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A COMPUTATIONAL METHOD FOR THE TRANSPORTATION PROBLEM ON A NETWORK

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INTRODUCTION

A computational method, based on a modified method of Ford-Fulkerson's algorithm for finding maximum network flows [1] and on the duality theorem of linear programming [2], is proposed for the following transportation problem on a network: Consider a network connecting the sources and the sinks by way of a number of intermediate nodes, and suppose that the arcs can handle certain designated amounts of traffic per unit time. Further, suppose that the cost is prescribed for the unit flow through each arc, and suppose that the net flow out of each source node or the net flow into each sink node is prescribed. Assuming a steady state condition, find a traffic flow of minimum cost from the sources to the sinks.

In this paper the author will give an extension of Ford-Fulkerson's primal-dual algorithm [1],[3] for the uncapacitated or capacitated Hitchcock problem [4] to the transportation problem on a general network [5].

FORMULATION OF THE PROBLEM

A network is a finite linear graph [6], in which any two nodes are connected by a single arc. This means that the graph is complete [6]. However, this assumption will result no real restriction, because arcs of zero capacity are admissible and many arcs connecting two nodes can be reduced to an arc [7] and the definition of arc capacities undermentioned allows us to include directed arcs.

The nodes are classified into three classes: the sources $P_1, \cdots, P_l$, the intermediate nodes $P_{l+1}, \cdots, P_m$, and the sinks $P_{m+1}, \cdots, P_n$. Two nonnegative integers c_{ij} and c_{ji} are prescribed to the arc connecting P_i and P_j. $c_{ij}(c_{ji})$ represents the prescribed capacity of flow from P_i

(P_j) to $P_j(P_i)$. $d_{ij}(d_{ji})$ stands for the prescribed cost of unit flow P_i (P_j) to $P_j(P_i)$, where $d_{ij}=\infty$ if $c_{ij}=0$, otherwise d_{ij} are nonnegative integers. It is clear that the definition $d_{ij}=\infty$ for ij such as $c_{ij}=0$ results no real restriction. The net flow out of the source P_i $(i=1, \cdots, l)$ is required to be equal to a prescribed positive integer a_i, and the net flow into the sink P_j $(j=m+1, \cdots, n)$ is required to be equal to a positive integer b_j. Two sets of positive integers $(a_1, \cdots, a_l)$ and $(b_{m+1}, \cdots, b_n)$ are given to satisfy the following equality.

$$\sum_{i=1}^{l} a_i = \sum_{j=m+1}^{n} b_j = K \tag{1}$$

Needless to say, the net flow out of the intermediate node P_k $(k=l+1, \cdots, m)$ is required to be zero.

It is known [7),8)] that the above network G with many sources and sinks can be reduced to a network G' with a single source and a single sink. It is obtained as follows : Consider a network G' consisting of G plus two additional nodes P_0 and P_{n+1}, where

$$c_{0i}=a_i,\quad d_{0i}=0,\quad c_{i0}=0,\quad d_{i0}=\infty \qquad (i=1, \cdots, l)$$

$$c_{j,n+1}=b_j,\quad d_{j,n+1}=0,\quad c_{n+1,j}=0,\quad d_{n+1,j}=\infty \qquad (j=m+1, \cdots, n)$$

$$c_{0k}=c_{k0}=0,\quad d_{0k}=d_{k0}=\infty \qquad (k=l+1, \cdots, n+1)$$

$$c_{n+1,p}=c_{p,n+1}=0,\quad d_{n+1,p}=d_{p,n+1}=\infty \qquad (p=0, 1, \cdots, m)$$

In the network N', P_0 is a single source and P_{n+1} is a single sink. Furthermore, the sum $\sum_{i=1}^{l} a_i$ of the capacities of the source arcs P_0P_i $(i=1, \cdots, n+1)$ is equal to the sum $\sum_{j=m+1}^{n} b_j$ of the capacities of the sink arcs P_jP_{n+1} $(j=0, 1, \cdots, n)$.

According to the above paragraph, without loss of generality we may assume that the network G has a single source P_1 and a single sink P_n, and that the equality

$$\sum_{j=1}^{n} c_{1j} = \sum_{i=1}^{n} c_{in} = K \tag{2}$$

holds. $c_{ii}=0$ $(i=1, \cdots, n)$ by the usual convention.

If we let x_{ij} be the flow through the arc P_iP_j from P_i to P_j, the constraints for flow are represented in the following way :

$$\sum_{j=1}^{n}(x_{ij}-x_{ji}) = \begin{cases} K & (i=1) \\ -K & (i=n) \\ 0 & (i \neq 1, n) \end{cases} \tag{3}$$

$$0 \leqq x_{ij} \leqq c_{ij} \tag{4}$$

where $x_{ii}=0$ by the usual convention.

Thus the primal problem is to minimize the total transportation cost

$$\sum_{i,j} d_{ij}x_{ij} \tag{5}$$

subject to the constraints (3) and (4). In the sequel, we will call (x_{ij}) satisfying (3) and (4) to be feasible to the primal problem.

DUAL PROBLEM

The dual problem is to maximize

$$K(\alpha_1-\alpha_n)+\sum_{i,j} c_{ij}\gamma_{ij} \tag{6}$$

subject to

$$d_{ij} \geqq \alpha_i-\alpha_j+\gamma_{ij} \tag{7 a}$$

$$\gamma_{ij} \leqq 0 . \tag{7 b}$$

We will call (α_i, γ_{ij}) satisfying (7a) and (7b) to be feasible to the dual problem. Notice that a feasible solution to the dual is immediately available, *e. g.*, a feasible solution is given by $\alpha_i=\gamma_{ij}=0$.

Lemma 1. Given a feasible solution (x_{ij}), if it exists, and given a feasible solution (α_i, γ_{ij}), the following inequality necessarily holds.

$$\sum_{i,j} d_{ij}x_{ij} \geqq K(\alpha_1-\alpha_n) + \sum_{i,j} c_{ij}\gamma_{ij} \tag{8}$$

The above inequality is clear from the duality theorem. However, we will prove the lemma and derive a formula useful for later purpose.

Proof. Remembering that (x_{ij}) satisfies (3),

$$\sum_{i,j} d_{ij}x_{ij} - K(\alpha_1-\alpha_n) - \sum_{i,j} c_{ij}\gamma_{ij}$$

$$= \sum_{i,j} d_{ij}x_{ij} - \sum_{i,j} \alpha_i(x_{ij}-x_{ji}) - \sum_{i,j} c_{ij}\gamma_{ij}$$

$$= \sum_{i,j} d_{ij}x_{ij} - \sum_{i,j} \alpha_i x_{ij} + \sum_{i,j} \alpha_j x_{ij} - \sum_{i,j} \gamma_{ij}x_{ij}$$

$$+ \sum_{i,j} \gamma_{ij}x_{ij} - \sum_{i,j} c_{ij}\gamma_{ij}$$

$$= \sum_{i,j} (d_{ij}-\alpha_i+\alpha_j-\gamma_{ij})\,x_{ij} - \sum_{i,j} (c_{ij}-x_{ij})\,\gamma_{ij} \tag{9}$$

From (4) and (7), (8) follows. (Q. E. D.)

In the sequel, we shall denote the set of indices $i=1, \cdots, n$ by N, and the set of ordered pairs ij by NN. A subset of N will be denoted by S, and the notations $\cup$, $\cap$ and $\subseteqq$ will be used for the set union, intersection and inclusion respectively. Complement will be denoted by barring the symbol, *e. g* the complement of S in N will be denoted by $\bar{S}$.

We will call a feasible solution to be proper if it has the property that $\gamma_{ij}<0$ implies $d_{ij}=\alpha_i-\alpha_j+\gamma_{ij}$.

For a proper feasible solution to the dual (α_j, γ_{ij}) define the sets

$$A=\{ij \in NN \mid \gamma_{ij}<0\} \tag{10 a}$$

$$B=\{ij \in NN \mid d_{ij}=\alpha_i-\alpha_j+\gamma_{ij} \text{ and } \gamma_{ij}=0\} \tag{10 b}$$

$$C=A \cup B \tag{10 c}$$

Lemma 2. If $ij \in A$, then $ji \in \overline{C}$.

Proof. If $ij \in A$, then

$$d_{ij}=\alpha_i-\alpha_j+\gamma_{ij}<\alpha_i-\alpha_j$$

since $\gamma_{ij}<0$. Therefore we obtain an inequality

$$\alpha_j-\alpha_i+\gamma_{ji}\leqq\alpha_j-\alpha_i<-d_{ij}\leqq 0\leqq d_{ji}$$

which imples $ji\in\overline{C}$. (Q. E. D.)

Lemma 3. If $c_{ij}=0$, then $ij\in\overline{C}$.

Proof. It is clear from the fact $c_{ij}=0$ implies $d_{ij}=\infty$. (Q. E. D.)

With a proper feasible solution $(\alpha_i,\ \gamma_{ij})$ we will associate a flow problem: Maximize

$$\sum_{j\in N}(x_{1j}-x_{j1}) \tag{11}$$

subject to

$$\sum_{j\in N}(x_{1j}-x_{j1})\leqq K \tag{12}$$

$$0\leqq x_{ij}\leqq c_{ij} \qquad (ij\in NN) \tag{13a}$$

$$x_{ij}=0 \qquad (ij\in\overline{C}) \tag{13b}$$

$$x_{ij}=c_{ij} \qquad (ij\in A) \tag{13c}$$

$$\sum_{j\in N}(x_{ij}-x_{ji})=0 \qquad (i\neq 1,\ n) \tag{14}$$

Lemma 4. If a maximizing solution to the flow problem yields the possibe limit $\sum_{j\in N}(x_{1j}-x_{j1})=K$, then (x_{ij}) is a minimizing solution to the primal problem.

Proof. Since $\sum_{j\in N}(x_{1j}-x_{j1})=K$, $\sum_{j\in N}(x_{nj}-x_{jn})=-K$ follows from (14) Hence (x_{ij}) is a feasible solution to the primal and the equation (9) holds.

The right hand side of (9) vanishes, since $d_{ij}-\alpha_i+\alpha_j-\gamma_{ij}>0$ implies $x_{ij}=0$ from (13b) and $\gamma_{ij}<0$ implies $c_{ij}-x_{ij}=0$ from (13c). Thus the inequality (8) holds with equality sign and it is clear from lemma 1 that (x_{ij}) is a minimizing solution to the primal. (Q. E. D.)

FLOW PROBLEM

Consider a network G' which differs from G only in the definitions of arc-capacities. The arc-capacities of G' will be defined as follows :

$$c_{ij}'=\begin{cases} c_{ij} & (ij \in C) \\ 0 & (ij \in \overline{C}) \end{cases} \tag{15}$$

Then, a maximizing solution (x_{ij}) to the flow problem (11) ~ (14) will be obtained by finding a maximum flow in the network G', where each x_{ij} such as $ij \in A$ is fixed to be equal to c_{ij}. To see this, it suffices to notice that the flow value is equal to $\sum_{j \in N} (x_{1j}-x_{j1})$.

The simplicity of the Hitchcock transportation network has allowed Ford-Fulkerson to delete the fixed flows $c_{ij} (ij \in A)$ simply. However, in our general case, the complexity of the network structure should require a somewhat different consideration. It will be found that lemmas 2 and 5 are useful for our purpose. The algorithm for finding a maximum flow, which is restricted in the sense that the flows $x_{ij}=c_{ij}$ $(ij \in A)$ are unchanged, will be a modified method of Ford-Fulkerson's by virtue of lemmas 2 and 5.

Assume that a flow (x_{ij}) satisfying (12), (13) and (14) is given. We shall introduce some auxiliary variables

$$a_{ij}=c_{ij}'-x_{ij}+x_{ji} \tag{16}$$

which satisfies the inequality

$$0 \leqq a_{ij} \tag{17}$$

by (13) and (15).

Lemma 5. If $a_{ij}>0$, then

$$\{ij,\ ji\} \cap C \neq \phi \quad \text{(void set)}$$

Proof. If $ij \in \overline{C}$, then $c_{ij}'=0$ and $x_{ij}=0$.
Therefore

$$0 < a_{ij} = c_{ij}' - x_{ij} + x_{ji} = x_{ji}$$

which implies $ji \in C$. (Q. E. D.)

We shall mention the modified algorithm for finding a restricted maximum flow in G'.

For certain values of $i=1, \cdots, n$, we shall define labels μ_j recursively as follows: Let $\mu_1=0$. For those j, such as $a_{1j}>0$ and $\{1j, j1\} \cap B \neq \phi$, define $\mu_j=1$. In general, from those i which have received labels μ_i, but which have not previously been examined, select an i and scan for all j, such as

$$a_{ij}>0, \quad \{ij, ij\} \cap B \neq \phi \tag{18}$$

and μ_j have not been defined. For those j, define $\mu_j=i$. Continue this process until μ_n have been defined, or until no further labelling may be made and μ_n have not been defined.

In the latter case which we will call case (b), the computation ends. In the former case which we will call case (a), proceed to obtain an increased flow (x'_{ij}).

In case (a), we will obtain a path connecting the source P_1 and the sink P_n

$$P_{i_0} P_{i_1} P_{i_2} \cdots P_{i_{m-1}} P_{i_m} \qquad (i_0=1, \ i_m=n)$$

such as

$$a_{i_0 i_1}>0, \ a_{i_1 i_2}>0, \ \cdots, \ a_{i_{m-1} i_m}>0.$$

Define

$$h=\min (a_{i_0 i_1}, \ a_{i_1 i_2}, \ \cdots, \ a_{i_{m-1} i_m})>0,$$

and a set

$$D=\{i_0 i_1, \ i_1 i_2, \ \cdots, \ i_{m-1} i_m, \ i_m i_{m-1}, \ \cdots \ i_2 i_1, \ i_1 i_0\}$$

We will define a new increased flow (x_{ij}') in the following way
If $i_k i_{k+1} \in B$,

$$x'_{i_k i_{k+1}} = x_{i_k i_{k+1}} + \min(h,\ c_{i_k i_{k+1}} - x_{i_k i_{k+1}})$$
$$x'_{i_{k+1} i_k} = x_{i_{k+1} i_k} - \max(0,\ h - c_{i_k i_{k+1}} + x_{i_k i_{k+1}}) \tag{19a}$$

If $i_k i_{k+1} \in \overline{B}$,

$$x'_{i_k i_{k+1}} = x_{i_k i_{k+1}}$$
$$x'_{i_{k+1} i_k} = x_{i_{k+1} i_k} - h \tag{19b}$$

If $ij \in \overline{D}$,

$$x_{ij}' = x_{ij} \tag{19c}$$

Lemma 6. (x_{ij}') is a flow satisfying (12)～(14) and

$$\sum_{\in N} (x_{1j}' - x_{j1}') = \sum_{j \in N} (x_{1j} - x_{j1}) + h \tag{20}$$

Proof. Since $A \cap D = \phi$ from (18) and lemma 2, it suffices to prove the following facts :

(I) If $i_k i_{k+1} \in B$ and $i_{k+1} i_k \in B$, then $x'_{i_{k+1} i_k} \geqq 0$.

(II) If $i_k i_{k+1} \in B$ and $i_{k+1} i_k \in \overline{C}$, then $x'_{i_{k+1} i_k} = 0$.

(III) If $i_k i_{k+1} \in \overline{C}$ and $i_{k+1} i_k \in B$, $x'_{i_{k+1} i_k} \geqq 0$.

(I) is clear from the fact that $h \leqq a_{i_k i_{k+1}} = c_{i_k i_{k+1}} - x_{i_k i_{k+1}} + x_{i_{k+1} i_k}$.
(II) is clear from the fact that $h \leqq a_{i_k i_{k+1}} = c_{i_k i_{k+1}} - x_{i_k i_{k+1}}$ since $x_{i_{k+1} i_k} = 0$.
(III) is clear from the fact that $h \leqq a_{i_k i_{k+1}} = x_{i_{k+1} i_k}$ since $c'_{i_k i_{k+1}} = x_{i_k i_{k+1}} = 0$. (Q. E. D.)

If x_{ij} are integers, *i. e.*, the flow is integral, then h is a positive integer and the flow (x'_{ij}) is also integral. Therefore the flow value increases by $h \geqq 1$ in passing from (x_{ij}) to (x'_{ij}). Since the flow value cannot be increased indefinitely by (12), we shall obtain case (b) after a finite number of iterations of the above procedure.

Lemma 7. In case (b), (x_{ij}) is a restricted maximum flow.

Proof. Define the set

$$S = \{i \in N \mid P_i \text{ has been labelled}\}$$

Then $1\in S$, $n\in\bar{S}$, and therefore $(S, \bar{S})$ forms a cut [8]. It is easily seen [8] that

$$\sum_{j\in N}(x_{1j}-x_{j1})=\sum_{ij\in S\bar{S}}(x_{ij}-x_{ji})$$

In our flow problem, where $x_{ij}=c_{ij}$ such as $ij\in A$ are fixed, the restricted maximum flow will be obtained if

$$\sum_{ij\in S\bar{S}}(x_{ij}-x_{ji})=\sum_{ij\in S\bar{S}}c'_{ij}-\sum_{kl\in \bar{S}S\cap A}c_{kl}=\sum_{ij\in S\bar{S}\cap C}c_{ij}-\sum_{kl\in \bar{S}S\cap A}c_{kl}=V \qquad (21)$$

In case (*b*) the fact that the above equality holds can be shown in the following way.

From lemma 2 it is clear that pairs $ij\in S\bar{S}$ falls into four mutually exclusive and exhaustive classes: (I) $ij\in S\bar{S}$, $ij\in A$ and $ji\in\bar{C}$. (II) $ij\in S\bar{S}$, $ij\in\bar{C}$ and $ji\in A$. (III) $ij\in S\bar{S}$, $\{ij, ji\}\cap B\neq\phi$ and $\{ij, ji\}\cap A=\phi$. (IV) $ij\in SS$ and $\{ij, ji\}\cap C=\phi$.

Case (I) $x_{ij}-x_{ji}=c_{ij}=c'_{ij}$, since $x_{ij}=c_{ij}$ and $x_{ji}=0$.

Case (II) $x_{ij}-x_{ji}=-c_{ji}$ since $c'_{ij}=0$, $x_{ij}=0$ and $x_{ji}=c_{ji}$.

Case (III) $a_{ij}=0$, since $P_i\in S$ and $P_j\in\bar{S}$. Therefore $x_{ij}-x_{ji}=c'_{ij}$ since $0=a_{ij}=c'_{ij}-x_{ij}+x_{ji}$.

Case (IV) $x_{ij}-x_{ji}=c'_{ij}$ since $c'_{ij}=c'_{ji}=0$ and $x_{ij}=x_{ji}=0$. (Q.E.D.)

We shall derive some results for later use.

Lemma 8. In case (*b*), if $ij\in\bar{S}S\cap B$, then $x_{ij}=0$.

Proof. This follows from the fact that the relations $x_{ij}>0$ and $j\in S$ would imply by (18) that $i\in S$, since $a_{ji}=c_{ji}-x_{ji}+x_{ij}\geqq x_{ij}>0$. (Q.E.D.)

Lemma 9. In case (*b*), if $ij\in\bar{S}S\cap B$, then $x_{ij}=c_{ij}$.

Proof. This follows from the fact that the relations $x_{ij}<c_{ij}$ and $i\in S$ would imply by (18) that $j\in S$, since $a_{ij}=c_{ij}-x_{ij}+x_{ji}\geqq c_{ij}-x_{ij}>0$.

NEW DUAL SOLUTION

Let us assume that a maximizing solution (x_{ij}) to the flow problem is obtained and an inequality

$$\sum_{j\in N}(x_{1j}-x_{j1})<K$$

holds. It is equivalent to

$$K-V>0 \tag{22}$$

Define new dual variables by

$$\alpha_i'=\begin{cases}\alpha_i+k & (i\in S)\\ \alpha_i & (i\in \bar{S})\end{cases} \tag{23a}$$

$$\gamma_{ij}'=\begin{cases}\gamma_{ij}-k & (ij\in S\bar{S}\cap C)\\ \gamma_{ij}+k & (ij\in \bar{S}S\cap A)\\ \gamma_{ij} & \text{(otherwise)}\end{cases} \tag{23b}$$

where

$$0<k=\min\left[\min_{ij\in S\bar{S}\cap C}(d_{ij}-\alpha_i+\alpha_j-\gamma_{ij}),\ \min_{ij\in \bar{S}S\cap A}|\gamma_{ij}|\right] \tag{23c}$$

Lemma 10. $(\alpha_i', \gamma_{ij}')$ is a proper feasible solution to the dual and

$$K(\alpha_1'-\alpha_n')+\sum_{ij\in NN}c_{ij}\gamma_{ij}'=K(\alpha_1-\alpha_n)+\sum_{ij\in NN}c_{ij}\gamma_{ij}+k(K-V)$$

Proof. Table 1 accounts for all cases. We see from the table that with k determined by (23c) $(\alpha_i', \gamma_{ij}')$ satisfies (7) and is proper.

From (23),

$$K(\alpha_1'-\alpha_n')+\sum_{i,j}c_{ij}\gamma_{ij}'-K(\alpha_1-\alpha_n)-\sum_{i,j}c_{ij}\gamma_{ij}$$

$$=k[K+\sum_{ij\in \bar{S}S\cap A}c_{ij}-\sum_{ij\in S\bar{S}\cap C}c_{ij}]$$

$$=k(K-V) \qquad \text{(Q.E.D.)}$$

With the new dual proper feasible solution $(\alpha_i', \gamma_{ij}')$, we will associate a new flow problem. Sets A, B and C for the new flow problem are designated by A', B', and C' respectively.

We shall prove the fact that (x_{ij}), which is a maximizing solution

Index sets	$\gamma_{ij}'-\gamma_{ij}$	$-\alpha_i'+\alpha_j'-\gamma_{ij}'$ $-(-\alpha_i+\alpha_j-\gamma_{ij})$
$\bar{C}\cap SS$	0	0
$\bar{C}\cap\overline{SS}$	0	0
$\bar{C}\cap S\bar{S}$	0	$-k$
$\bar{C}\cap\bar{S}S$	0	k
$A\cap SS$	0	0
$A\cap\overline{SS}$	0	0
$A\cap S\bar{S}$	$-k$	0
$A\cap\bar{S}S$	k	0
$B\cap SS$	0	0
$B\cap\overline{SS}$	0	0
$B\cap S\bar{S}$	$-k$	0
$B\cap\bar{S}S$	0	k

Table 1

to the old flow problem, is also an admissible flow to the new flow problem.

Lemma 11 If $ij\in A'$, then $x_{ij}=c_{ij}$.

Proof. If $ij\in A$, this follows from the fact that the computation leaves x_{ij} unchanged. On the other hand the table shows that any $ij\in A'$ that is not in A is in $B\cap S\bar{S}$, and the conclusion follows from lemma 9. (Q.E.D.)

Lemma 12. If $ij\in\bar{C}'$, then $x_{ij}=0$.

Proof. If $ij\in\bar{C}$, the computation does not change x_{ij}. If $ij\in C$, it follows from the table that $ij\in B\cap\bar{S}S$, and lemma 8 applies. (Q.E.D.)

Lemmas 11 and 12 give the desired

Lemma 13, The maximizing solution (x_{ij}) to the old flow problem may be taken as a starting flow for the new flow problem.

COMPUTATIONAL PROCEDURE AND A PROOF OF COVERGENCE

A computational procedure proceeds as follows : First, find a proper feasible solution (α_i, γ_{ij}) such as α_i are integers and $\gamma_{ij}=0$. The associated flow problem has zero flow as a starting flow. It should be remarked that the flow is always integral and that α_i, γ_{ij} are always integers since k is a positive integer. If a maximum flow yields the value equal to K, then (x_{ij}) is a minimizing solution to the primal. Otherwise, define new dual variables. The associated new flow problem has (x_{ij}), which is a maximum flow in the preceding flow problem, as a starting flow. Continue the process until a maximum flow which value is equal to K will be obtained.

According to lemma 10, the value of the dual objective function (6) increases by $k(K-V)\geqq 1$ in passing from a cycle to a next cycle. Therefore, it is clear from lemma 1 that the computation terminates after a finite number of iterations whenever a feasible solution to the primal problem exists. Summarizing these results, we obtain

Lemma 14. If a feasible solution to the primal problem exists, then a minimizing solution to the primal problem is obtained after a finite number of iterative computations.

Gale has given a necessary and sufficient condition for the feasibility of requirement on network flows [8]. It is very simple from the theoretical viewpoint, but it is impossible to apply his criterion to large-scale problems. Therefore we shall have to prove the convergence, *i. e.*, the fact that the computation also terminates after a finite number of iterations when the requirement is not feasible.

Termination occurs only when we shall have obtained $k=\infty$ or $\sum_{j\in N}(x_{1j}-x_{j1})=K$. But the latter case cannot occur in the infeasible case. $k=\infty$ means from lemma 10 that the dual objective function has no upper bound, and this means from lemma 1 that no feasible solution exists to the primal problem. Hence, for our purpose it suffices to prove that $k=\infty$ occurs after a finite number of iterations if the given requirement is not feasible.

$k=\infty$ is equivalent to say that

$$\bar{S}S\cap A=\phi,$$

and either

$$S\bar{S}\cap C=\phi$$

or

$$d_{ij}-\alpha_i+\alpha_j-\gamma_{ij}=\infty \quad \text{if} \quad ij\in S\bar{S}\cap\bar{C}.$$

We notice the fact that the requirement is not feasible is equivalent to the fact that the maximum flow value of G is less than K.

According to the above, let us assume that the maximum flow value of G is equal to $K'<K$.

Lemma 15. If the maximum flow value of a flow problem in a cycle is equal to the maximum value of the flow problem in the next cycle, then $S\subseteq S'$ *i. e.*, each labelled node in the preceding flow problem is also labelled in the next flow problem.

Proof. If $P_i\in S$, there exists a path

$$P_{i_0}P_{i_1}\cdots P_{i_{m-1}}P_{i_m} \qquad (i_0=1, \quad i_m=i) \tag{24a}$$

such as

$$a_{i_0i_1}>0, \ \cdots, \ a_{i_{m-1}i_m}>0 \tag{24b}$$

$$\{i_0i_1, \ i_1i_0\}\cap B\neq\phi, \ \cdots, \ \{i_{m-1}i_m, \ i_mi_{m-1}\}\cap B\neq\phi \tag{24c}$$

(24c) means that either $i_ki_{k+1}\in B$ or $i_ki_{k+1}\in\bar{C}$. It is clear from table 1 that $i_ki_{k+1}\in B'$ or $i_ki_{k+1}\in\bar{C}'$ respectively, since $i_ki_{k+1}\in B\cap SS$ or $i_ki_{k+1}\in\bar{C}\cap SS$ respectively. As the same way $i_{k+1}i_k\in B$ $(\bar{C})$, which follows from (24c), implies that $i_{k+1}i_k\in B'$ $(\bar{C}')$. Therefore the following relations hold.

$$a_{i_0i_1}=a'_{i_0i_1}>0, \ \cdots, \ a_{i_{m-1}i_m}=a'_{i_{m-1}i_m}>0 \tag{25a}$$

$$\{i_0i_1,\ i_1i_0\}\cap B'\neq\phi,\ \cdots,\ \{i_{m-1}i_m,\ i_mi_{m-1}\}\cap B'\neq\phi \tag{25b}$$

which imply $P_i\in S'$, *i. e.*, P_i will have received a label in the next flow problem. (Q.E.D.)

Lemma 16. If a flow which value K'' is less than K' is given, we shall obtain an increased flow after a finite number of iterations.

Proof. Since the value of the flow (x_{ij}) is less than K', from Ford-Fulkerson's theory [1] there is a path connecting P_1 and P_n

$$P_{i_0}P_{i_1}P_{i_2}\cdots P_{i_{m-1}}P_{i_m} \qquad (i_0=1,\ i_m=n) \tag{26}$$

such that

$$c_{i_ki_{k+1}}-x_{i_ki_{k+1}}+x_{i_{k+1}i_k}>0 \qquad (k=0,\ 1,\ \cdots,\ m-1) \tag{27}$$

The left hand side of the inequality coincides with a_{ij} only when $i_ki_{k+1}\in C$.

First notice that $i_ki_{k+1}\in\overline{A}_1$ $(k=0,\ 1,\ \cdots,\ m-1)$ because $i_ki_{k+1}\in A$ means $c_{i_ki_{k+1}}=x_{i_ki_{k+1}}$ and $x_{i_{k+1}i_k}=0$ from lemma 2.

Let us assume that the flow value could not be increased in any finite number of iterations. Then the desired contradiction would be obtained if we will have given P_n a label in a flow problem associated with a dual solution which is obtained after a finite number of iterations.

Let t be a positive integer such as

$$i_0\in S,\ \cdots,\ i_{t-1}\in S \text{ and } i_t\in\overline{S}. \tag{28}$$

Let us assume that $i_t\in S^{(q)}$ for any q which is a positive integer. From the invariance assumptiom of flow it follows that

$$i_0\in S^{(q)},\ \cdots,\ i_{t-1}\in S^{(q)} \text{ and } i_t\in\overline{S}^{(q)} \tag{29}$$

for any q, since $\{S, S',\ S'',\ \cdots,\ S^{(q)},\ \cdots\}$ is a nondecreasing sequence of sets from lemma 15.

We notice that

$$i_{t-1}i_t\in\overline{C}\cap S\overline{S} \tag{30}$$

because $i_{t-1}i_t \in B$ would imply $i_t \in S$ from the fact $a_{i_{t-1}i_t}=c_{i_{t-1}i_t}-x_{i_{t-1}i_t}+x_{i_ti_{t-1}}>0$ and the fact $\{i_{t-1}i_t,\ i_ti_{t-1}\}\cap B\neq\phi$.

It is clear from (30) that $x_{i_{t-1}i_t}=0$. Hence (27) reduces to

$$c_{i_{t-1}i_t}+x_{i_ti_{t-1}}>0 \tag{31}$$

If $c_{i_{t-1}i_t}>0$, *i*, e., $d_{i_{t-1}i_t}<\infty$, then table 1 shows that there exists a positive integer u such as $i_{t-1}i_t \in B^{(u)}$. This implies $i_t \in S^{(u)}$ and this contradicts (29).

If $c_{i_{t-1}i_t}=0$, *i. e.*, $d_{i_{t-1}i_t}=\infty$, then (31) reduces to

$$x_{i_ti_{t-1}}>0 \tag{32}$$

This implies $i_ti_{t-1} \in A\cap\bar{S}S$ because $i_ti_{t-1} \in B$ would imlpy $i_t \in S$ from the fact that $i_{t-1} \in S$, $a_{i_{t-1}i_t}=x_{i_ti_{t-1}}>0$ and $\{i_{t-1}i_t,\ i_ti_{t-1}\}\cap B\neq\phi$.

$i_ti_{t-1} \in A\cap\bar{S}S$ shows from table 1 that there exists a positive integer u' such as $i_ti_{t-1} \in B^{(u')}$. This implies $i_t \in S^{(u')}$ and this contradicts (29).

According to the above paragraph, there exists a positive integer v such as $q\geqq v$ implies $i_k \in S^{(q)}$ $(k=0, \cdots, t)$. Therefore applying mathematical induction it is easily seen that there exists a positive integer w such as

$$P_{i_m}=P_n \in S^{(w)}.$$

This means that P_n can be labelled after $(w-1)$-times iterations. This is the desired contradiction. (Q. E. D.)

Lemma 17. If the given requirement is not feasible, we shall obtain $k=\infty$ in a finite number of iterations.

Proof. We remark that an increment of flow value is not less than 1 since flows in consideration are integral. Therefore by the conclusion of lemma 16 we shall obtain the maximum flow value K' after a finite number of iterations. Hence let us assume that we are given a flow (x_{ij}), which value is equal to K', associated with a dual solution $(\alpha_i,\ \gamma_{ij})$.

Let us assume that $k=\infty$ does not occur in any finte number of iterations. Then the flow value is always K' in any cycle hereafter, and we shall obtain a nondecreasing sequence of sets

$$S \subseteqq S' \subseteqq S'' \subseteqq \cdots \subseteqq S^{(q)} = S^{(q+1)} = \cdots \tag{33}$$

from lemma 15 and the finiteness of the network G. Here $S^{(q)} = S^{(q+r)}$ for any positive integer r.

We shall show that

$$\overline{S^{(q)}} S^{(q)} \cap A^{(q)} = \phi \tag{34}$$

and either

$$S^{(q)} \overline{S^{(q)}} \cap \overline{C^{(q)}} = \phi \tag{35a}$$

or

$$d_{ij} - \alpha_i^{(q)} + \alpha^{j(q)} - \gamma_{ij)}^{q)} = \infty \quad \text{if} \quad ij \in S^{(q)} \overline{S^{(q)}} \cap \overline{C^{(q)}} \tag{35b}$$

If (34) is not satisfied, then a pair $ij \in \overline{S^{(q)}} S^{(q)} \cap A^{(q)}$ exists. From table 1 and $j \in S^{(q+r)}$ for any nonnegative integer r it is known that there exists a positive integer w such as $ij \in B^{(q+w)}$. This implies $i \in S^{(q+w)}$ since $j \in S^{(q+w)}$ and $a_{ji} = c_{ji} - x_{ji} + x_{ij} \geqq x_{ij} = c_{ij} > 0$. This means that

$$S^{(q)} \neq S^{(q+w)}$$

which contradicts (33).

If (35a) is not satisfied, then a pair $ij \in S^{(q)} \overline{S^{(q)}} \cap \overline{C^{(q)}}$ exists. Furthermore if $c_{ij} > 0$ for the pair ij, then from $d_{ij} < \infty$ and table 1 it is known that there exists a positive intgeer w such as $ij \in B^{(q+w)}$. This implies $j \in S^{(q+w)}$ since $i \in S^{(q+w)}$ and $a_{ij} = c_{ij} - x_{ij} + x_{ji} = c_{ij} + x_{ji} \geqq c_{ij} > 0$. This means

$$S^{(q)} \neq S^{(q+w)}$$

which contradicts (33). Therefore $c_{ij} = 0$, i. e., $d_{ij} = \infty$ and (35b) holds. (Q.E.D.)

CONCLUSIONS

The problem treated in this paper is a linear programming problem, but the usual linear programming approach will be inefficient. Alternative approach is either a primal-dual method presented here or a tree-simplex method. The author has used a primal-dual method owing to Dantzig-Ford-Fulkerson [1), 3), 9)] though Watanabe [5)] has used a tree-simplex method owing to Hitchcock-Koopmans-Flood [4), 10), 11)].*

The present method will be more efficient if we could find a general method which gives a proper feasible dual solution more adequate rather than $\alpha_i = \gamma_{ij} = 0$.

There is no loss of generality in assuming that c_{ij} (d_{ij}) are integers rather than rational numbers, since the problem is essentially unchanged if c_{ij} (d_{ij}) is replaced by cc_{ij} (dd_{ij}), where c (d) is any positive integer. The effect of irrationality of prescribed constants is a possible lack of convergence of the iterative process. However such a consideration is not of importance in the usual applications.

Careful reading will show that the method is also applicable to the case where some of the capacities of arcs excluding source arcs and sink arcs are infinite.

It is easy to see that a little modification of the method gives an algorithm for finding the path of minimum cost between any two distinct points of a network. However in such a case, the computational method will be much simpler than the transportation problem on a network discussed above.

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