J. Op. Res. Japan, Vol. 1, No. 3, 1958.

DETERMINATION OF THE OPTIMUM NUMBER OF SEATS OF A. PASSENGER TRANSPORT PLANE

JIRO KONDO

University of Tokyo

(Received February 7, 1958)

INTRODUCTION

In recent years, we are *able* to design a large transport airplane. But it is not always economical to *use* a large plane for an air route having only a small traffic density. Therefore, "How many may be the most suitable number of passengers' seats for a certain air-route?" is a problem to be studied by Operations Research.

The purpose of this paper is to provide an airline company and/or aircraft manufacturer with a general method of making decision on the number of passengers' seats of a transport airplane. The number of seats must be determined so as to make maximum the profit of an airline company. This profit is defined as the surplus of income or passengers' fare over the operating cost which depends on the number of seats.

Among the performance characteristics of a transport airplane, the cruising speed, the range and the take-off distance are closely related to the block time, the stage length and the landing field of a particular airport; and these are specified by the policy of the airline company or by the aircraft designer.

The gross weight of a plane is also related to these performance on one hand and closely connected to the number of passengers'seats on the other hand.

There have been several methods published for the estimation of the direct operating costs.[1]

The expectation of profit for a company is studied. Assuming the probability density of passengers by some mathematical model, the optimum numbers are obtained for various values of the cost-profit ratio.

The method is an extension of the famous newsboy problem.[2)]

SYMBOLS

Symbol	Definition	Unit
a	net profit per one passenger carried	yen/passenger, flight
b	direct operating cost per hour	yen/hour
c	direct operating cost per one flight	yen/flight
g_0	the part of gross weight independent of the number of passengers' seats	Kg
$k_1=\frac{\partial b}{\partial G}$	rate of increase of direct operating cost due to the unit increase of gross weight	yen/hour, Kg
$k_2=\frac{\partial b}{\partial V_b}$	rate of increase of direct operating cost due to the unit increase of block speed	yen/hour, Km/h yen/Km
k_3	the part of direct operating cost per hour independent of the increase of gross weight or of block speed	yen/hour
s	number of passengers' seats	
s_c	number of crew's seats	
s_*	optimum number of passengers' seats	
t	block time	
x	number of passengers proposed for a flight	
G	gross weight	Kg
L	stage length	Km
$\alpha=\frac{k_1\eta L}{V_b}$,	rate of increase of direct operation cost per one flight due to the increase of one passenger's	yen/passenger, flight

	seat	
β	the part of direct operation cost per one flight independent of the number of passengers' seats	yen/flight
η	growth factor, *i.e.* rate of increase of gross weight due to the increase of one passenger's seat	Kg/passenger
$e(P\|s)$	reduced expectation of profit	yen/flight
$E(P\|s)$	expectation of profit of the airline company when the number of passengers' seats is s	yen/flight
$p(x)$	probability density function	
$P(h)$	$\sum_0^h p(x)$ cumulative distribution function	
$P(x\|s)$	net profit of the airline company when x passengers appear and s seats are prepared	yen/flight

PROFIT ANALYSIS

The profit per one flight obtained by the airline company is $ax-bt-c$ when $x\leq s$ passengers appear and $as-bt-c$ when there are $x>s$ passengers:

$$P(x|s)=\begin{cases} ax-bt-c, & x\leq s, \\ as-bt-c, & x>s, \end{cases} \tag{1.1}$$

where a is the net profit per one passenger carried; b is the direct operating cost per hour; t is the block time; and c is the direct operating cost per one flight.

Net profit a is the difference between the fare per one passenger and the costs of refreshments and others served for one passenger during the flight.

The direct operating cost per hour b is estimated to include the following costs: pilot's and co-pilot's cost, flight engineer's cost, radio operator's cost, navigator's cost, cost of fuel and oil consumed, depreciation, and maintenance of the flight equipment. All these are accounted for one hour. Landing fee and other costs which are natural to be accounted for one flight, are all included into c.

According to the Locheed Method the direct operating cost is assumed to be

$$b=k_1G+k_2V_b+k_3 \tag{1.2}$$

where the coefficients k_1, k_2 and k_3 are constants statistically determined for several ranges of V_b and L.

The block time t is the ratio of the stage length L and the block speed V_b, and we have

$$t=L/V_b. \tag{1.3}$$

The gross weight G is composed of structural weight, weight of power plant, weight of fuel and oil, pay load and weight of fixed equipments.

Pay load is the sum of the following weights: passengers, passengers' baggages and cargo, which are estimated as 110 Kg per one passenger. The weight of fixed equipments is composed of the following: the crew, all the weight directly assignable to the crew, furnishings and services for passengers and hydraulic and electric systems, etc., which is $75s+105s_c+0.032G$. Kg where s_c represents the number of crews.[3)]

Therefore, by taking $x=s$, the gross weight is expressed as

$$G=185s+105s_c+0.032G+F(V_b,\ L,\ G). \tag{1.4}$$

where the function F is determined by the design data. This is a certain function related in a complicated fashion to block speed, stage length, fuel consumption ratio, etc. But at the stage of lay-out, the gross weight may be simply assumed as

$$G=\eta s+g_o. \tag{1.5}$$

The growth factor η is defined by

$$\eta=\frac{\partial G}{\partial s}/G=185/G. \tag{1.6}$$

Among parameters, stage length L, block speed V_b, etc. are determined by the policy of the airline company, and we shall discuss this policy-making problem on some other occasion.

We have

$$bl+c=\{k_1(\eta s+g_0)+k_2V_b+k_3\}L/V_b+C\equiv\alpha s+\beta, \tag{1.7}$$

where

$$\alpha=\frac{k_1\eta L}{V_b} \tag{1.7a}$$

and

$$\beta=(k_1g_0+k_2V_b+k_3)L/V_b+c. \tag{1.7b}$$

The α is the rate of increase of direct operation cost per one flight due to the increase of passenger's seats by unity.

Hence we have

$$P(x|s)=\begin{cases}ax-\alpha s-\beta, & x\leq s;\\ as-\alpha s-\beta, & x>s.\end{cases} \tag{1.8}$$

EXPECTATION OF PROFIT

If $p(x)$ is the probability density of passengers' frequency x, then we have the expectation of profit of the airline company for one flight when the number of passengers' seats is s as follows:

$$\begin{aligned}E(P|s)&=\sum_{x=0}^{\infty}P(x|s)p(x)\\ &=\sum_{x=0}^{s}(ax-\alpha s-\beta)p(x)+\sum_{s+1}^{\infty}(as-\alpha s-\beta)p(x)\\ &=\sum_{0}^{s}(ax-\alpha s-\beta)\,p(x)+(as-\alpha s-\beta)\Big[1-\sum_{0}^{s}p(x)\Big]\\ &=a\sum_{0}^{s}xp(x)+as\Big[1-\sum_{0}^{s}p(x)\Big]-\alpha s-\beta\\ &=a\sum_{0}^{s}xp(x)-as\sum_{0}^{s}p(x)+(a-\alpha)s-\beta.\end{aligned} \tag{2.1}$$

Optimum number of passengers' seats is the value of s giving the maximum value of $E(p|s)$.

The optimum number of passengers' seats s_* which gives the maximum profit to the airline company must satisfy the following conditions:

$$E(P|s_*) - E(P|s_*-1) \geq 0 \tag{2.2a}$$

and

$$E(P|s_*) - E(P|s_*+1) \geq 0. \tag{2.2b}$$

From (2.1),

$$E(P|s_*) - E(P|s_*-1)$$

$$= aP(s) - a\sum_0^s p(x) + (a-\alpha)$$

$$= a\{1-\alpha/a-P(s_*-1)\}. \tag{2.3}$$

We have

$$1-\alpha/a-P(s_*-1) \geq 0, \tag{2.4a}$$

and similarly,

$$1-\alpha/a-P(s_*) \leq 0. \tag{2.4b}$$

Therefore, to find the optimum s_*, for the given passenger cost-fare ratio α/a, we have simply to find the value s_* from the table of cumulative distribution of passengers' frequency which satisfies the above two conditions, (2.4a) and (2.4b).

POISSON DISTRIBTION

If the probability density law of passengers' frequency is a Poisson distribution;

$$p(x) = e^{-\lambda}\frac{\lambda^x}{x!},$$

then the cumulative distribution function is

$$P(h|\lambda)=\sum_0^h p(x)$$

$$=\sum_0^h e^{-\lambda}\frac{\lambda^x}{x!}, \tag{3.2}$$

of which the numerical value is given by the statistical table. Then we have

$$\sum_0^s xp(x)=\sum_1^s e^{-\lambda}\frac{\lambda^{x-1}}{(x-1)!}\cdot\lambda=\lambda\sum_0^{s-1}e^{-\lambda}\frac{\lambda^x}{x!}$$

$$=\lambda P(s-1|\lambda).$$

The expectation of profit is

$$E(P|s)=a\lambda P(s-1|\lambda)-asP(s|\lambda)+(a-\alpha)s-\beta. \tag{3.3}$$

The optimum number of seats s_* gives the maximum value of

$$e(P|s)=\lambda P(s-1|\lambda)-sP(s|\lambda)+(1-\alpha/a)s. \tag{3.4}$$

Table 1 and figure 1 give $e(P|s)$ for various values of s, when $\lambda=10$, $\alpha/a=0.1$, 0.5 and 0.9.

Table 1. Reduced expectation $e(P|s)$ of profit per one flight for various number of passengers' seats

s	$\alpha/a=0.1$	$\alpha/a=0.5$	$\alpha/a=0.9$
1	0.900	0.500	0.100
2	1.799	0.999	0.199
3	2.697	1.497	0.297
4	3.586	1.986	0.396
5	4.457	2.457	0.457
6	5.290	2.890	0.490
7	6.060	3.260	0.460
8	6.740	3.540	0.340
9	7.307	3.707	0.107
10	7.749	3.747	-0.251

s	$\alpha/a=0.1$	$\alpha/a=0.5$	$\alpha/a=0.9$
11	8.066	3.666	-0.734
12	8.269	3.469	-1.331
13	8.378	3.178	-2.022
14	8.413	2.813	-2.787
15	8.397	2.397	-3.603
16	8.345	1.945	-4.455
17	8.272	1.472	-5.328
18	8.187	0.987	-6.213
19	8.094	0.494	-7.106
20	7.997	-0.003	-8.003
21	7.899	-0.501	-8.901
22	7.800	-1.000	-9.800
23	7.660	-1.540	-10.740
24	7.600	-2.000	-11.600
25	7.500	-2.500	-12.500

(Poisson Distribution, $\lambda=10$).

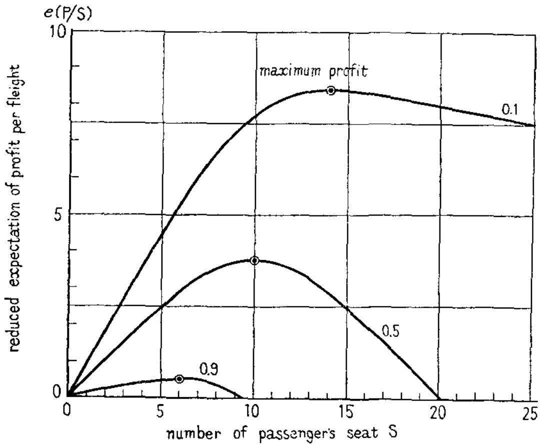

Fig. 1. Variation of profit of flight for various number of passengers' seats. $\lambda=10$, $\alpha/a=0.1$, and 0.9. (Poisson distribution)

The curve of the reduced expectation of profit per one flight $e(P|s)$ increases rapidly and then declines slowly after it reaches the maximum point when cost-profit ratio is low, while the curve of high cost-profit ratio increases steadily and falls down quickly. It is recommended for a profitable airline to use a plane having larger capacity of passengers' seats while for a line with less profit to use smaller capacity than the average number of passengers. The risks for any unsuitable operation are shown by these curves.

From (2.4a) and (2.4b), we have

$$(1-\alpha/a)-P(s_*-1|\lambda)\geq 0, \tag{3.5a}$$

and

$$(1-\alpha/a)-P(s_*|\lambda)\leq 0. \tag{3.5b}$$

Table 2. Optimum number of passengers' seats s_* for various α/a and λ.*)

α/a	λ									
	10	20	30	40	50	60	70	80	90	100
0.05	15	28	38	50	61	72	83	94	105	116
0.1	14	26	36	48	59	69	80	91	102	112
0.2	13	24	34	45	55	66	77	87	97	108
0.3	12	22	32	43	53	64	74	84	94	105
0.4	11	21	31	41	51	61	72	82	92	102
0.5	10	20	30	40	50	60	70	80	90	100
0.6	9	19	28	38	48	58	67	77	87	97
0.7	8	18	27	36	46	55	65	75	85	94
0.8	7	16	25	34	44	53	63	72	82	91
0.9	6	14	23	31	41	50	59	68	77	87
1.0	0	0	0	0	0	0	0	0	0	0

From these conditions, we have

*) Approximation by a normal curve for Poisson's distribution is adopted for λ above 30.

$$(1-\alpha/a)-P(s_*-1|\lambda)\geqq 0, \qquad (3.69)$$

and

$$(1-\alpha/a)-P(s_*|\lambda)\leq 0. \qquad (3.66)$$

The optimum values s_* for various α/a are given in Table 2, and are shown in Figures 2, 3 and 4.

ACKNOWLEDGEMENTS

The author wishes to express his appreciation to Prof. H. Nakaguchi for the estimation formulae of gross weight, to Mr. S. Kikuhara and Mr. K. Tokuda for their valuable comments and to Mr. E. Komabayashi and Dr. H. Kimura for their encouragement throughout these studies. He is also indebted to Mr. Y. Koyanagi for his cooperation during this analysis. This work is financially supported by the Transport Aircraft Development Association of Japan.

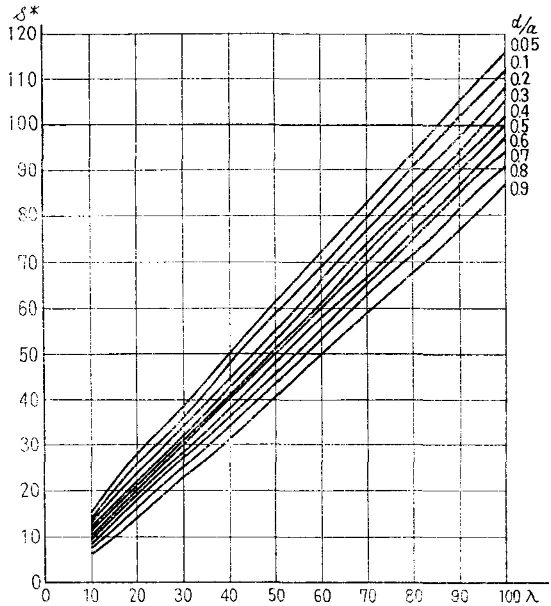

Fig. 2. Optimum number of passengers' seats s_* for various traffic densities.

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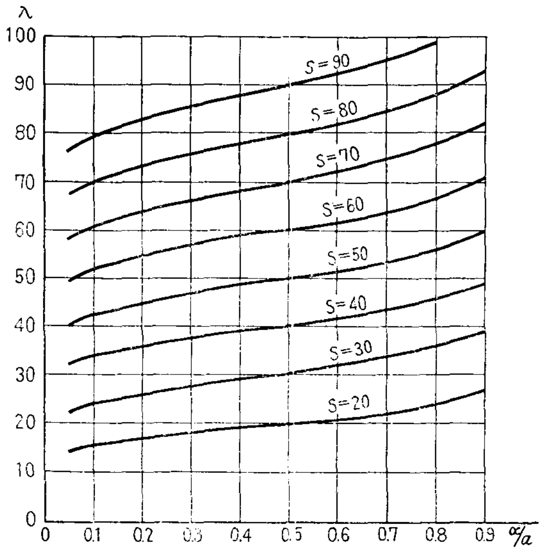

Fig. 3. Relation between λ and α/a maximizing profit of flight for various number of passengers' seats.

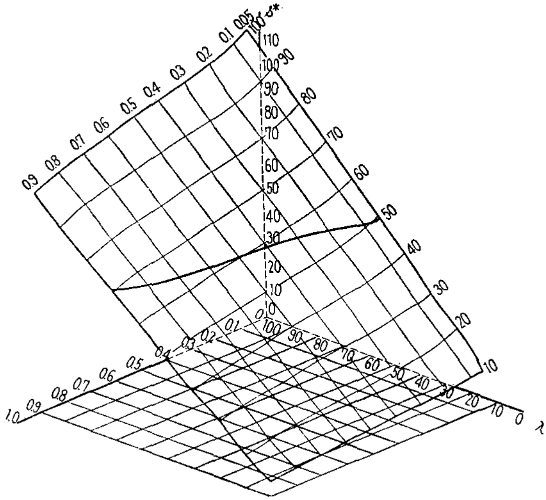

Fig. 4. Surface of optimizing number of passengers' seats s_*

$$G=\frac{A}{1+\left[(a_1+b_1fn_A)+\dfrac{a_2a_3V^3}{8813\,\rho^0/\rho_z}+\dfrac{1}{\dfrac{1}{2}+\dfrac{146}{R}\dfrac{\eta}{b}\dfrac{C_z}{C_x}}\right]}$$

G: gross weight, Kg.
A: pay load, Kg (by Cornings' estimation formula)
a_i, b_i: statistical constants
V: maximum speed, kt
ρ_0/ρ_z: air density ratio
R: range, sea miles
η: propeller efficiency
b: consumption rate of fuel oil, Kg/HP, hr
C_z/C_x: lift-drag ratio at cruising speed
n_A: load factor at A case
f: safety factor

Abstract

HIROSHI KASUGAI,* TADAMI KASEGAI: Some Considerations on Uncertainty Models with an Application to Inventory Problem,** (*Keiei-Kagaku,* 2 No. 2 (1958)).

Most of the stochastic models in today's OR, especially in inventory problem, are solved to optimize the mathematical expectation of loss, expense, or revenue. This idea to optimize the mathematical expectation is called the principle of mathematical expectation, which is one of the most famous decision criteria. The mathematical expectation, however, has a meaning only to the long run of operation, and when, it is applied to the decision of each one operation, the principle of mathematical expectation is risky. We would like to propose the other decision criterion, which is, in a game situation, to find the safety region (set) of our strategies that is able to prevent our loss within the tolerable maximum loss value, assuming that the enemy's strategies (demand of market) have a certain set (finite interval). This criterion can be regarded as a modification of the Minimax principle, but has some differences to the so called ε-Minimax solutions.

* *School of Science and Engineering, Waseda University*
** *Graduate School of Science and Engineering, Waseda University*

KATSURO MAEDA: Convertion Program of Timbers by L. P. (Trim Program), (*Keiei-Kagaku,* 2 No. 2 (1958)).

The number of available timbers and the orders of boards to be cut out of timbers are given. Adopting a certain method of cutting, a certain number of boards of several sorts are taken out of a timber. In this case, the convertion program of timbers is determined by Linear Programming so as to minimize the amount of timbers used. There are two features with repect to this problem. One is that a feasible solution can be derived easily. A program is taken as the first basic feasible solution by which all timbers are converted by a certain method of cutting. Thus a circuitous method that the first solution is formed from slack variables, can be avoided. Another feature is that the problem is a discrete one i.e. not a fractional cut but the integer solution is necessary. If the solution from Linear Programming is rounded off, it may not be feasible.

Taking several combinations of the rounded solutioms into consideration, the feasible solution is elected out of these ones. But such program is not assured to be the discrete optimal solution. Generally speaking, the rounded optimal solution is not necessarily a discrete optimum under circumstances. More theoretical studies are desirable with respect to this point.

Railway Technical Research Institute

EIICHI SUZUKI : On the Weather Service by Telephone and OR., (*Keiei-Kagaku*, **2** No. 2 (1958)).

The weather service by telephone was begun from Sep. 15. in 1954, and the more effective use of it is to be probably achieved by the application of the extended traffic and process theory and Operations Research.

In this paper, the author tried to analyse statistically and calculate several numerical examples for the following two problems.

(1). The most efficient method to increase the network circuit in telephone, was given in the direction how to gain statistic maximum benefit.

(2). In the case of call number by telephone varying with time, statistical adjustment of network circuit for each time interval under the restriction of constant cost, has to be made by the method of minimum loss of call number for a given total time interval.

As the results, we could get the ordinary fact that 460 cct (which is used in telephonic weather service today) is considered to be sufficiently suitable from the view of socially various standpoint, and the number of network circuit is to be correspondingly varied with the diurnal fluctuation of call number by telephone.

Meteorological Research Institute

JIRO KONDO : Estimation of Mortality, (*Keiei-Kagaku*, **2** No. 2 (1958)).

In this paper we consider an estimation of mortality of industrial materials (machine parts, tools etc.) from informations of the market, such as sequencial data of replacement and total quantities of consume. The fundamental equation

of this problem is the integral equation of Volterra type. The method of an analytical solution of this equation and the existence of a solution have been well established. We consider a numerical analysis of an estimation of mortality from discrete time series of replacement and consumption. Newton's interpolation formula is applied. Another problem of determination of parameters of probability distribution functions which are assumed to fit the mortality, are treated. The analysis is carried out by the operational calculus.

Tokyo University

TAKAKAZU YAMADA: A Queuing Problem in Camera Repair Service, (*Keiei-Kagaku,* **2** No. 2 (1958)).

Repair service is one of the the important fields of after services for camera makers. We should offer good services to the customers by means of repairing their cameras as fast as possible.

Therefore we should consider and make decision for the following problems:

1. If the queuing time of repair service is limited within some reasonable and constant limit by the policy of our company, how many repair workers are optimum against the variable number of cameras bringing to our repair service department by the customers?
2. If the number of repair workers were constant, how does the queuing time of repair service vary according to the variable number of cameras?
3. And then in the 2nd problem, how many cameras should be subcontracted to repair for the purpose of limiting the queuing time within some reasonable and constant limit?

We studied the types of distribution of repair service time and number of received cameras. It was known that the former was exponential distribution and the latter was Poisson distribution, and the functions of distribution were respectively

$$P(t) = P_o e^{-1.15t}$$

$$P_n = \frac{(8.19t)^n}{n!} e^{-8.19t}$$

So, let

a: number of repair workers

λ: mean number of received cameras at the repair service department/hour

μ: mean number of repaired cameras/man hour

n: total number of repairing cameras and queuing cameras

$\overline{\tau}$: mean value of queuing time

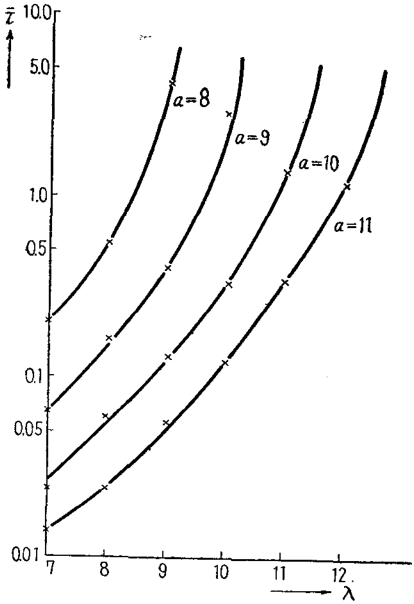

Fig. 1

We have

$$\bar{\tau}=\frac{1}{\mu(a-\lambda/\mu)^2}\cdot\frac{(\lambda/\mu)^a}{(a-1)!}P_o$$

where

$$P_o=\left[e^{\lambda/\mu}+\frac{1}{(a-\lambda/\mu)}\cdot\frac{(\lambda/\mu)^a}{(a-1)!}-\sum_{n=a}^{\infty}\frac{(\lambda/\mu)^n}{n!}\right]^{-1}$$

We took λ and a for parameter, and calculated $\bar{\tau}$ for the various values of λ and a, and then expressed the relations of $\bar{\tau}$, λ, and a graphically. (Fig. 1) Now we can readily plan and decide the above 3 problems graphically, and offer good services to the customers.

Konishiroku Photo-Industry Co., Ltd.

HIDEYUKI MAEKAWA, CHOOICHIRO : The Optimum Allocation of Newspaper Advertising Expenses, (*Keiei-Kagaku*, **1** No. 1 (1956)).

Introduction

This paper describes an optimum allocation of the fixed advertising expenses on several newspapers.

We conducted a sales drive of a multivitamin preparation by means of a prize competition by newspaper advertising. We received empty cases of the preparation in which the customer's name, address and the name of the newspaper in which the advertisement was observed were written, and we sent lot-cards in return for the empty cases. Therefore we regarded the number of empty cases as the amount of the responses.

Of course, we can not consider the response of this sales drive as the true proceeds of sales. But, at least, we can regard that the response has an influence in proportion to the proceeds approxmately.

b_j: the advertising expense spent on B_j

A_{ij}: $A_{ij} = R_{ij}/\sum_i R_{ij}$

Then, solve the problem of λ_i in order to make the expected responses of all, $\left(\frac{\sum Rij}{\lambda^*_i} \cdot \lambda_i\right)$, maximum.

Conclusion

After making an input-output table, the solution is calculated by the Simplex Method of Linear Programming. From this result, the expenses are divided into four newspapers, and remaining papers have zero expense on allocation. However, each paper has a different class of readers. Then, we reconsider the above allocations in every locality B_j, and adjust some expenses in order to be divided into the remaining papers.

Department of Q C, Shionagi & Co. Ltd.