2,\ 3,\ \cdots) \tag{7}$$

To compute the inverse transforms of Eq. (7), expanding $V_{k+1}(s)$ in the power series,

$$\begin{aligned}V_{k+1}(s)&=\frac{v_0e^{-kTs}/(nT)^ks^{k+1}}{(1+e^{-Ts}/nTs)^k}\\&=\frac{v_0e^{-kTs}}{(nT)^ks^{k+1}}-\frac{v_0ke^{-(k+1)Ts}}{(nT)^{k+1}s^{k+2}}+\frac{v_0k(k+1)e^{-(k+2)Ts}}{2(nT)^{k+2}s^{k+3}}\\&\quad-\frac{v_0k(k+1)(k+2)e^{-(k+3)Ts}}{3!(nT)^{k+3}s^{k+4}}+\cdots .\end{aligned} \tag{8}$$

Therefore the indicial responses required are given by

$$\begin{aligned}\frac{v_{k+1}(t)}{v_0}=\frac{1}{n^k(k-1)!}\Big[&\left(\frac{t}{T}-k\right)^k\Big/k-\left(\frac{t}{T}-k-1\right)^{k+1}\Big/n(k+1)\\&+\left(\frac{t}{T}-k-2\right)^{k+2}\Big/2n^2(k+2)-\left(\frac{t}{T}-k-3\right)^{k+3}\Big/3!\,n^3(k+3)\\&+\left(\frac{t}{T}-k-4\right)^{k+4}\Big/4!\,n^4(k+4)-\cdots\Big],\end{aligned} \tag{9}$$

where $t-kT$ or $t/T-k$ is zero for $t\leqq kT$ by the translation theorem of the Laplace transformation.

Hence the indicial responses for $k=1,\ 2$ are the following

$$\begin{aligned}\frac{v_2(t)}{v_0}&=\frac{1}{n}\left(\frac{t}{T}-1\right)-\frac{1}{2n^2}\left(\frac{t}{T}-2\right)^2+\frac{1}{6n^3}\left(\frac{t}{T}-3\right)^3\\&\quad-\frac{1}{24n^4}\left(\frac{t}{T}+4\right)^4+\frac{1}{120n^5}\left(\frac{t}{T}-5\right)^5\\&\quad-\frac{1}{720n^6}\left(\frac{t}{T}-6\right)^6+\frac{1}{5040n^7}\left(\frac{t}{T}-7\right)^7-\cdots,\\\frac{v_3(t)}{v_0}&=\frac{1}{2n^2}\left(\frac{t}{T}-2\right)^2-\frac{1}{3n^3}\left(\frac{t}{T}-3\right)^3+\frac{1}{8n^4}\left(\frac{t}{T}-4\right)^4\end{aligned} \tag{10}$$

$$-\frac{1}{30n^5}\left(\frac{t}{T}-5\right)^5+\frac{1}{144n^6}\left(\frac{t}{T}-6\right)^6$$

$$-\frac{1}{840n^7}\left(\frac{t}{T}-7\right)^7+\cdots.$$

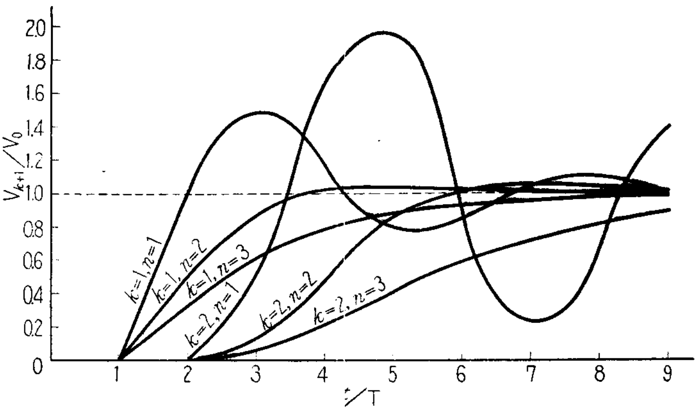

Fig. 2.

Fig. 2 shows the indicial responses expressed by Eq. (10). As shown in Fig. 2 the movements (responses) of the $(k+1)$th vehicle will require the dead time of kT before actual starting, and will accompany oscillations if n is small.

These oscillations are called cycling in this paper. As time elapses this cycling will generally weaken and finally fade and v_{k+1} approaches its final value v_0.

Now using Eq. (6) consider the movement of a queue of traffic after a sudden stop of the top vehicle. Since all the vehicles in the queue of traffic are initially moving with a uniform velocity v_0, the initial conditions required are

$$v_1(t)=0 \quad \text{for} \quad t\geqq 0, \quad v_{k+1}(0)=v_0, \qquad (k=1,\ 2,\ 3,\ \cdots).$$

Hence we have from Eq. (6)

$$V_2(s)=\frac{nTv_0}{nTs+e^{-Ts}},$$

$$V_3(s)=\frac{nTv_0e^{-Ts}}{(nTs+e^{-Ts})^2}+\frac{nTv_0}{nTs+e^{-Ts}}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \tag{11}$$

$$V_{k+1}(s)=\frac{nTv_0e^{-(k-1)Ts}}{(nTs+e^{-Ts})^k}+\frac{nTv_0\,e^{-(k-2)Ts}}{(nTs+e^{-Ts})^{k-1}}+\cdots+\frac{nTv_0}{nTs+e^{-Ts}}$$

and $V_1(s)$ vanishes. Then we can write the inverse transformations of Ep. (11) for $k=1$, 2 in the following forms

$$\begin{aligned}\frac{v_2(t)}{v_0}=&1-\frac{1}{n}\left(\frac{t}{T}-1\right)+\frac{1}{2n^2}\left(\frac{t}{T}-2\right)^2-\frac{1}{6n^3}\left(\frac{t}{T}-3\right)^3\\&+\frac{1}{24n^4}\left(\frac{t}{T}-4\right)^4-\frac{1}{120n^5}\left(\frac{t}{T}-5\right)^5\\&+\frac{1}{720n^6}\left(\frac{t}{T}-6\right)^6-\frac{1}{5040n^7}\left(\frac{t}{T}-7\right)^7+\cdots,\\\frac{v_3(t)}{v_0}=&1-\frac{1}{2n^2}\left(\frac{t}{T}-2\right)^2+\frac{1}{3n^2}\left(\frac{t}{T}-3\right)^3\\&-\frac{1}{8n^4}\left(\frac{t}{T}-4\right)^4+\frac{1}{30n^5}\left(\frac{t}{T}-5\right)^5\\&-\frac{1}{144n^6}\left(\frac{t}{T}-6\right)^6+\frac{1}{840n^7}\left(\frac{t}{T}-7\right)^7-\cdots\end{aligned} \tag{12}$$

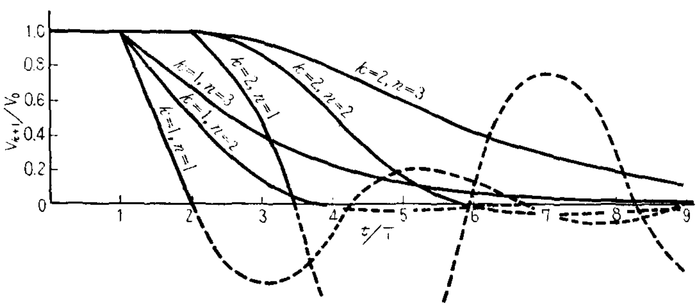

Fig. 3.

The result of indicial responses of each vehicle calculated by Eq. (12) is shown in Fig. 3. The actual responses are shown by full line in Fig. 3, since the velocity can not be negative. The responses are meaningless for the dotted line.

Replacing v in Eq. (10) by v_{start} and v in Eq. (12) by v_{stop}, we find

$$v_{2,start}(t)+v_{2,stop}(t)=v_0,\ \ v_{3,start}(t)+v_{3,stop}(t)=v_0 \tag{13}$$

These relations are self-evident if Eq. (12) is compared with Eq. (10). Hence the following relations are generally suggested,

$$v_{k+1,start}(t)+v_{k+1,stop}(t)=v_0, \qquad (k=1,\ 2,\ 3,\ \cdots) \tag{14}$$

Eq. (14) should be proved here by means of mathematical induction as follows.

Let $G(s)$ denote the transfer function of Eq. (6). Then we have

$$G(s)=\frac{e^{-Ts}}{nTs+e^{-Ts}} \tag{15}$$

Hence we obtain from Eq. (7)

$$V_{k,start}(s)=v_0[G(s)]^{k-1}/s. \tag{16}$$

If we put

$$F(s)=\frac{nT}{nTs+e^{-Ts}}+nTe^{Ts}G(s) \tag{17}$$

We have from Eq. (11)

$$V_{k,stop}(s)=v_0H_{k-1}(s), \tag{18}$$

where

$$\begin{aligned} H_{k-1}(s) &= F(s)+F^2(s)e^{-Ts}/nT+F^3(s)e^{-2Ts}/(nT)^2 \\ &\quad +\cdots+F^k(s)e^{-(k-1)Ts}/(nT)^{k-1} \\ &= nTe^{Ts}G(s)[1+G(s)+G^2(s)+\cdots+G^{k-1}(s)]. \end{aligned} \tag{19}$$

Now let us assume

$$v_{k,start}(t)+v_{k,stop}(t)=v_0 \tag{20}$$

or

$$G^{k-1}(s)+sH_{k-1}(s)=1, \tag{21}$$

Then we can prove the relation $G^k(s)+sH_k(s)=1$ as follows.

This relation becomes

$$G^k(s)+sH_k(s)=G^k(s)+nTse^{Ts}G(s)\,[1+G(s)+G^2(s)+\cdots+G^k(s)].$$

However, under the assumption of Eq. (21) we have from Eq. (19)

$$nTse^{Ts}[1+G(s)+G^2(s)+\cdots+G^k(s)]=1-G^{k-1}(s)+nTse^{Ts}.$$

Hence we find

$$\begin{aligned} G^k(s)+sH_k(s) &= G^k(s)+G(s)\,[1-G^{k-1}(s)+nTse^{Ts}] \\ &= G(s)\,[1+nTse^{Ts}] \\ &= 1. \end{aligned} \tag{22}$$

The inverse transformation of Eq. (22) becomes

$$v_{k+1,start}(t)+v_{k+1,stop}(t)=v_0. \tag{23}$$

Thus, whenever we assume Eq. (20) we can find the relation of Eq. (23). The validity of Eq. (23) for $k=1$, 2 is clear as was in Eq. (13), so the relation of Eq. (23) is generally proved for all k.

Therefore, if we know either v_{start} or v_{stop}, we can calculate another unknown.

STABILITY OF INDICIAL RESPONSE UNDER STEP FUNCTION DISTURBANCE

If the indicial response approaches its final value v_0 as time elapses, the response is stable, but that the indicial response shows a permanent continuation of cycling the response is unstable.

As the transfer function of Eq. (6) is given by

$$G(s) = \frac{e^{-Ts}}{nTs+e^{-Ts}} = \frac{e^{-Ts}/nTs}{1+e^{-Ts}/nTs}, \tag{24}$$

it is clear that the queue of traffic flow constitues a feedback system with feedback transfer function [3] e^{-Ts}/nTs as shown in Fig. 4, which is its block diagram showing a servo system.

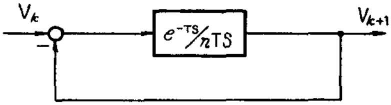

Fig. 4.

A practical and simplest method of obtaining the response is Power Series Method such as described in the foregoing paragraph, but if the stability problems are to be considered a method by characteristic roots will be more proper.

The characteristic equation of Eq. (24), that is

$$1+e^{-sT}/nTs=0$$

has the following roots whose form is now assumed;

$$S_i = -\sigma_i + j\omega_i\,; \qquad (i=1,\ 2,\ 3,\ \cdots). \tag{25}$$

Then these roots may be calculated from

$$\sigma_i = \frac{\omega_i}{\tan \omega_i T}, \quad \omega_i = \pm\sqrt{\frac{e^{2\sigma_i T}}{(nT)^2} - \sigma_i^2}. \tag{26}$$

The indicial response of the 2nd vehicle is

$$\begin{aligned} v_2(t) &= \frac{v_0}{2\pi j}\int_{c-j\infty}^{c+j\infty} \frac{e^{-Ts}e^{st}}{s(nTs+e^{-Ts})}\,ds \\ &= v_0 \sum \text{Residues of } \left[\frac{e^{-Ts}e^{st}}{s(nTs+e^{-Ts})}\right]. \end{aligned} \tag{27}$$

If we place

$$\frac{P(s)}{Q(s)}=\frac{e^{-Ts}}{s(nTs+e^{-Ts})},$$

$e^{st}P(s)/Q(s)$ behaves regularly except $s=0$ and $s=S_i$, accordingly its residue is unit at the simple pole $s=0$, and the residues at the simple poles $s=S_i$ are described by

$$e^{S_i t}P(S_i)/Q'(S_i)=[-1/(1+TS_i)]e^{S_i t},\quad (i=1,\ 2,\ 3,\ \cdots)$$

Hence we have from Eq. (27)

$$v_2(t)=v_0\Big[1+\sum_{i=1}^{\infty}\Big(\frac{-1}{1+TS_i}\Big)e^{S_i t}\Big]. \tag{28}$$

If we write

$$R_i=-1/(1+TS_i),\qquad (i=1,\ 2,\ 3,\ \cdots) \tag{29}$$

corresponding to a conjugate imaginary pair of roots: [4]

$$S_i=\sigma_i+j\omega_i,\quad \bar{S}_i=-\sigma_i-j\omega_i, \tag{30}$$

then we have

$$v_2(t)=v_0[1+2\sum|R_i|e^{-\sigma_1 t}\cos\ (\omega_i t+\arg R_i)]. \tag{31}$$

Denoting S_1 the nearest root from the imaginary axis of all the characteristic roots, the response is represented approximately by a component response of S_1, that is

$$v_2(t)=v_0[1+2|R_1|e^{-\sigma_1 t}\cos\ (\omega_1 t+\arg R_1)]. \tag{32}$$

As an example of the use of Eq. (32), we consider the responses for $n=1,\ 2$. The characteristc roots S_1 are calculated from Eq. (26) as

$$-0.318/T\pm1.338j/T,\quad -0.794/T\pm0.770j/T$$

and they correspond to the points J_1 and J_2 in Fig. 5.

Hence the indicial responses are given as follows

$$v_2(t)\begin{cases}=v_0[1+1.334e^{-0.318t/T}\cos(1.338t/T-4.24)] & \text{for } n=1\\ =v_0[1+2.51e^{-0.794t/T}\cos(0.770t/T-4.63)] & \text{for } n=2.\end{cases} \tag{33}$$

These responses for $t/T>1$ well coincide with the responses obtained by Power Series Method.

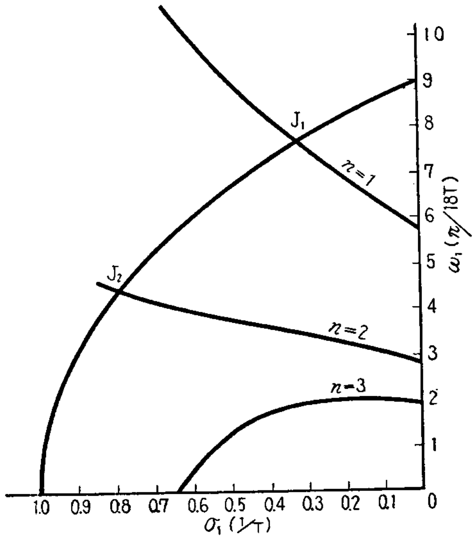

Fig. 5.

As shown in Eq. (31), the real parts of all the characteristic roots should be negative when the indicial respone of the 2nd vehicle is stable. Since $\sigma_1=0$ at the stability limit, we can derive directly from Eq. (26) the condition defining the limit of stability,

$$n=2/\pi. \tag{34}$$

On the other hand, the critical condition in which cycling disappears may be similarly derived from $\omega_1=0$, that is

$$n=e. \tag{35}$$

Next we consider about the general case. The transfer functions to the $(k+1)$th vehicle are given expilicitly as

$$G_{k+1}(s)=G^k(s)=\left(\frac{e^{-Ts}}{nTs+e^{-Ts}}\right)^k, \qquad (k=1,\ 2,\ 3,\ \cdots) \tag{36}$$

A block diagram of this servo system is shown in Fig. 6. The characteristic roots of this system are given by

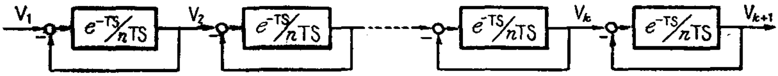

Fig. 6.

$$(nTs+e^{-Ts})^k=0$$

and are the roots of k th order of Eq. (26).

Since the indicial responses are represented by

$$\begin{aligned} v_{k+1}(t) &= \frac{v_0}{2\pi j}\int_{c-j\infty}^{c+j\infty}\frac{e^{-kTs}e^{st}}{s(nTs+e^{-Ts})^k}ds \\ &= v_0\sum \text{Residues of } \left[\frac{e^{-kTs}e^{st}}{s(nTs+e^{-Ts})^k}\right] \\ &= v_0\sum \text{Residues of } [J(s)e^{st}], \end{aligned} \tag{37}$$

we have in general

$$\begin{aligned} \frac{v_{k+1}(t)}{v_0} = 1+\sum_{i=1}^{\infty}\Bigg\{ & \frac{t^{k-1}e^{S_it}}{(k-1)!}\Big[(s-S_i)^kJ(s)\Big]_{s=S_i} \\ & +\frac{t^{k-2}e^{S_it}}{(k-2)!}\Big[\frac{d}{ds}(s-S_i)^kJ(s)\Big]_{s=S_i} \\ & +\frac{t^{k-3}e^{S_it}}{2(k-3)!}\Big[\frac{d^2}{ds^2}(s-S_k)^kJ(s)\Big]_{s=S_i} \\ & +\frac{t^{k-4}e^{S_it}}{3!(k-4)!}\Big[\frac{d^3}{ds^3}(s-S_i)^kJ(s)\Big]_{s=S_i} \\ & +\cdots+\frac{e^{S_it}}{(k-1)!}\Big[\frac{d^{k-1}}{ds^{k-1}}(s-S_i)^kJ(s)\Big]_{s=S_i}\Bigg\},\ (k=1,\ 2,\ 3,\cdots). \end{aligned} \tag{38}$$

As is known from Eq. (38), an indicial response has some component responses of the form $t^{k-1}e^{-\sigma_it}$ when the characteristic equation has the root of k th order. But so far as $\sigma_i>0$ these component responses will be damped as time elapses.

Since the indicial responses may be represented approximately in terms of S_1 as mentioned above, the stability condition in general is $n>2/\pi$, and the critical condition in which cycling vanishes is $n=e$. Hence we have the following conclusion.

If a spacing between two succssive vehicles is given by the linear form of velocity of the following vehicle, the indicial response of each vehicle in a queue of traffic is unstable for $n\leqq 2/\pi$, stable with cycling for $2/\pi<n<e$, and stable without cycling for $n\geqq e$.

Thus the queue traffic regarded as a servomechanism will be stable if and only if the characteristic equation has no poles in the right half of the s-plane or on the imaginary axis, and the stability will be lost as a characteristic root will come near the imaginary axis.

Therefore the stability of the indicial response with cycling may be evaluated by

$$\theta=\sigma_1/\sqrt{\sigma_1^2+\omega_1^2}, \tag{39}$$

which is zero for $n=2/\pi$ and unit for $n=e$.

STABILITY OF PROPAGATION OF A SINUSOIDAL DISTURBANCE

When the top vehicle in a queue of traffic was disturbed, a wave of the disturbance would travel down the queue of vehicles.

If the disturbance be a simple sinusoidal oscillation

$$a=A\sin\omega t, \tag{40}$$

then, it is possible to express a sinusoidal disturbance as the pure imaginary part of an exponential function

$$a=Ae^{j\omega t}, \tag{41}$$

but it must be noted the real solution is given by the pure imaginary part of the complex solution, devided by j.

After a sufficient time, the response of a stable system under such a sinusoidal disturbance will become sinusoidal with an altered amplitude and a phase shift. Considerations are given to the response in the steady state.

Denoting the amplitude z and the phase shift ϕ, we may write the response as

$$z=ze^{j(\omega t+\phi)}, \tag{42}$$

which is called the frequency response.

From the characteristics of frequency response, we have

$$\begin{aligned} z&=A|G_{k+1}(j\omega)|=A\{|G(j\omega)|\}^k, \\ \phi&=\tan^{-1}I_m[G_{k+1}(j\omega)]/Re[G_{k+1}(j\omega)], \end{aligned} \tag{43}$$

where

$$G_{k+1}(j\omega)=[G(j\omega)]^k=\left[\frac{e^{-j\omega T}}{jn\omega T+e^{-j\omega T}}\right]^k. \tag{44}$$

The initial sinusoidal oscillation is transmitted to the following vehicles magnified if $|G(j\omega)|>1$ and is transmitted damped if $|G(j\omega)|<1$. Hence the stability condition of the propagation is

$$|G(j\omega)|<1$$

or more explicitly

$$\left|\frac{e^{-j\omega T}}{jn\omega T+e^{-j\omega T}}\right|<1 \tag{45}$$

However since

$$\begin{aligned} |G(j\omega)|&=1/\sqrt{1+n^2\omega^2T^2-2n\omega T\sin\omega T}, \\ \arg G(j\omega)&=\tan^{-1}\frac{-n\omega T\cos\omega T}{1-n\omega T\sin\omega T}, \end{aligned} \tag{46}$$

the stability condition is given by

$$n^2\omega^2T^2-2n\omega T\sin\omega T>1$$

or

$$n>\frac{2\sin\omega T}{\omega T}. \tag{47}$$

It is clear that all the values of $\sin\omega T/\omega T$ are less than or equal to 1. Therefore if $n>2$ the propagation of a sinusoidal disturbance is stable for all ωT.

Fig. 7 shows the amplitudes of the frequency responses depending on ωT. As illustrated in Fig. 7, the amplitude of response increases as n becomes smaller and the high frequency componets of a disturbance disappear in a short period of time.

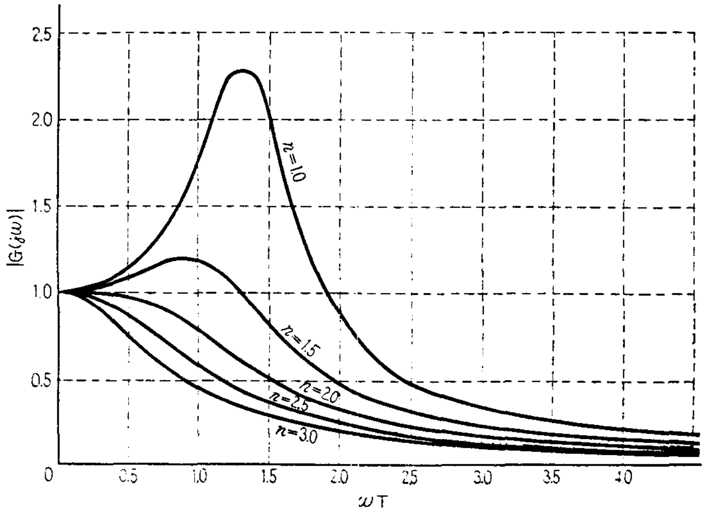

Fig. 7.

If the disturbance be no longer a simple sinusoidal oscillation, but of a non-periodic function of time, the solution is more difficult. However, such a complicated case may be reduced to the simple sinusoidal oscillation by use of Fourier theorem.

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